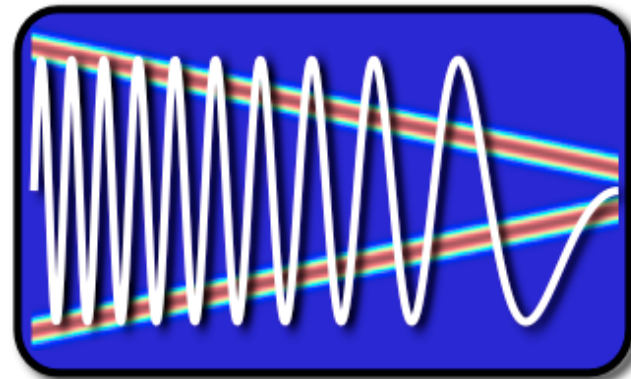


EE123



Digital Signal Processing

Lecture 15

Topics

- Last time

- Ideal Sampling model C/D
- Impulse sampling $x_c(t) \Rightarrow x_s(t)$
- Impulses to discrete samples $x_s(nT) \Rightarrow x[n]$
- Relationship $X_c(j\Omega) \Leftrightarrow X_s(j\Omega) \Leftrightarrow X(e^{j\omega})$
- Ideal reconstruction D/C

- Today

- D.T processing of C.T signals
- C.T processing of D.T signals (ha?????)
- Downsampling

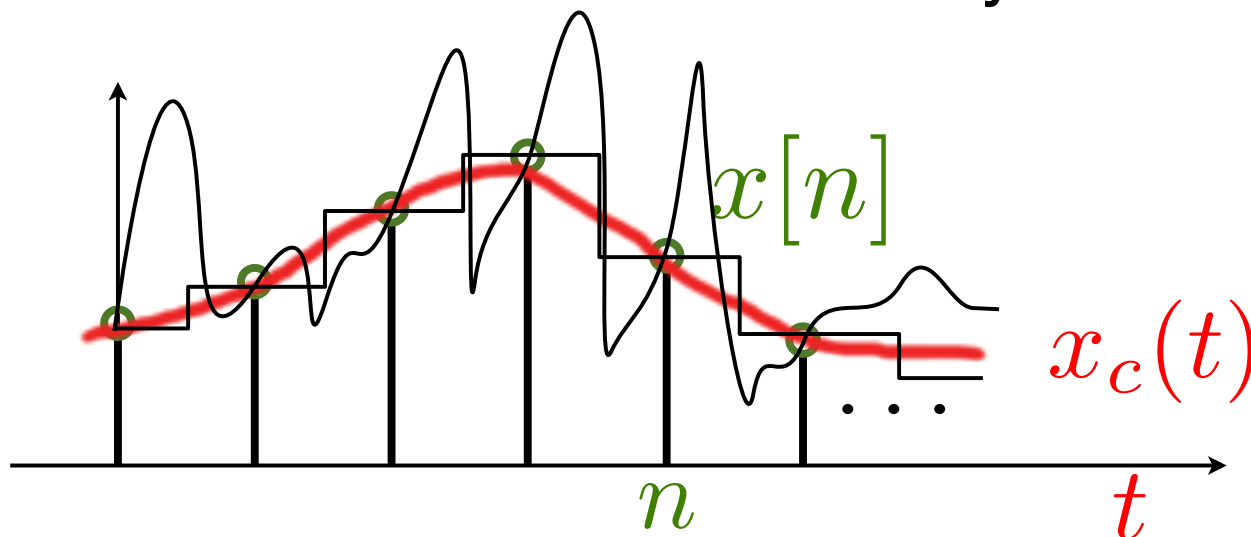
Reconstruction of Bandlimited Signals

- Nyquist Sampling Thm: suppose $x_c(t)$ is bandlimited

$$X_c(j\Omega) = 0 \quad \forall \quad |\Omega| \geq \Omega_N$$

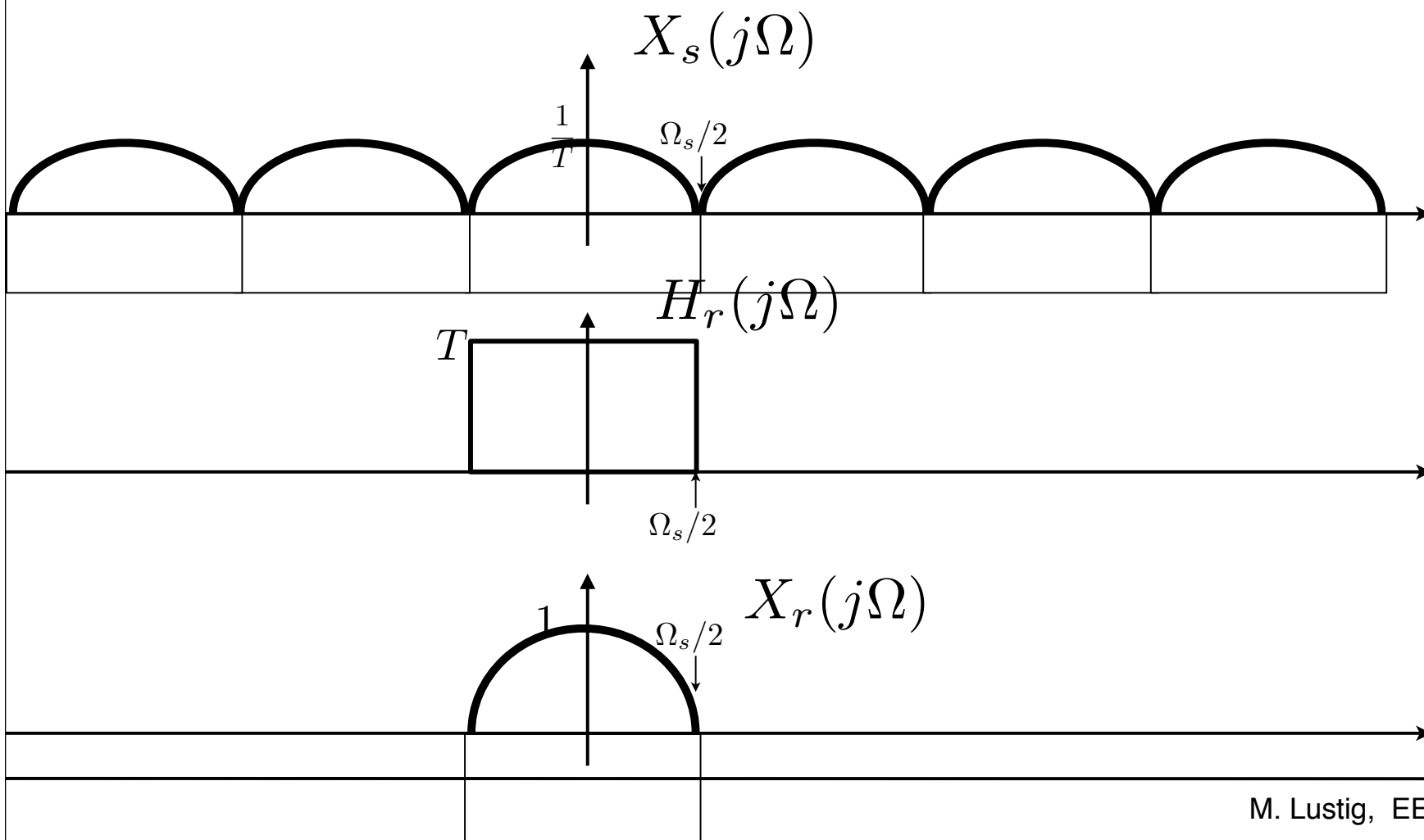
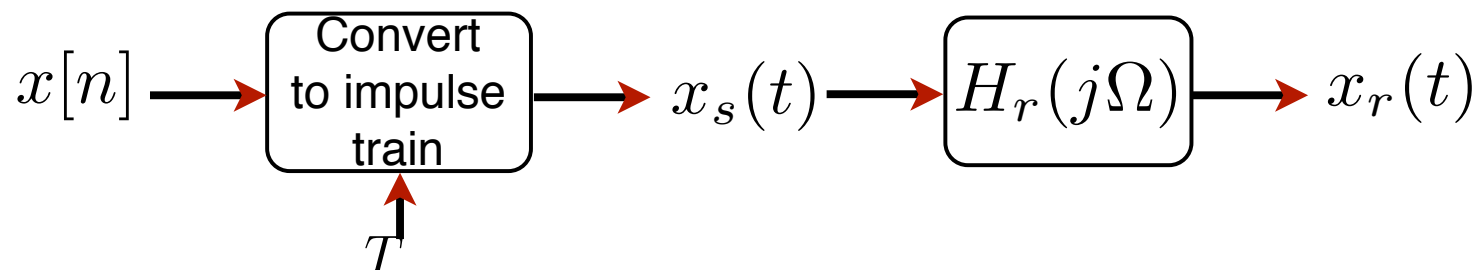
if $\Omega_s \geq 2\Omega_N$, then $x_c(t)$ can be uniquely determined from its samples $x[n] = x_c(nT)$

- Bandlimitedness is the key to uniqueness



multiple signals go through the samples, but only one is bandlimited!

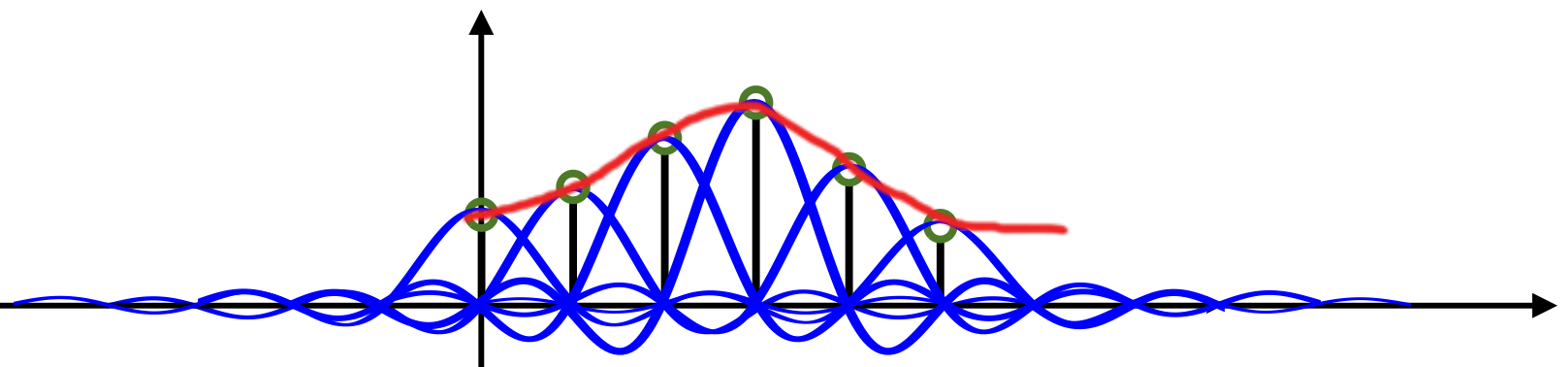
Reconstruction in Frequency Domain



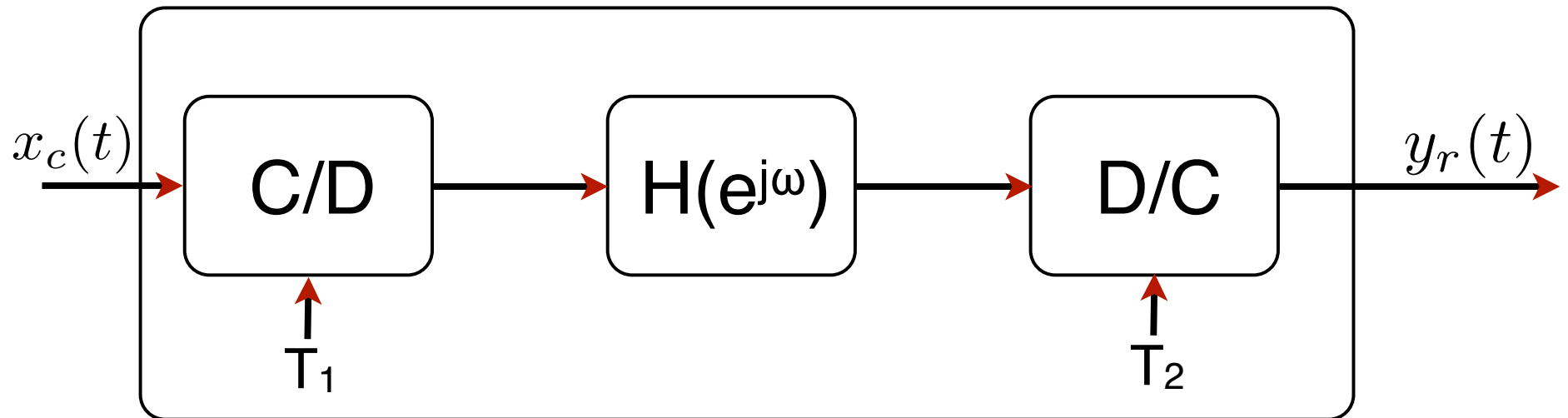
Reconstruction in Time Domain

$$\begin{aligned}x_r(t) = x_s(t) * h_r(t) &= \left(\sum_n x[n] \delta(t - nT) \right) * h_r(t) \\ &= \sum_n x[n] h(t - nT)\end{aligned}$$

The sum of “sincs” gives $x_r(t) \Rightarrow$ Unique signal
bandlimited by Ω_s

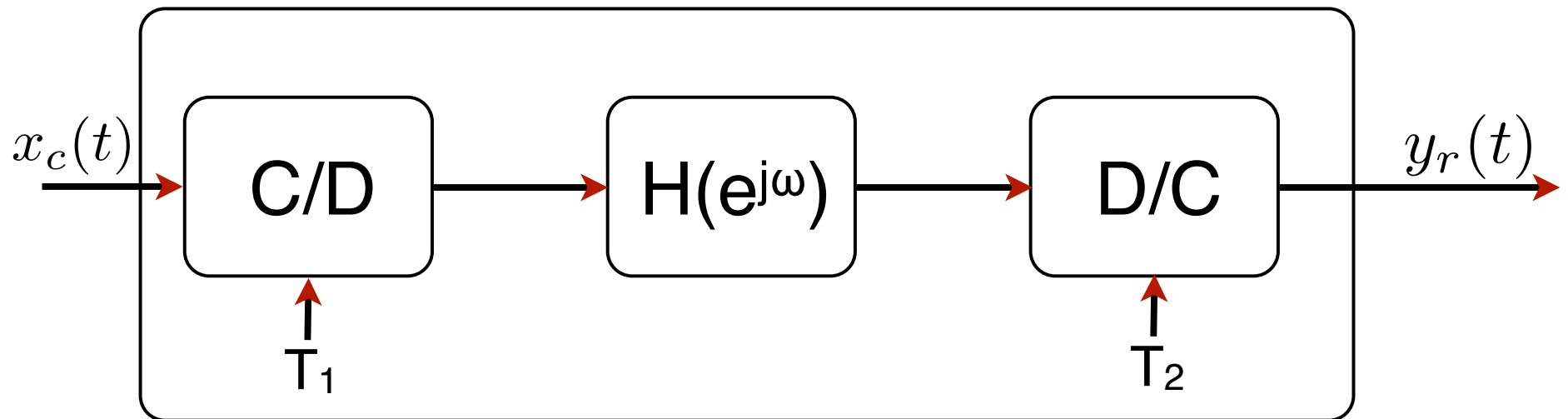


Discrete-Time Processing of C-T Signals



- Q: If $h[n]$ is LTI, $H(e^{j\omega})$ exists, Is the whole system LTI?

Discrete-Time Processing of C-T Signals

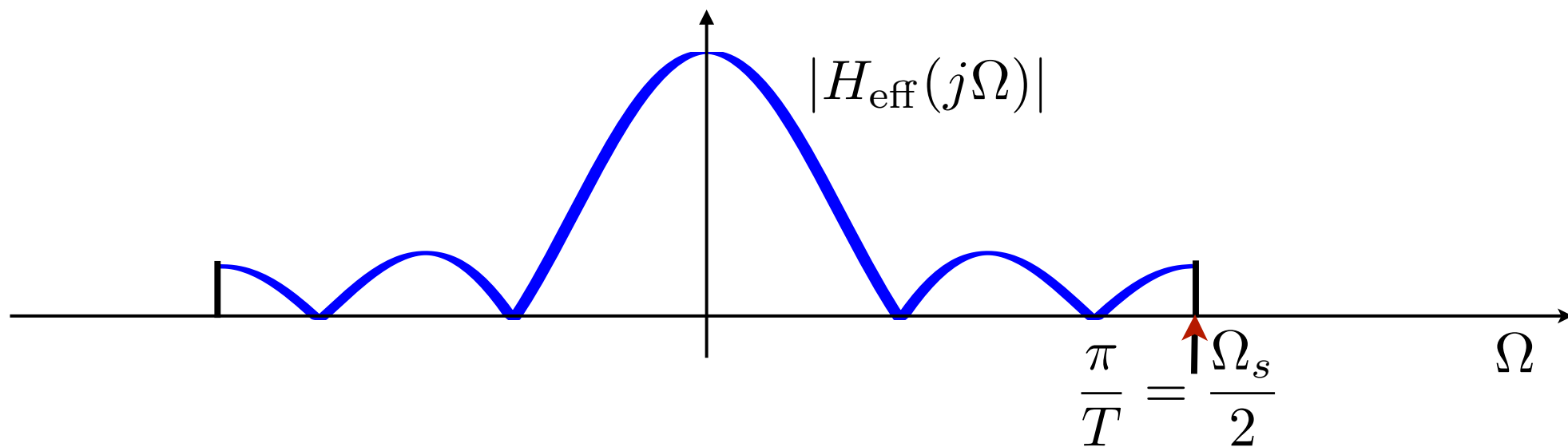
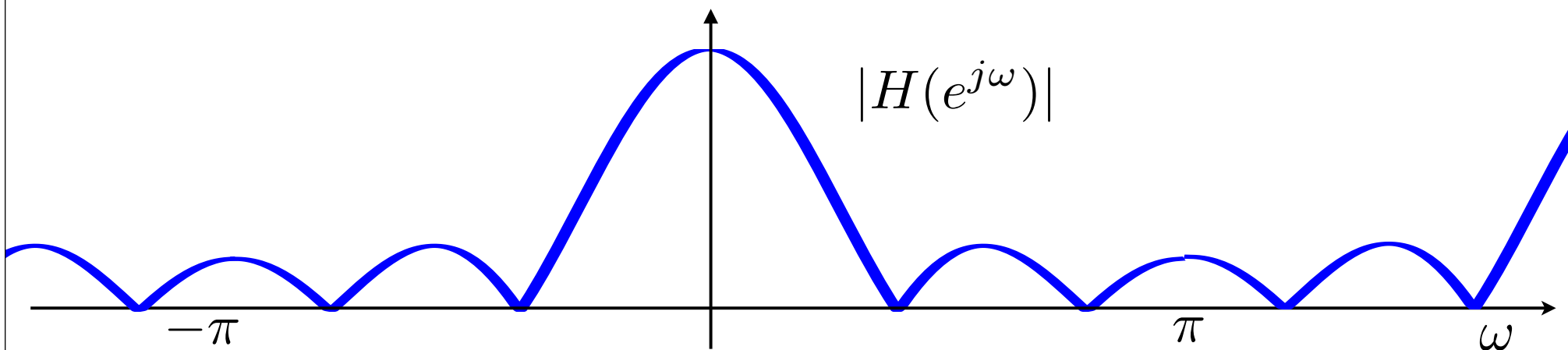


- Q: If $h[n]$ is LTI, $H(e^{j\omega})$ exists, Is the whole system LTI?
- A: If $x_c(t)$ is bandlimited by $\frac{\Omega_s}{2} = \frac{\pi}{T}$ then,

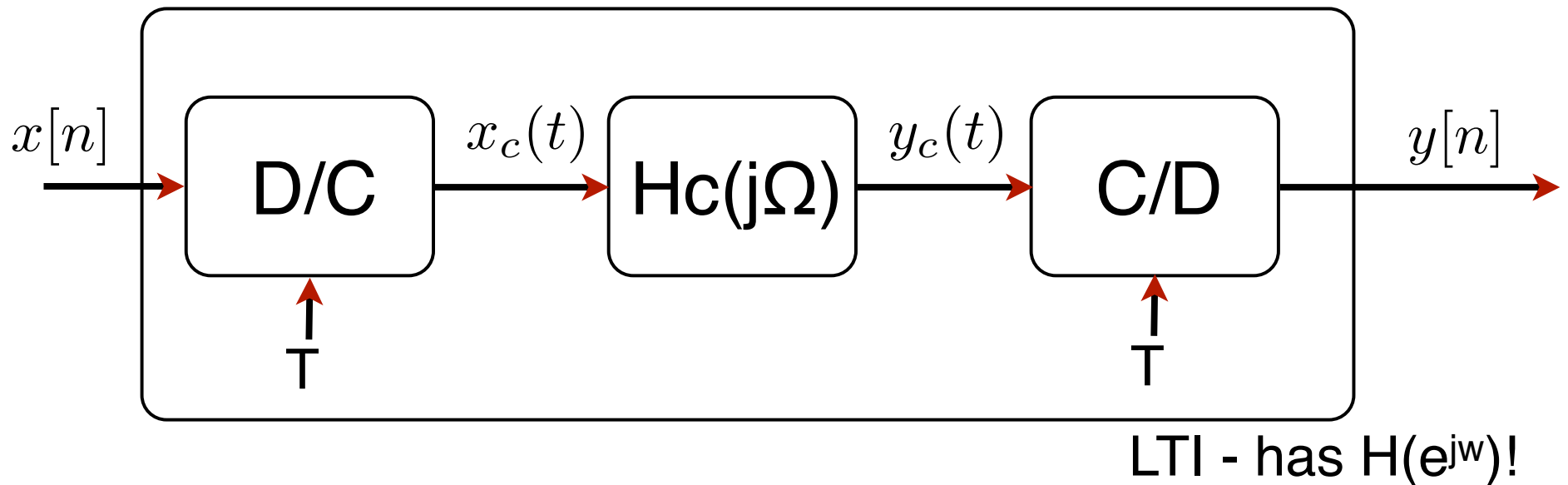
$$\frac{Y_r(j\Omega)}{X_c(j\Omega)} = H_{\text{eff}}(j\Omega) = \begin{cases} H(e^{j\omega})|_{\omega=\Omega T} & |\Omega| < \Omega_s/2 \\ 0 & \text{otherwise} \end{cases}$$

Example:

- Length 5 moving average



C.T Processing of D.T Signals



- Useful to interpret D.T. systems with no simple interpretation in discrete domain.

- Tool: recall:
$$x_c(t) = \sum_{n=-\infty}^{\infty} x[n] \operatorname{sinc}\left(\frac{t - nT}{T}\right)$$

Derivation

$$X_c(j\Omega) = \begin{cases} TX(e^{j\omega}) \Big|_{\omega=\Omega T} & |\Omega| \leq \Omega_s/2 \\ 0 & \text{otherwise} \end{cases}$$

$$Y_c(j\Omega) = H_c(j\Omega)X_c(j\Omega) \Rightarrow \text{also bandlimited}$$

so,

$$Y(e^{j\omega}) = \frac{1}{T} \sum_k Y_c(j(\Omega - k\Omega_s)) \Big|_{\Omega=\frac{\omega}{T}} = \frac{1}{T} Y_c(j\Omega) \Big|_{\Omega=\frac{\omega}{T}}$$

no aliasing!

Derivation

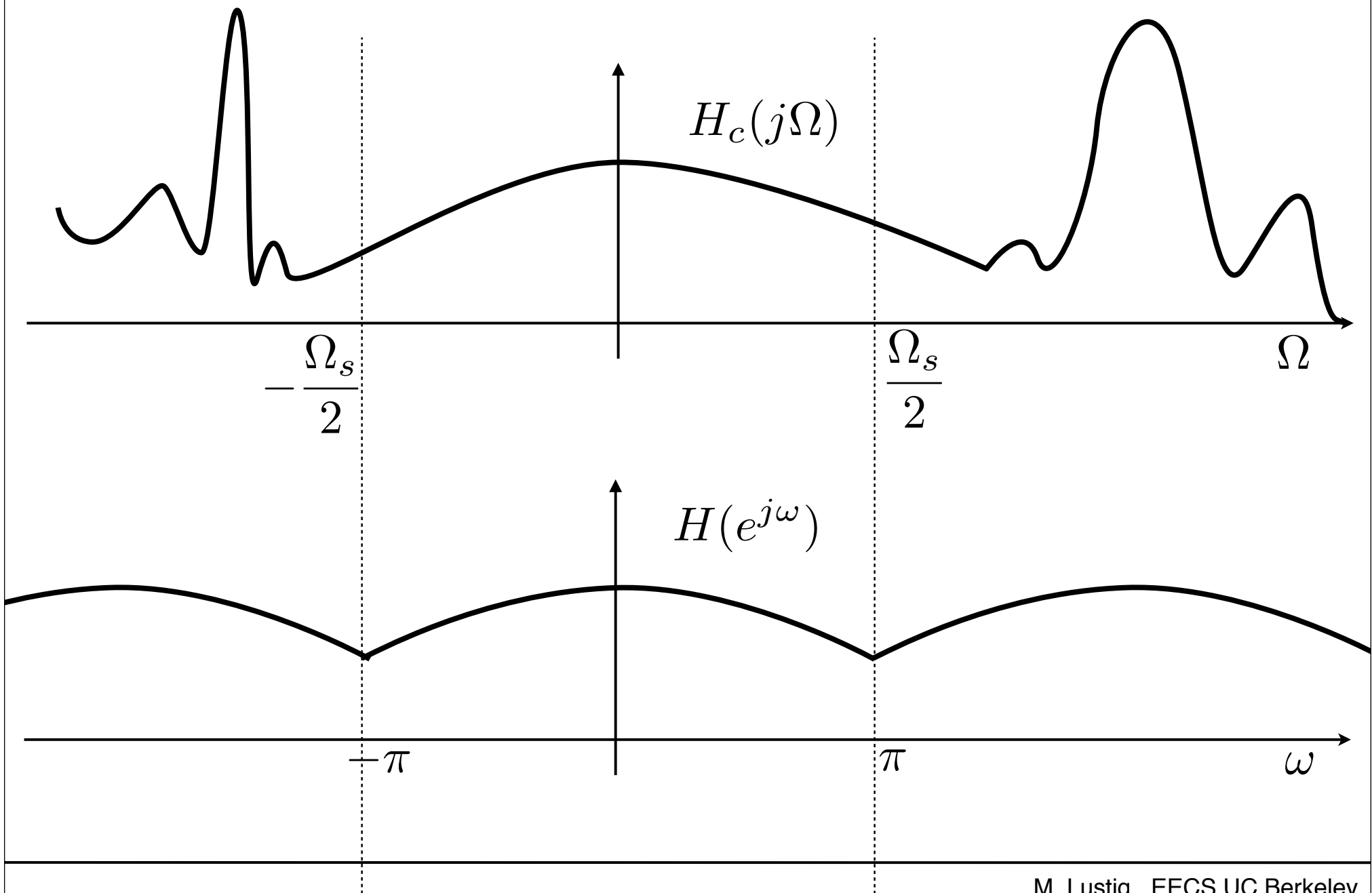
$$Y_c(j\Omega) = H_c(j\Omega)X_c(j\Omega)$$

$$Y(e^{j\omega}) = \frac{1}{T} \sum_k Y_c(j(\Omega - k\Omega_s)) \Big|_{\Omega = \frac{\omega}{T}} = \frac{1}{T} Y_c(j\Omega) \Big|_{\Omega = \frac{\omega}{T}}$$

Combining the result:

$$Y(e^{j\omega}) = \underbrace{H_c(j\Omega) \Big|_{\Omega = \frac{\omega}{T}}}_{H(e^{j\omega})} X(e^{j\omega}) \quad |\omega| < \pi$$

Example:

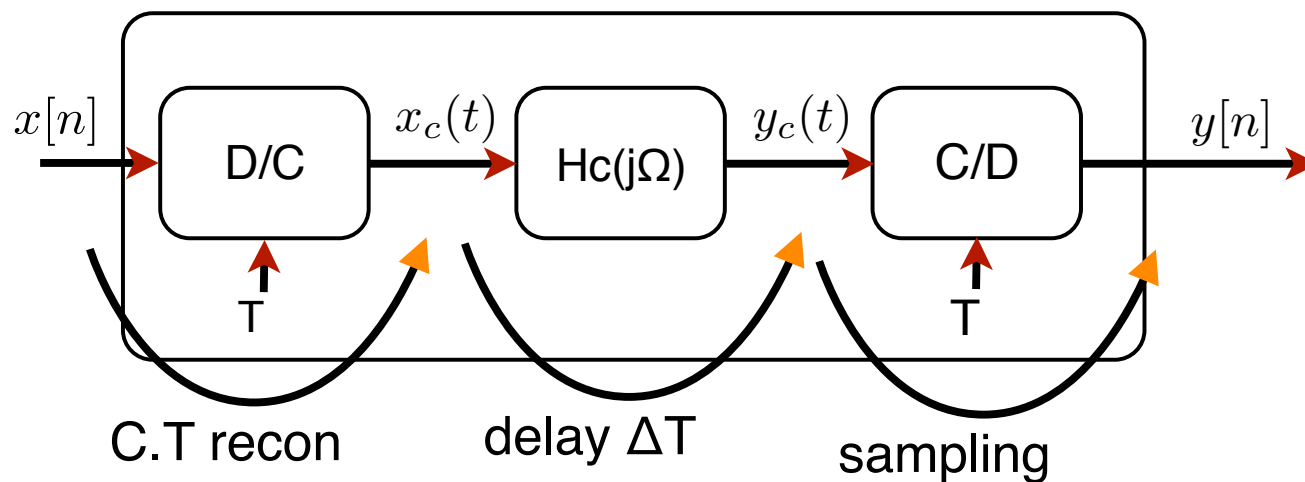


Example:

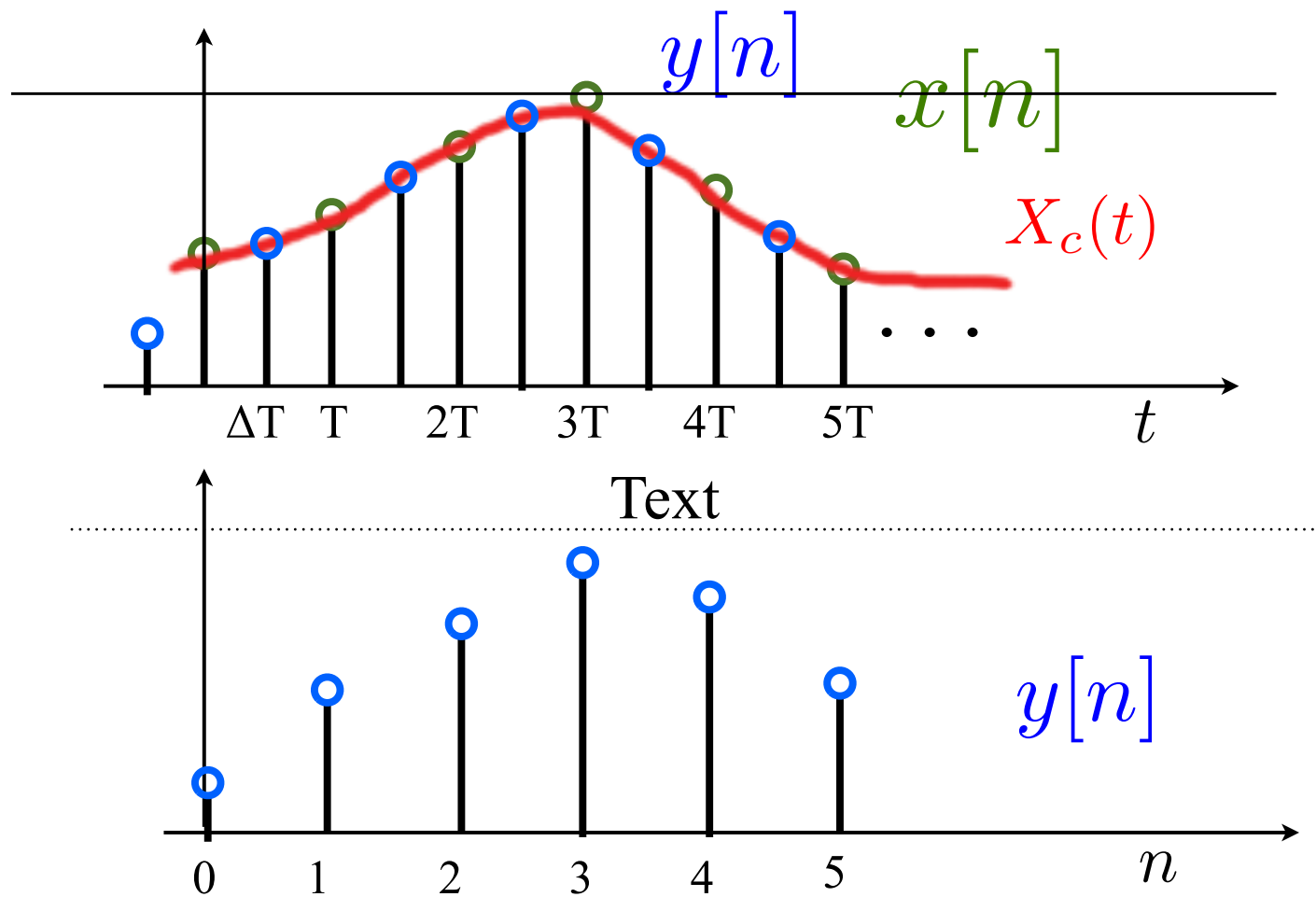
Non-integer delay: $H(e^{j\omega}) = e^{-j\omega\Delta}$

- What is the time-domain operation when Δ is not an integer ($\Delta=1/2$)?

Let: $H_c(j\Omega) = e^{-j\Omega\Delta T}$ delay of ΔT in time



Example: Non Integer Delay



Example: Non Integer Delay

- The block diagram is only for interpretation!

$$y_c(t) = x_c(t - \Delta)$$

$$\begin{aligned} y[n] &= y_c(nT) = x_c(nT - T\Delta) \\ &= \sum_k x[k] \operatorname{sinc} \left(\frac{t - kT - T\Delta}{T} \right) \Big|_{t=nT} \end{aligned}$$

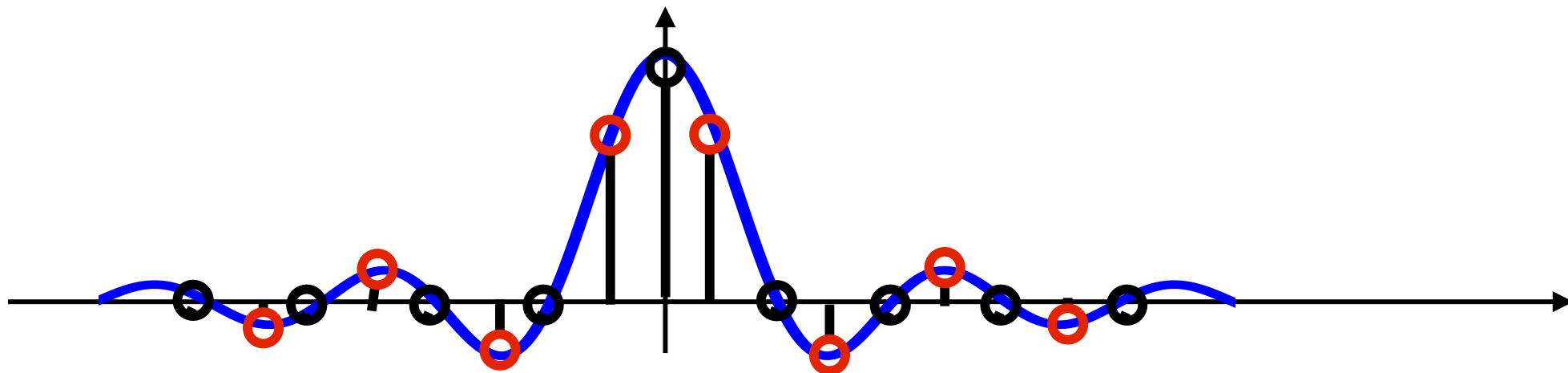
T's cancel!

$$= \sum_k x[k] \operatorname{sinc}(n - k - \Delta)$$

Example: Non Integer Delay

$$h[n] = \text{sinc}(n - \Delta)$$

Example: a discrete delta is a representation of a sampled sinc



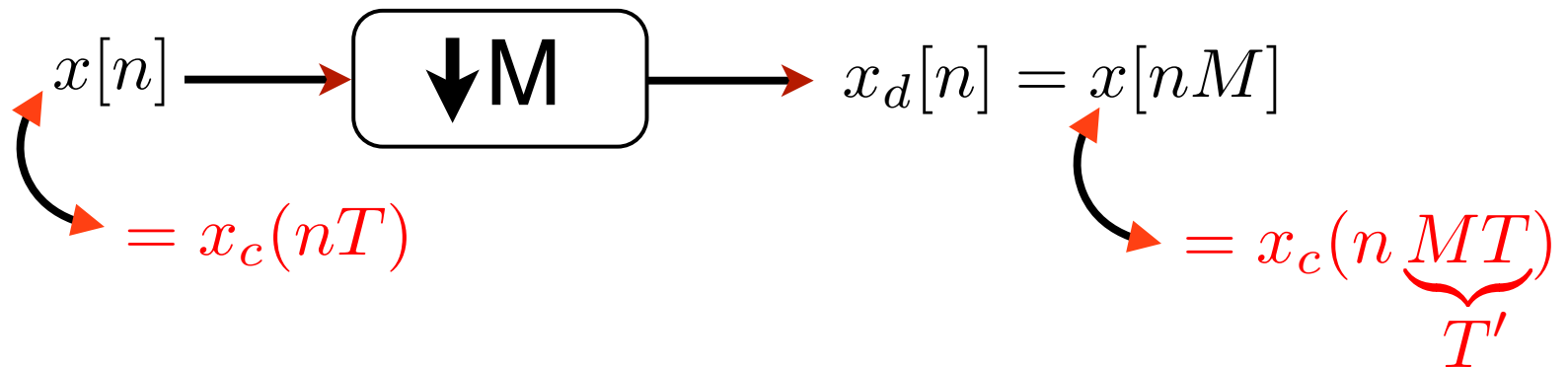
shifted by partial samples results in many coefficients!

DownSampling

- Much like C/D conversion
- Expect similar effects:
 - Aliasing
 - mitigate by antialiasing filter
- Finely sampled signal \Rightarrow almost continuous
 - Downsample in that case is like sampling!

Changing Sampling-rate via D.T Processing

Downsampling:



The DTFT:

$$X(e^{j\omega}) = \frac{1}{T} \sum_k X_c \left(j \left(\underbrace{\frac{\omega}{T}}_{\Omega} - \underbrace{\frac{2\pi}{T} k}_{\Omega_s} \right) \right)$$

$$X_d(e^{j\omega}) = \frac{1}{MT} \sum_k X_c \left(j \left(\frac{\omega}{MT} - \frac{2\pi}{MT} k \right) \right)$$

Changing Sampling-rate via D.T Processing

The DTFT:

$$X(e^{j\omega}) = \frac{1}{T} \sum_k X_c \left(j \left(\underbrace{\frac{\omega}{T}}_{\Omega} - \underbrace{\frac{2\pi}{T}}_{\Omega_s} k \right) \right)$$

$$X_d(e^{j\omega}) = \frac{1}{MT} \sum_k X_c \left(j \left(\frac{\omega}{MT} - \frac{2\pi}{MT} k \right) \right)$$

we would like to bypass X_c and go from $X(e^{j\omega}) \Rightarrow X_d(e^{j\omega})$

substitute $r = kM + i$ $i=0,1,\dots,M-1$
 $k=-\infty,\dots,\infty$

two counters

e.g., k : hours, i : minutes

Changing Sampling-rate via D.T Processing

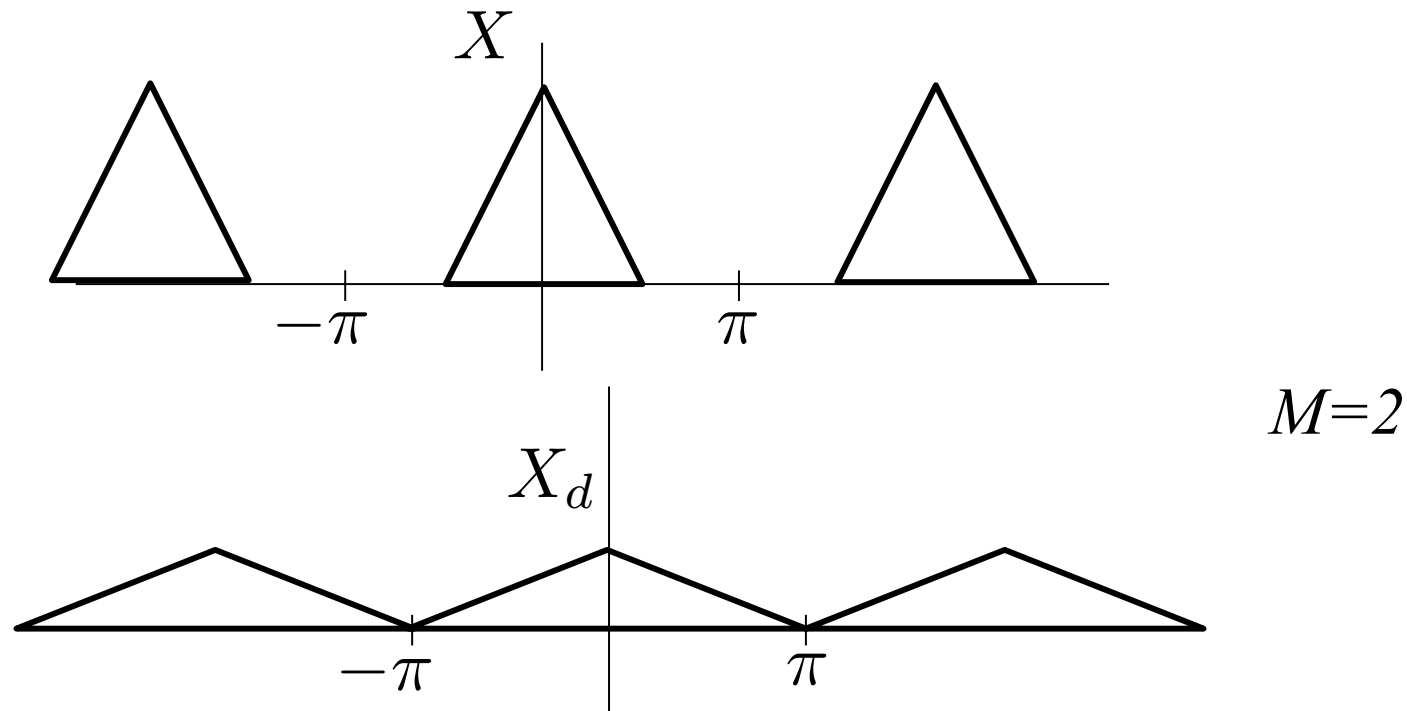
$$\begin{aligned} X_d(e^{j\omega}) &= \frac{1}{MT} \sum_k X_c \left(j \left(\frac{\omega}{MT} - \frac{2\pi}{MT} k \right) \right) \\ &= \frac{1}{M} \sum_{i=0}^{M-1} \underbrace{\frac{1}{T} \sum_{-\infty}^{\infty} X_c \left(j \left(\frac{\omega}{MT} - \frac{2\pi}{MT} i - \frac{2\pi}{T} k \right) \right)}_{X(e^{j(\frac{\omega}{M} - \frac{2\pi}{M} i)})} \end{aligned}$$

$$X_d(e^{j\omega}) = \frac{1}{M} \sum_{i=0}^{M-1} X \left(e^{j \left(\frac{\omega}{M} - \frac{2\pi}{M} i \right)} \right)$$

↑ ↑
stretch replicate
by M

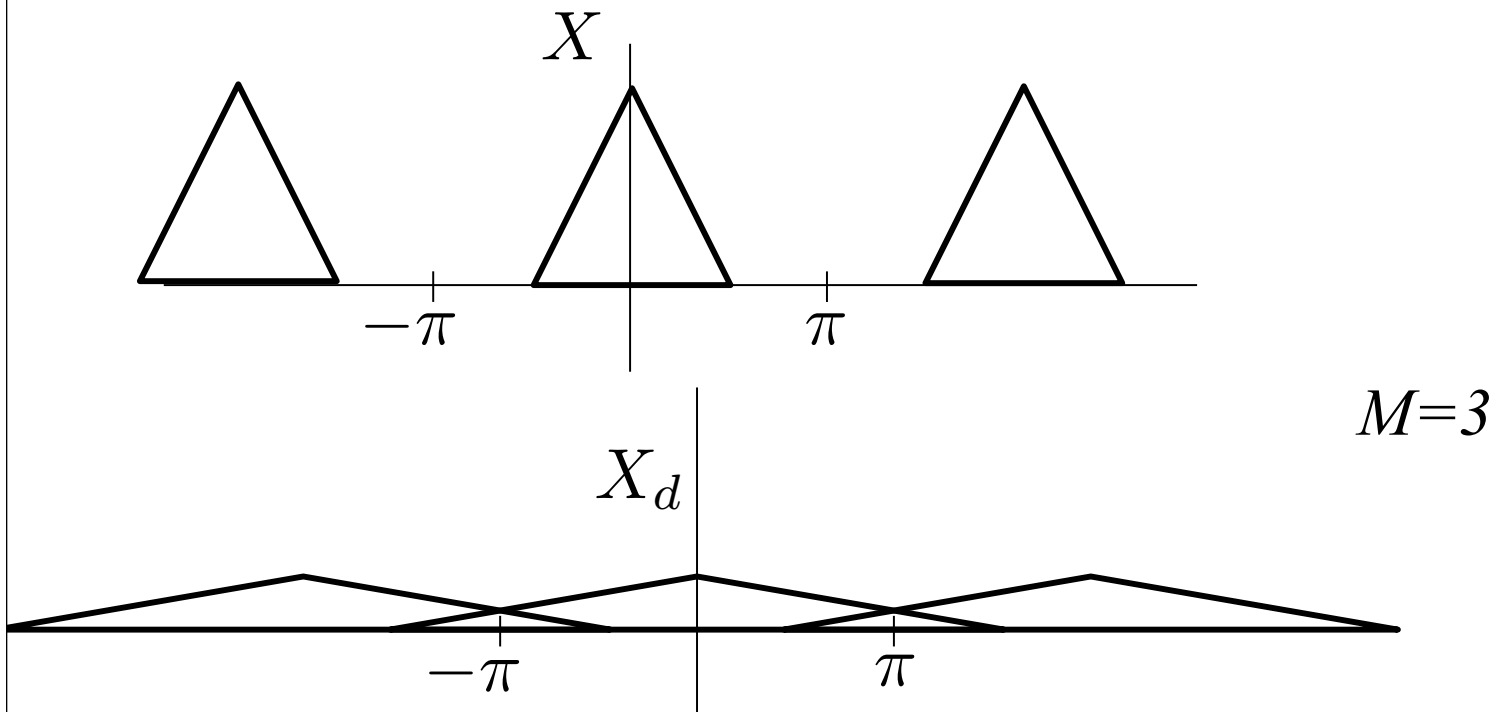
Changing Sampling-rate via D.T Processing

$$X_d(e^{j\omega}) = \frac{1}{M} \sum_{i=0}^{M-1} X \left(e^{j(\omega/M - 2\pi i/M)} \right)$$

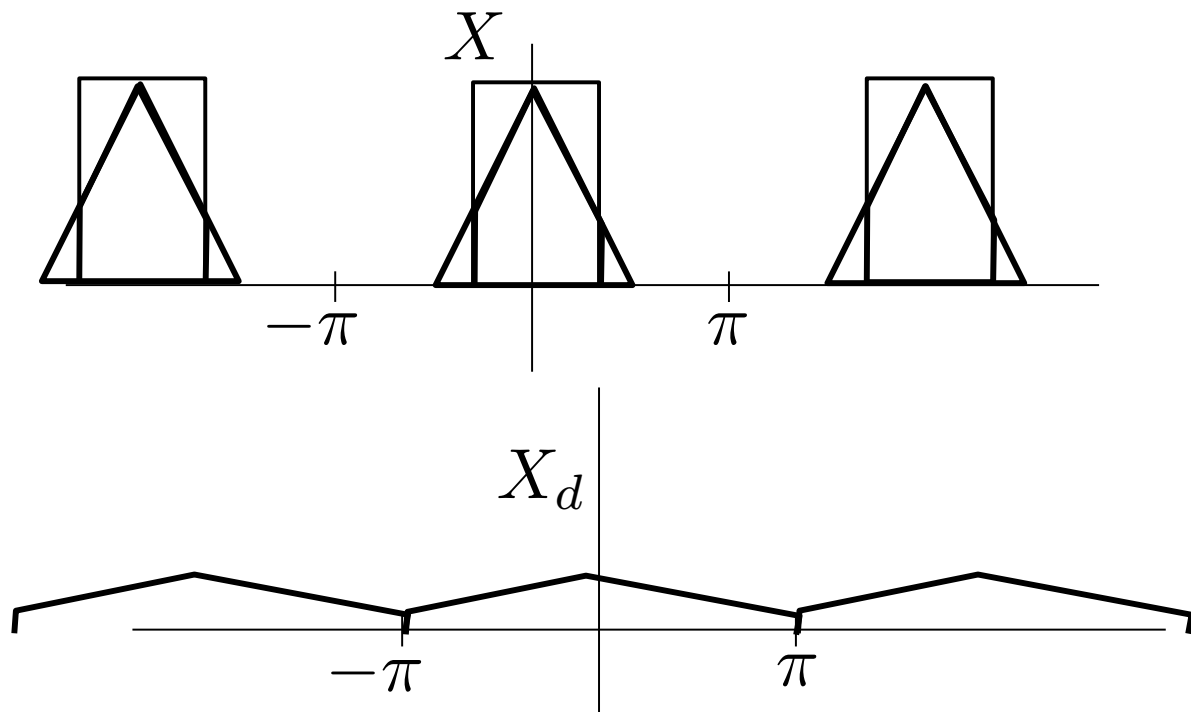
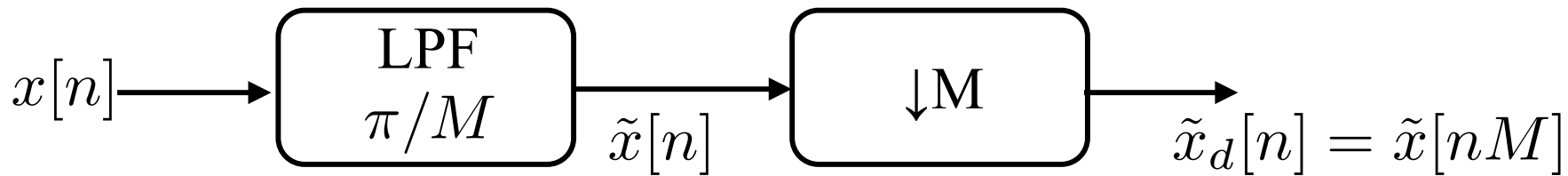


Changing Sampling-rate via D.T Processing

$$X_d(e^{j\omega}) = \frac{1}{M} \sum_{i=0}^{M-1} X \left(e^{j(\omega/M - 2\pi i/M)} \right)$$



Anti-Aliasing



$M=3$