EE123
Digital Signal Processing

Lecture 16
Resampling

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Topics

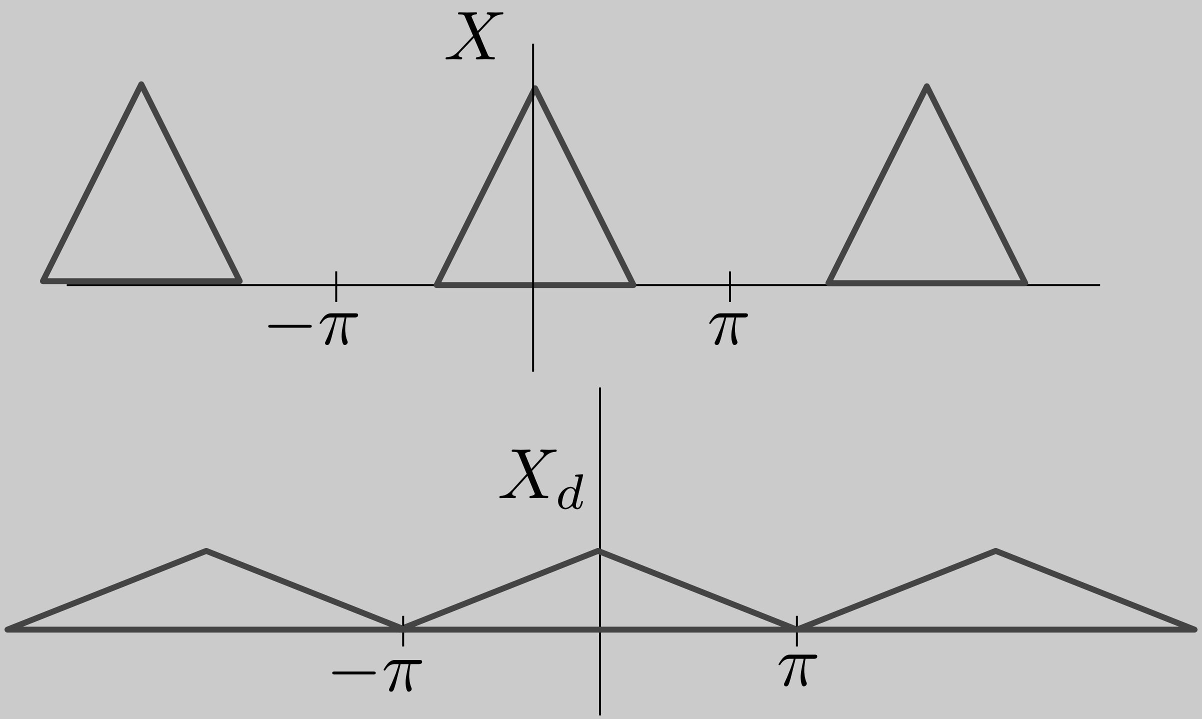
- Did you sign up for the ham exam?
- Last time
  - D.T processing of C.T signals
  - C.T processing of D.T signals (ha?????)
  - Downsampling
- Today
  - Changing Sampling Rate via DSP
  - Upsampling
  - Rational resampling
  - Interchanging operations

Review DownSampling

- Much like C/D conversion
- Expect similar effects:
  - –Aliasing
  - –mitigate by antialiasing filter
- Finely sampled signal $\Rightarrow$ almost continuous
  - –Downsample in that case is like sampling!

Changing Sampling-rate via D.T Processing

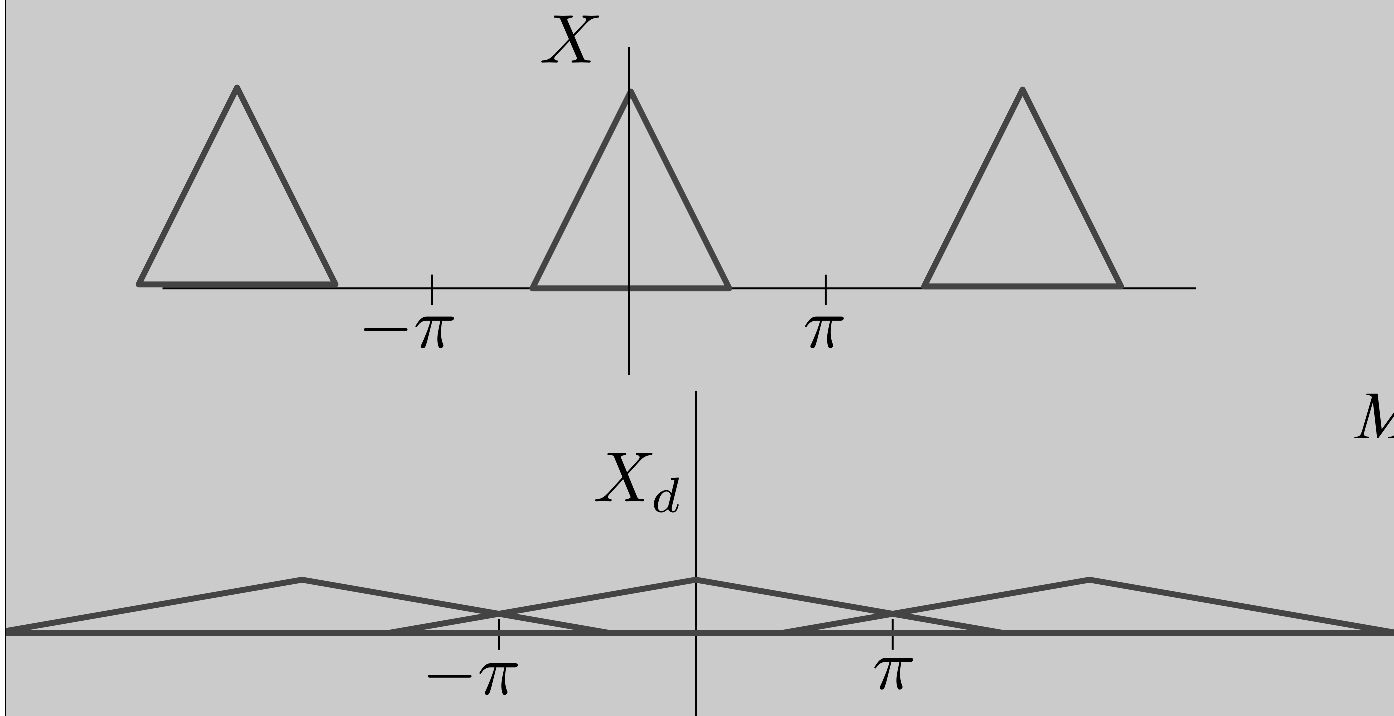
$$X_d(e^{j\omega}) = \frac{1}{M}\sum_{i=0} X\left(e^{j(w/M-2\pi i/M)}\right)$$



M=2

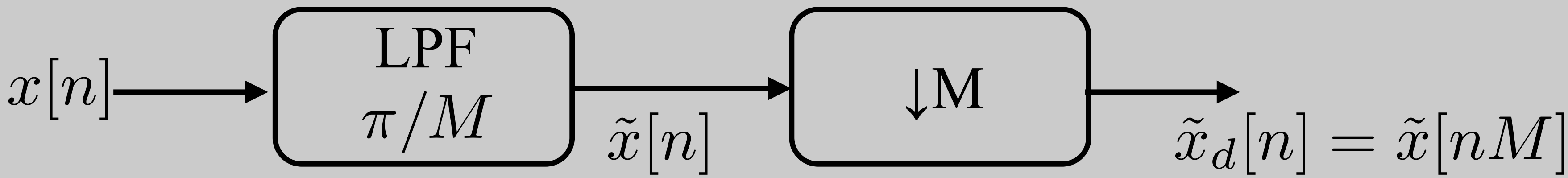
Changing Sampling-rate via D.T Processing

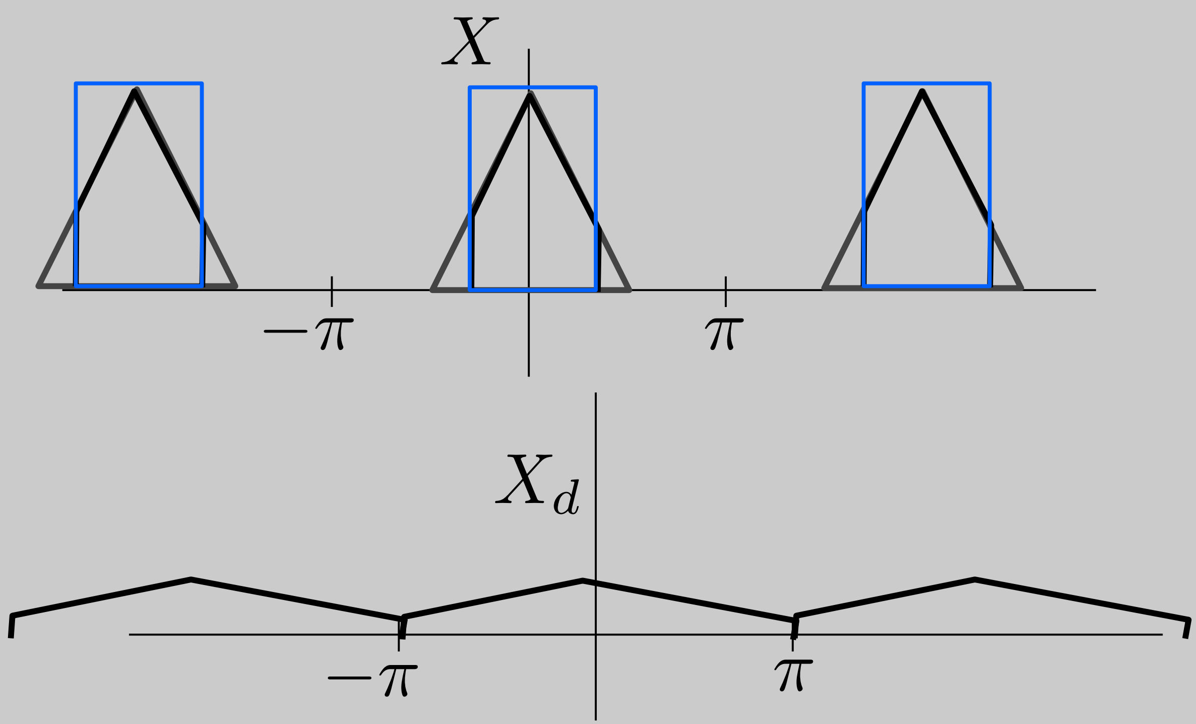
$$X_d(e^{j\omega}) = \frac{1}{M}\sum_{i=0} X\left(e^{j(w/M - 2\pi i/M)}\right)$$



M=3

Anti-Aliasing





M=3

UpSampling

- Much like D/C converter
- Upsample by A LOT $\Rightarrow$ almost continuous
- Intuition:
  - Recall our D/C model: $x[n] \Rightarrow x_s(t) \Rightarrow x_c(t)$
  - Approximate "$x_s(t)$" by placing zeros between samples
  - Convolve with a sinc to obtain "$x_c(t)$"

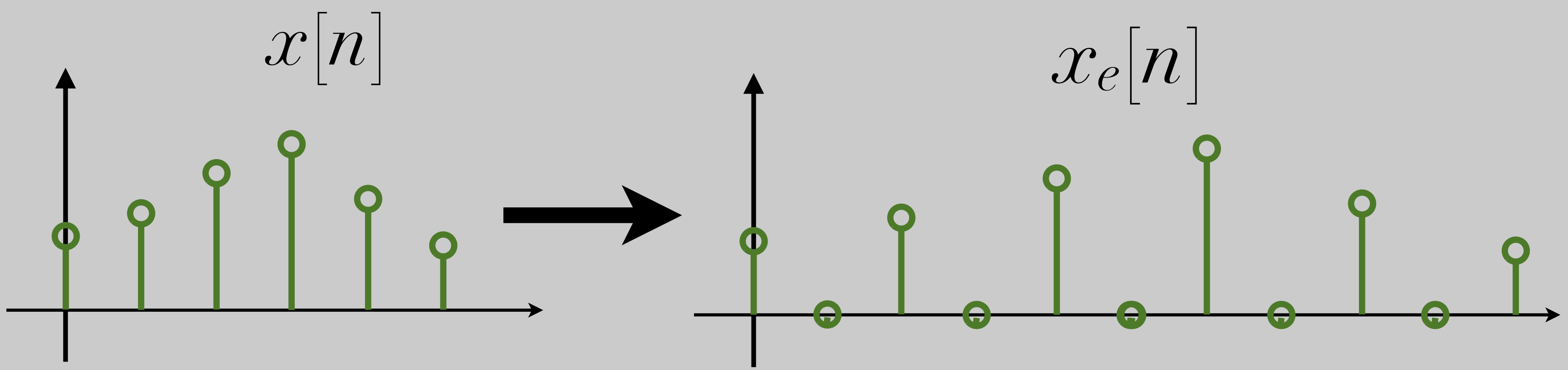
Up-sampling

$$x[n] = X_c(nT)$$

$$x_i[n] = X_c(nT') \quad \text{where } T' = \frac{T}{L} \qquad L \text{ integer}$$

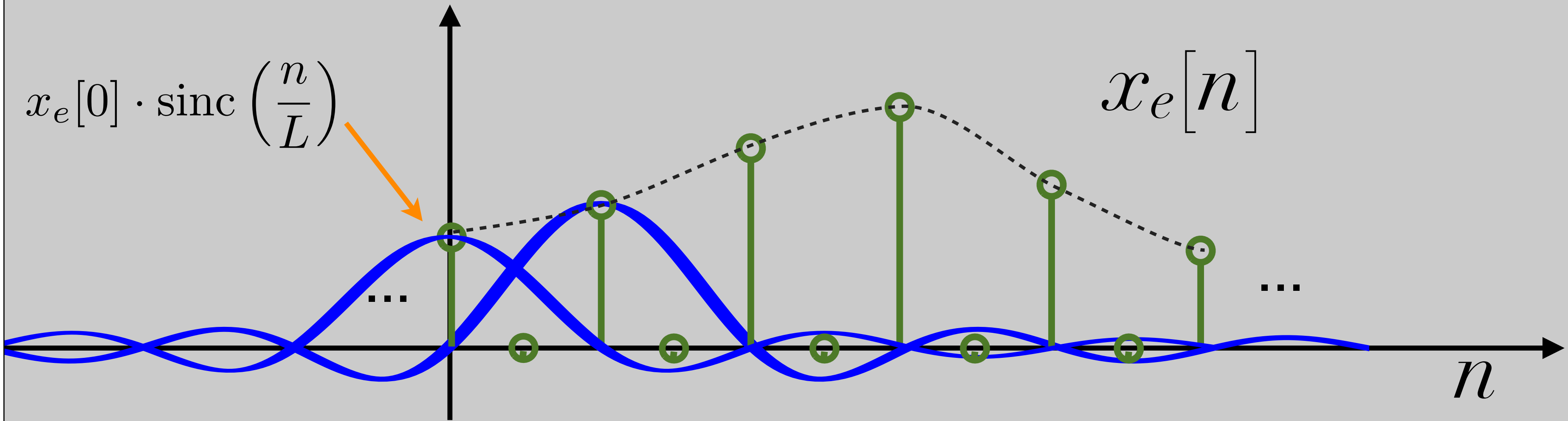
Obtain $x_i[n]$ from $x[n]$ in two steps:

(1) Generate: $$x_e = \begin{cases} x[n/L] & n = 0, \ \pm L, \ \pm 2L, \cdots \\ 0 & \text{otherwise} \end{cases}$$

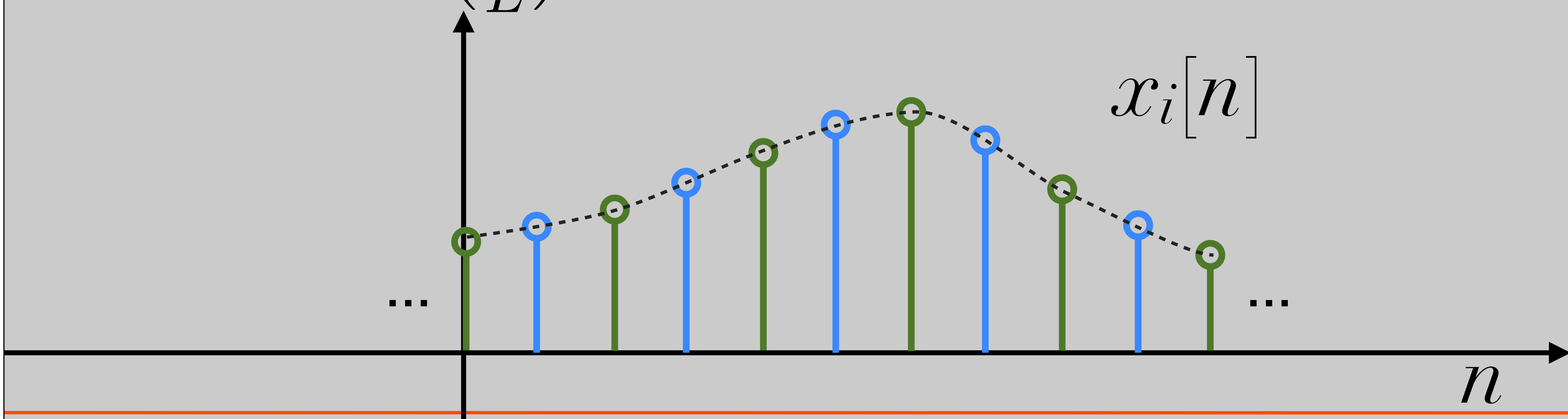


Up-Sampling

(2) Obtain $x_i[n]$ from $x_e[n]$ by bandlimited interpolation:



$$x_i = x_e[n] * \mathrm{sinc}\left(\frac{n}{L}\right)$$



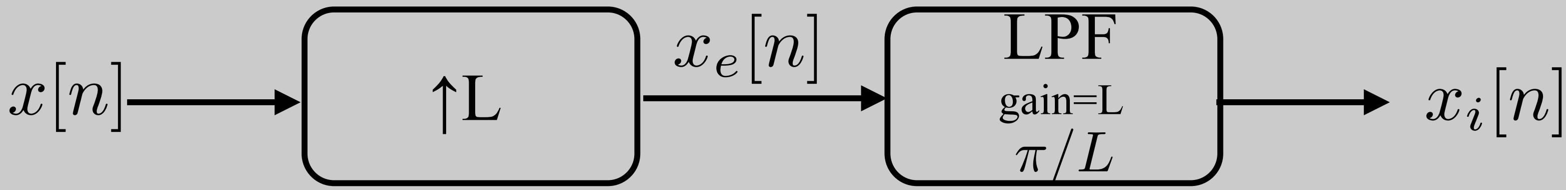
Up-Sampling

$$x_i[n] = x_e[n] * \mathrm{sinc}(n/L)$$

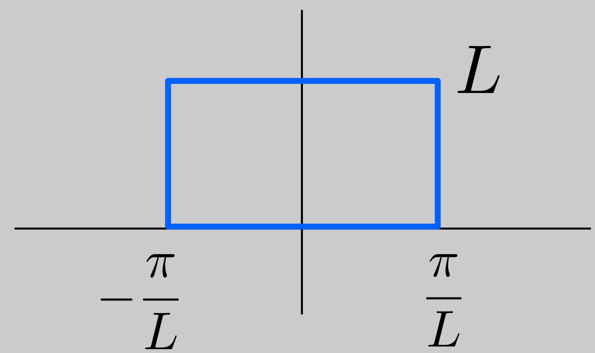
$$x_e[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-kL]$$

$$x_i[n] = \sum_{k=-\infty}^{\infty} x[k]\mathrm{sinc}(\frac{n-kL}{L})$$

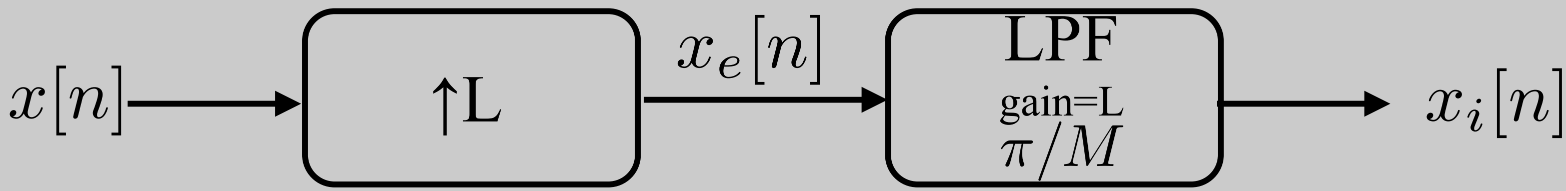
Frequency Domain Interpretation



$\text{sinc}(n/L)$ DTFT $\Rightarrow$



Frequency Domain Interpretation

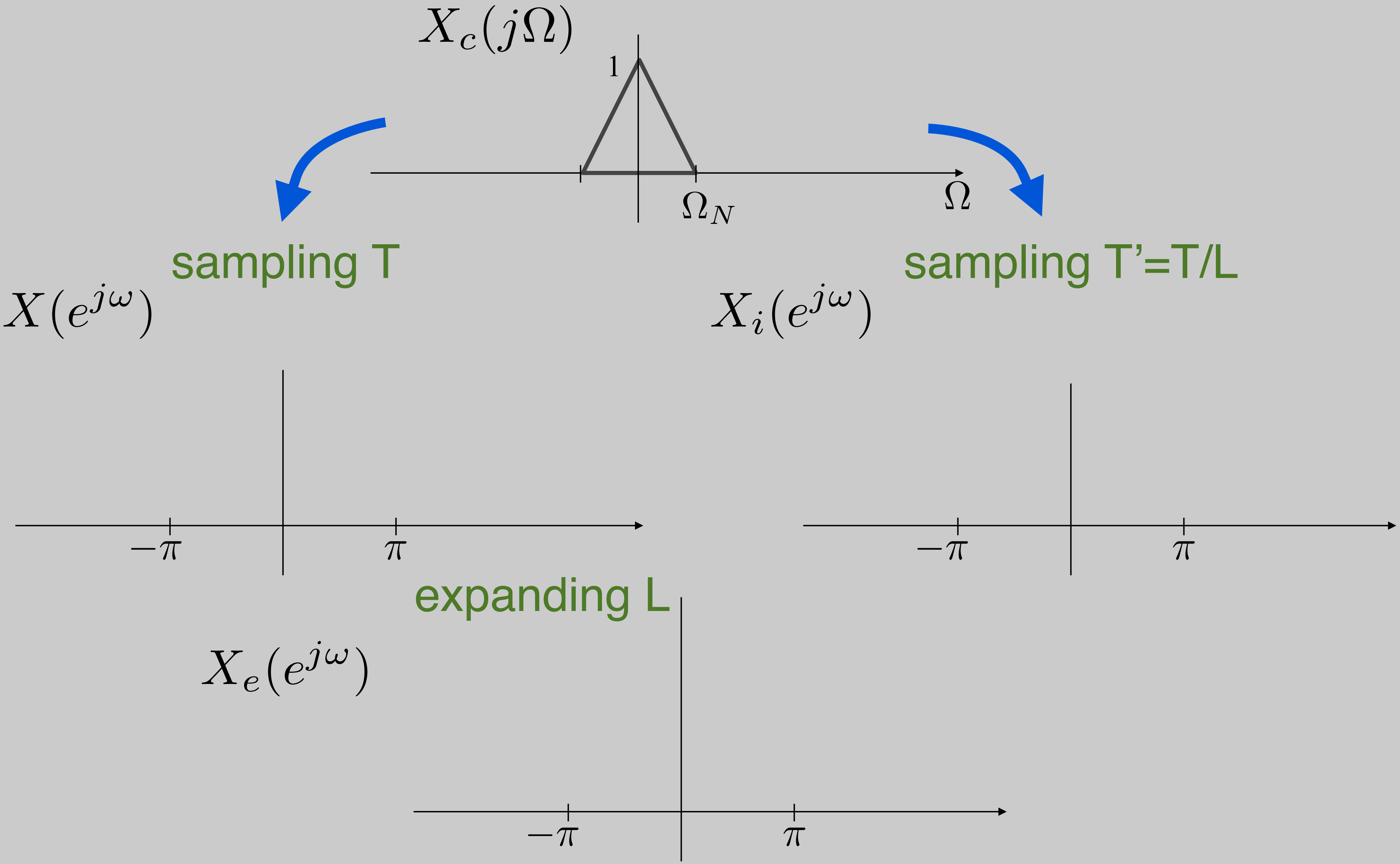


$$X_e(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \underbrace{x_e[n]}_{\neq 0 \text{ only for } n=mL \text{ (integer m)}} e^{-j\omega n}$$

$$= \sum_{m=-\infty}^{\infty} \underbrace{x_e[mL]}_{=x[m]} e^{-j\omega mL} \qquad = X(e^{j\omega L})$$

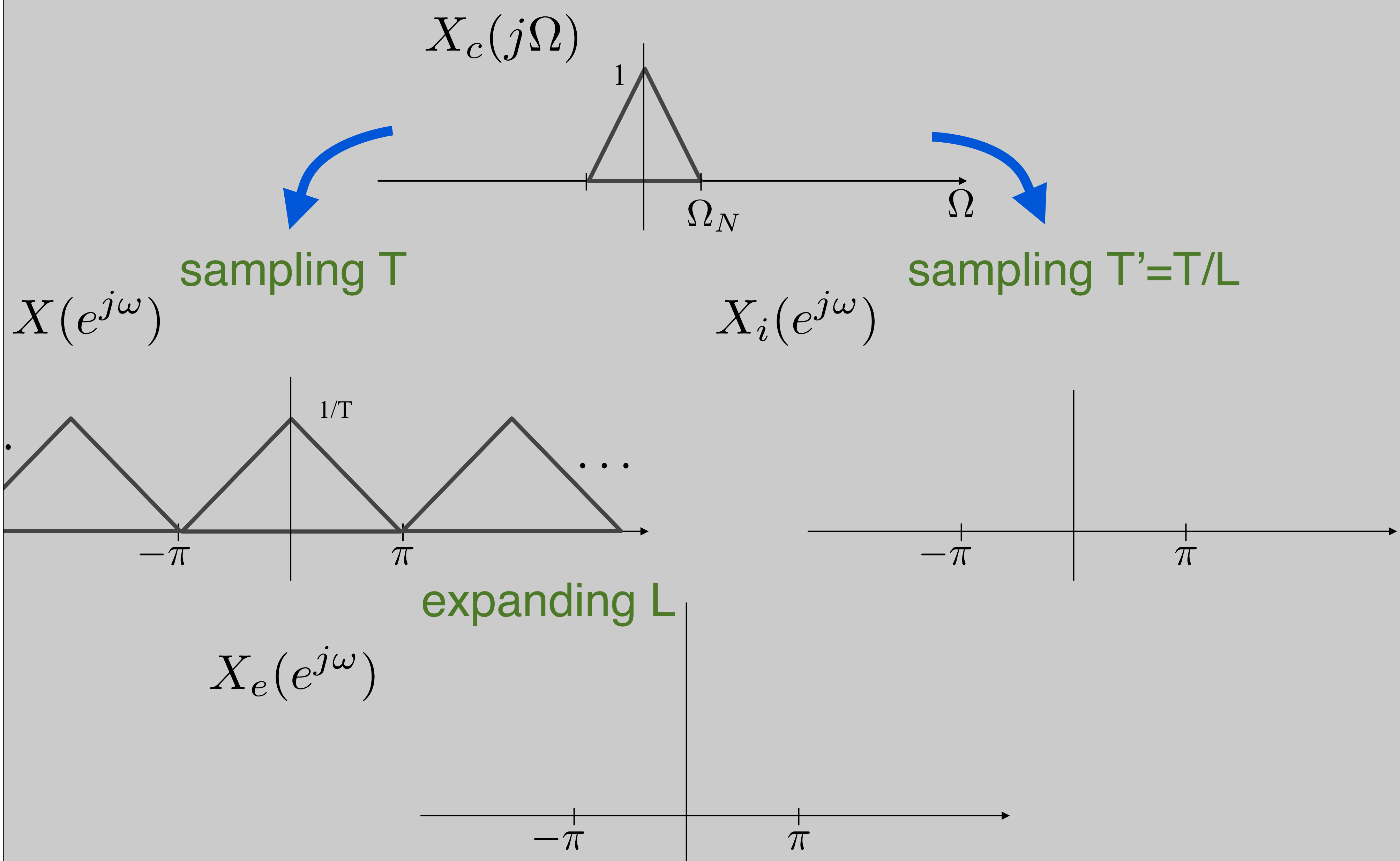
Compress DTFT by a factor of L!

Example:



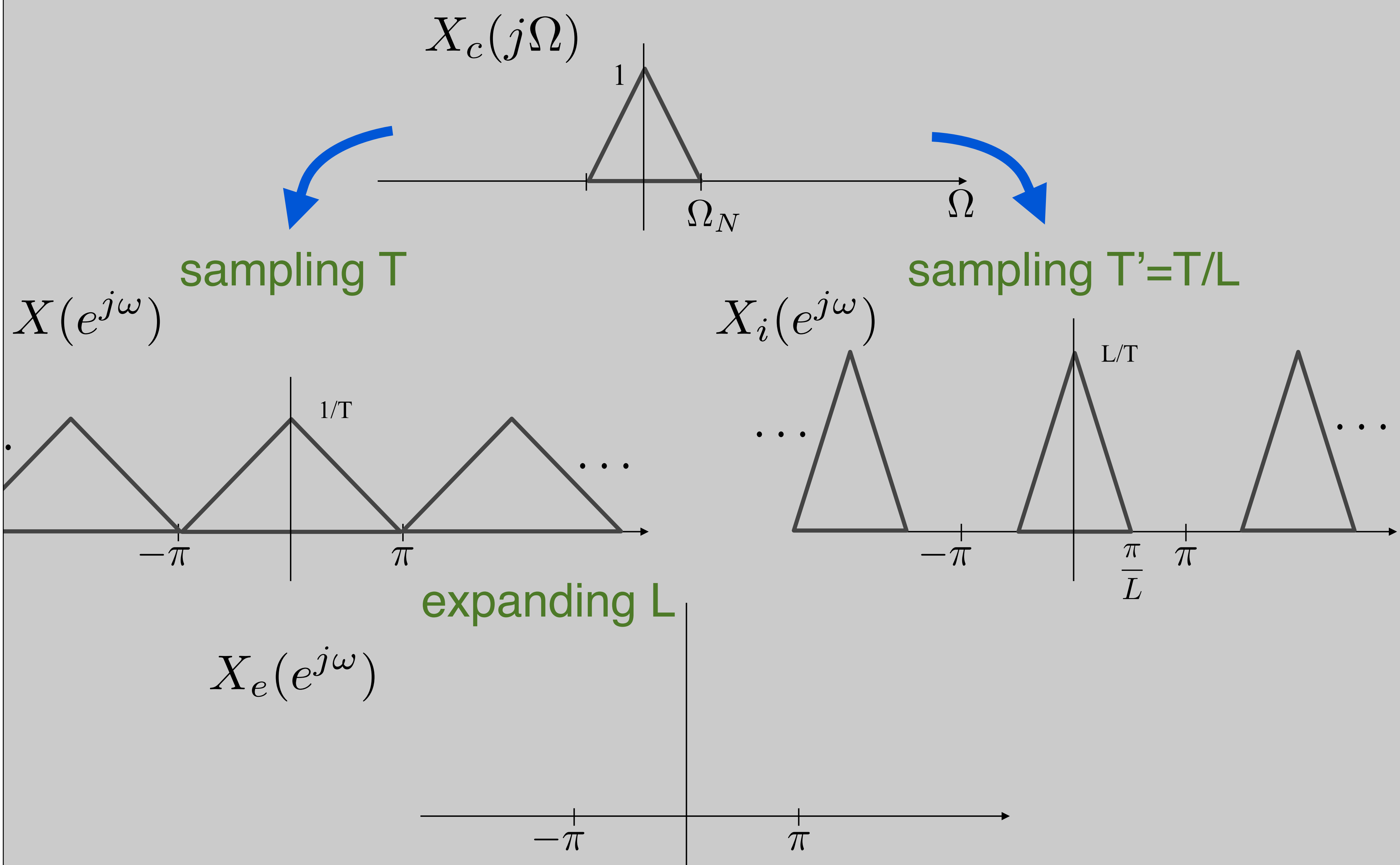
$X_c(j\Omega)$
1
Ω_N
Ω
sampling T
sampling T'=T/L
$X(e^{j\omega})$
$X_i(e^{j\omega})$
$-\pi$
π
$-\pi$
π
expanding L
$X_e(e^{j\omega})$
$-\pi$
π

Example:



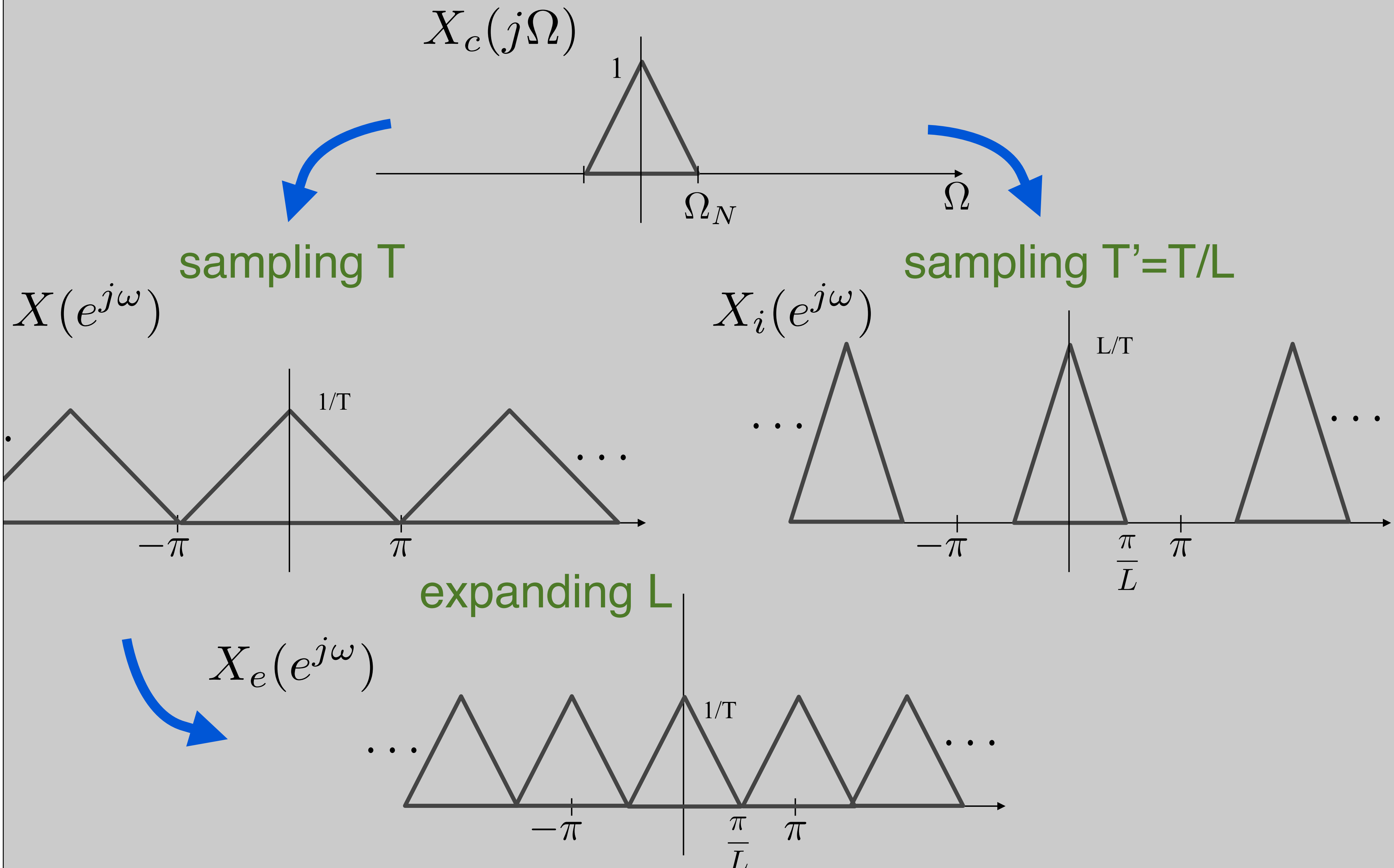
$X_c(j\Omega)$
1
Ω_N
Ω
sampling T
sampling T'=T/L
$X(e^{j\omega})$
1/T
$-\pi$
π
$X_i(e^{j\omega})$
$-\pi$
π
expanding L
$X_e(e^{j\omega})$
$-\pi$
π

Example:



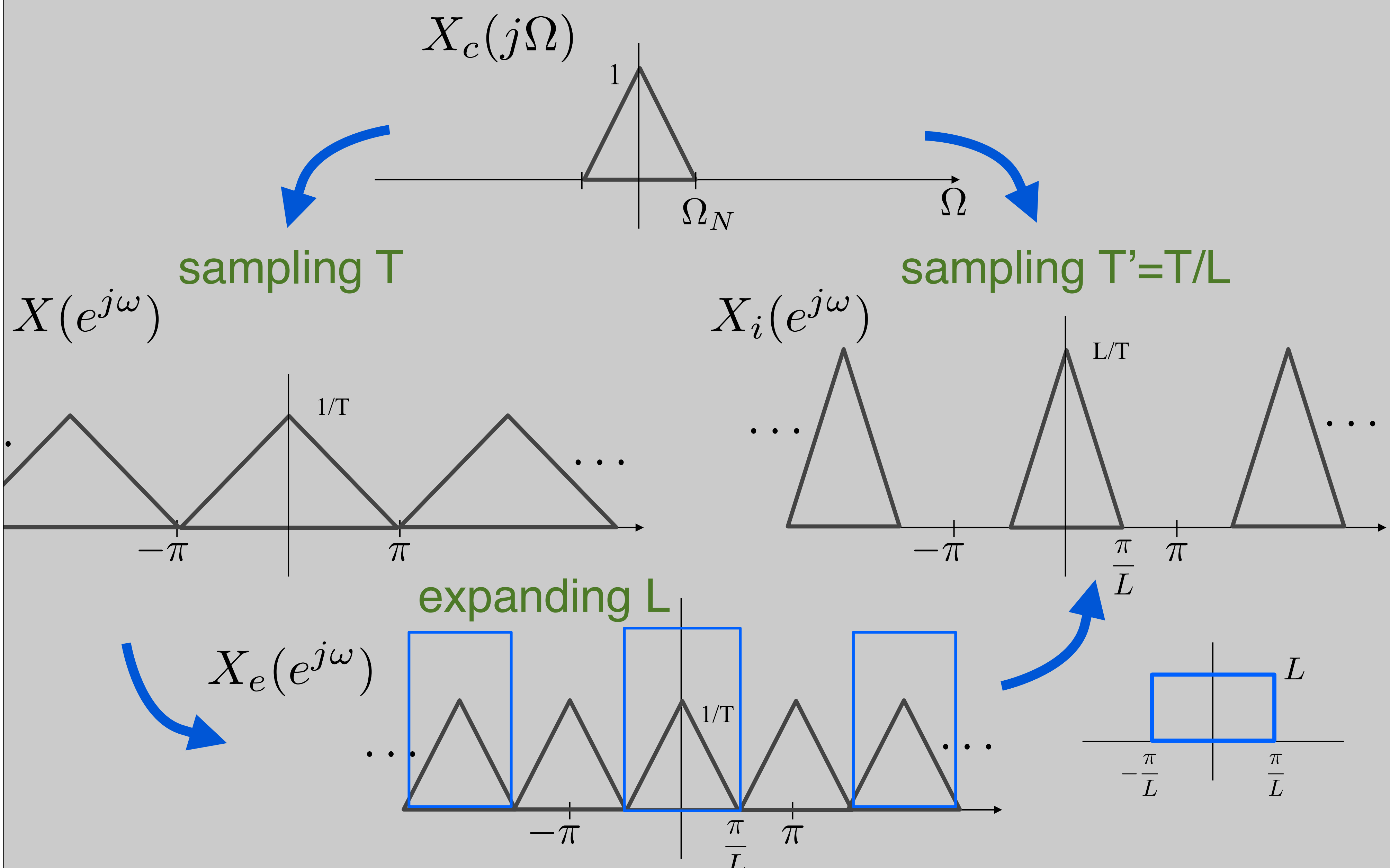
$X_c(j\Omega)$
1
Ω_N
Ω
sampling T
sampling T'=T/L
$X(e^{j\omega})$
1/T
$-\pi$
π
$X_i(e^{j\omega})$
L/T
$-\pi$
$\frac{\pi}{L}$
π
expanding L
$X_e(e^{j\omega})$
$-\pi$
π

Example:



$X_c(j\Omega)$
1
Ω_N
Ω
sampling T
sampling T'=T/L
$X(e^{j\omega})$
1/T
$\cdots$
$-\pi$
π
$X_i(e^{j\omega})$
L/T
$\cdots$
$-\pi$
$\frac{\pi}{L}$
π
expanding L
$X_e(e^{j\omega})$
1/T
$\cdots$
$-\pi$
$\frac{\pi}{L}$
π

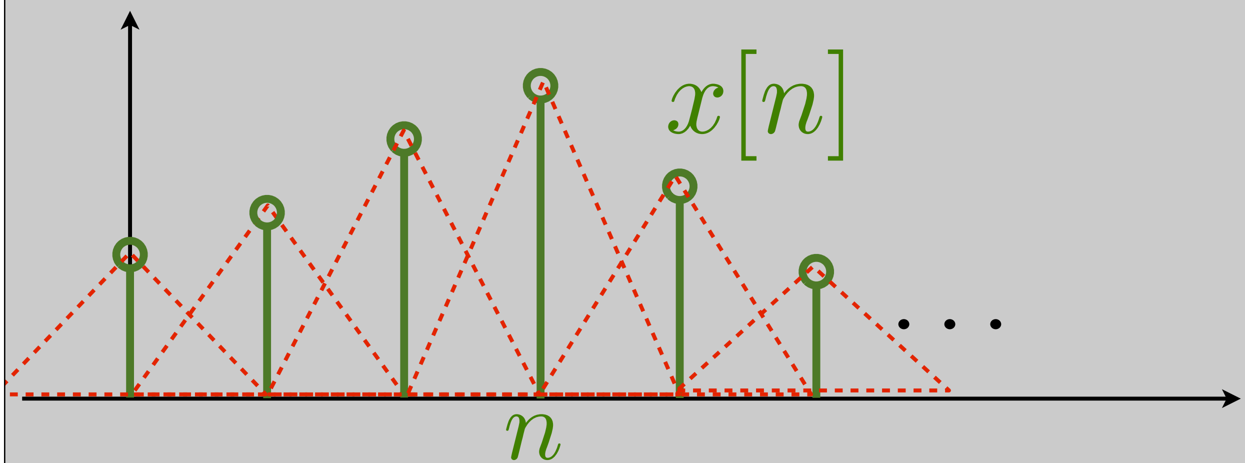
Example:

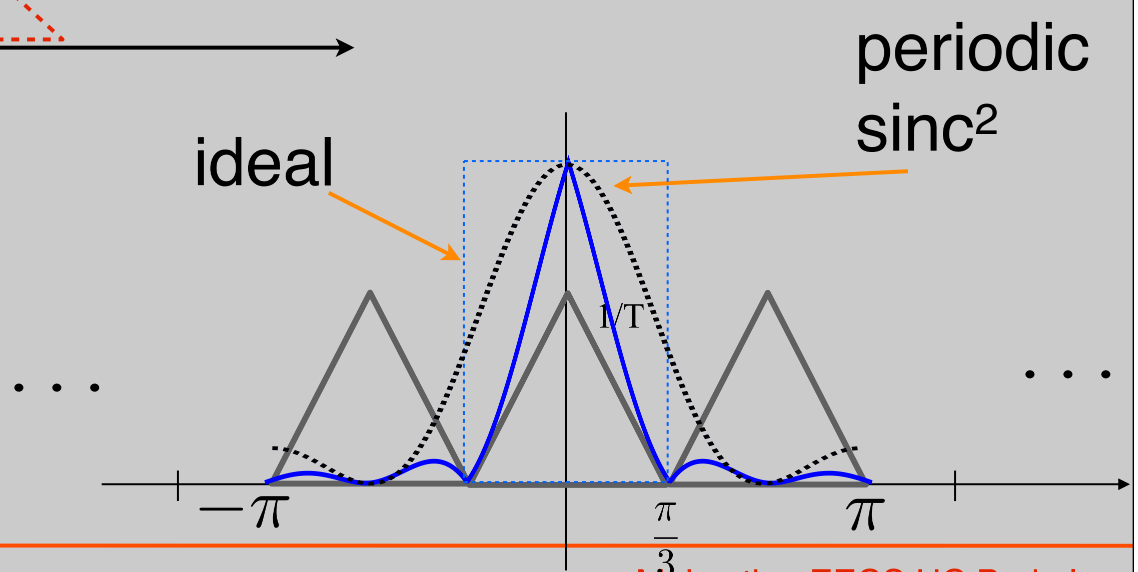


$X_c(j\Omega)$
1
Ω_N
Ω
sampling T
sampling T'=T/L
$X(e^{j\omega})$
1/T
$\cdots$
$-\pi$
π
$X_i(e^{j\omega})$
L/T
$\cdots$
$-\pi$
$\frac{\pi}{L}$
π
expanding L
$X_e(e^{j\omega})$
1/T
$\cdots$
$-\pi$
$\frac{\pi}{L}$
π
L
$-\frac{\pi}{L}$
$\frac{\pi}{L}$

Practical Upsampling

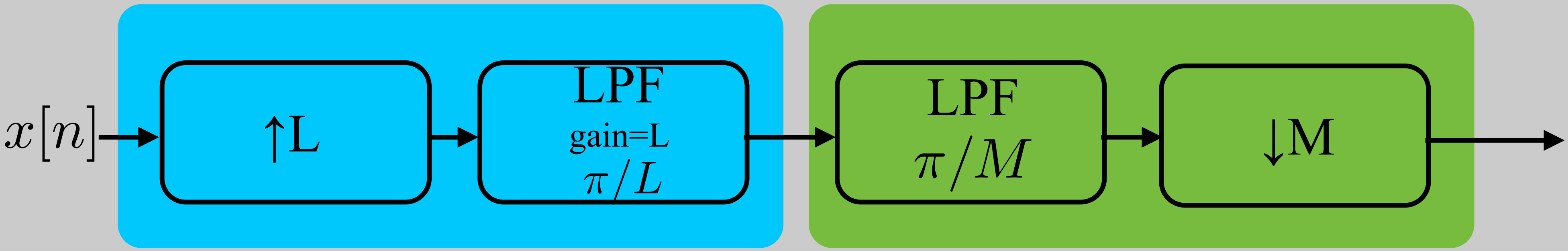
- Can interpolate with simple, practical filters. What's happening?
- Example: L=3, linear interpolation - convolve with triangle



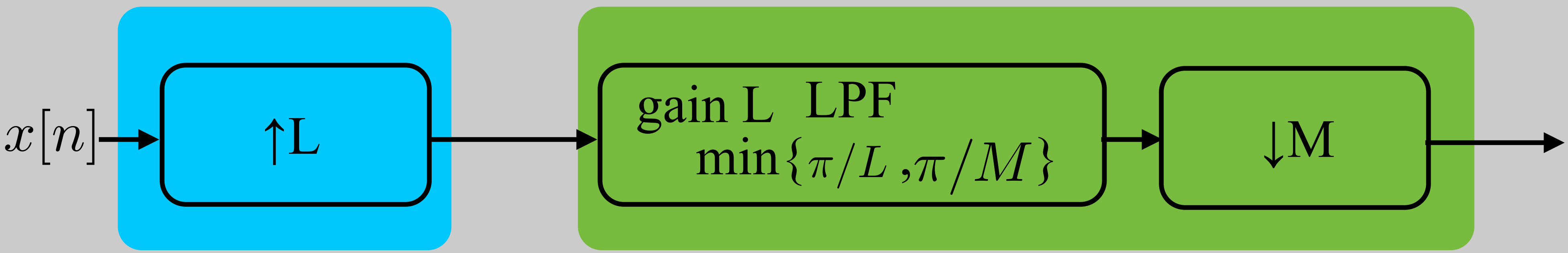


Resampling by non-integer

- T' = TM/L (upsample L, downsample M)



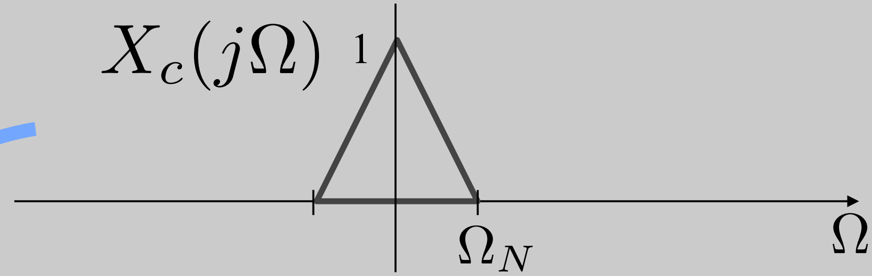
Or,



- What would happen if change order?

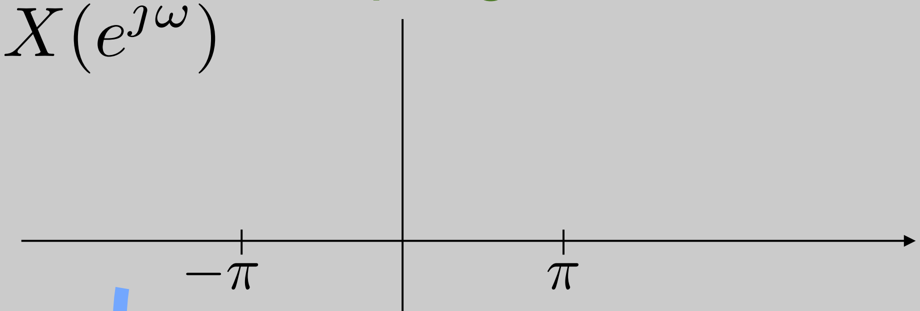
Example:

- L = 2, M=3,   T'=3/2T   (fig 4.30)



sampling T

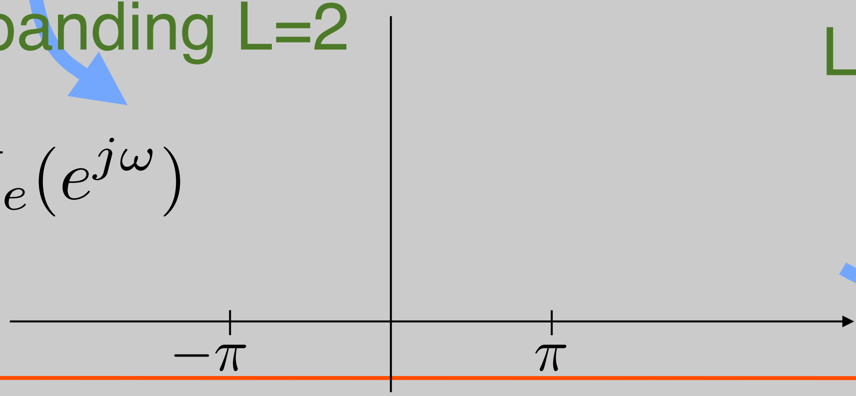
$X(e^{j\omega})$



Subsampling M=3

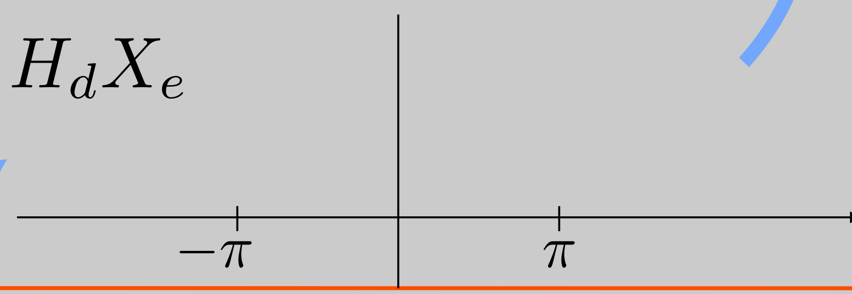
expanding L=2

$X_e(e^{j\omega})$



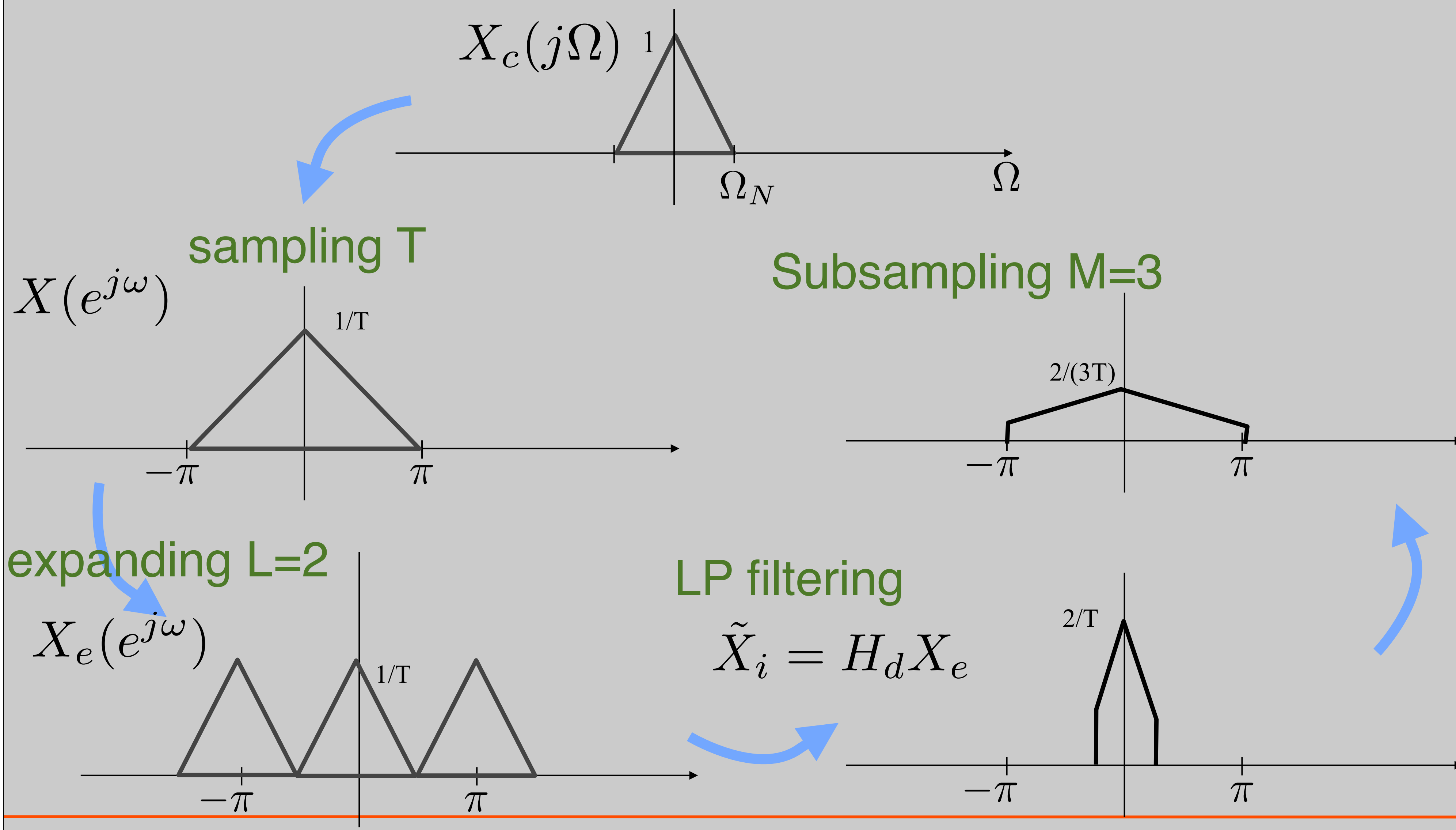
LP filtering

$$\tilde{X}_i = H_d X_e$$



Example:

- L = 2, M=3,   T'=3/2T   (fig 4.30)



Multi-Rate Signal Processing

- What if we want to resample by 1.01T?
  - Expand by L=100
  - Filter π/101 ($$$$$)
  - Downsample by M=101

- Fortunately there are ways around it!
  - Called multi-rate
  - Uses compressors, expanders and filtering

Interchanging Operations

$$\longrightarrow \boxed{\uparrow \mathrm{L}} \longrightarrow$$

"expander"

$$\longrightarrow \boxed{\downarrow \mathrm{M}} \longrightarrow$$

"compressor"

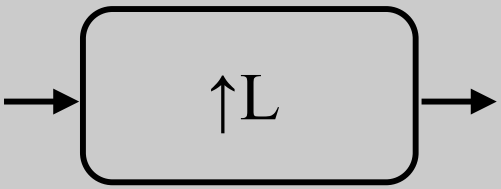
<u>not LTI!</u>

<u>Note:</u>

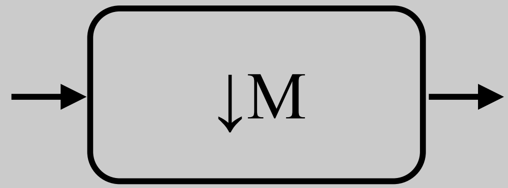
$$x[n] \longrightarrow \boxed{H(z)} \underset{H(e^{j\omega})X(e^{j\omega})}{\longrightarrow} \boxed{\uparrow \mathrm{L}} \longrightarrow \underset{H(e^{j\omega L})X(e^{j\omega L})}{y[n]}$$

$$x[n] \longrightarrow \boxed{\uparrow \mathrm{L}} \underset{X(e^{j\omega L})}{\longrightarrow} \boxed{H(z)} \longrightarrow \underset{H(e^{j\omega})X(e^{j\omega L})}{y[n]}$$

Interchanging Operations



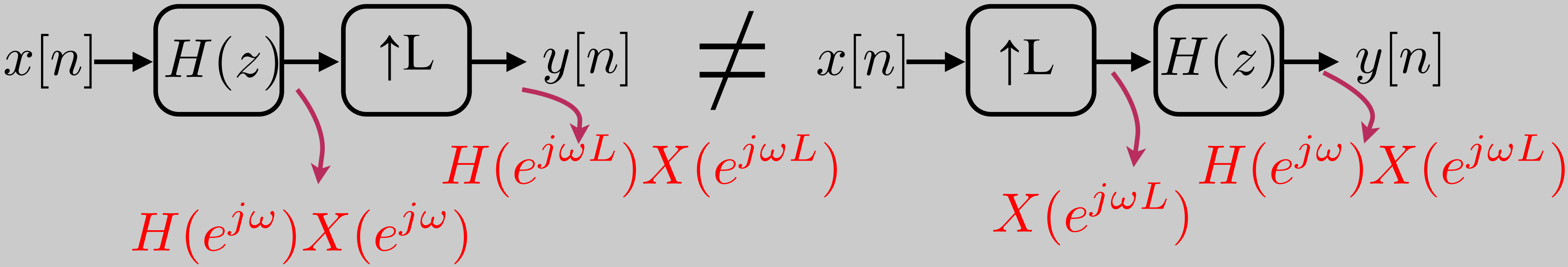
“expander”

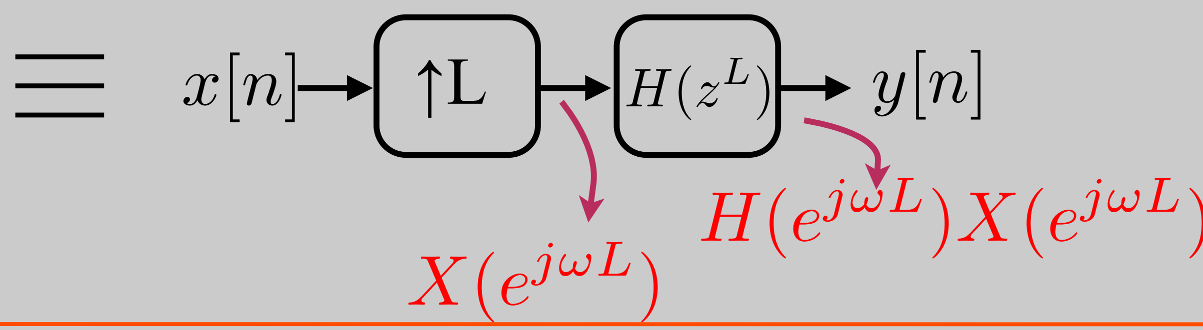


“compressor”

not LTI!

Note:





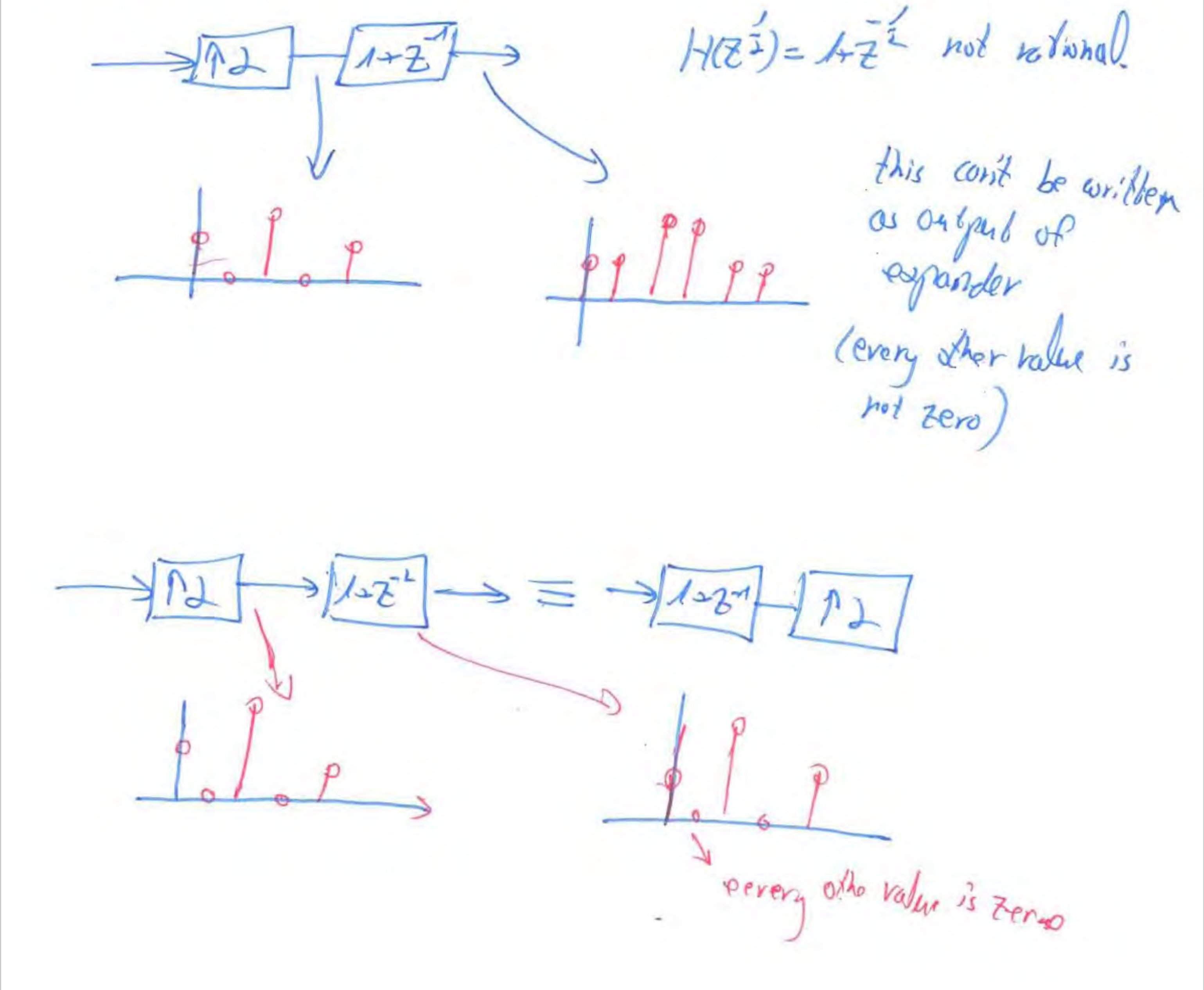
Interchanging Filter Expander

- Q: Can we move expander from Left to Right (with xform)?

$$\rightarrow \boxed{\uparrow L} \rightarrow \boxed{H(z)} \rightarrow \qquad \equiv ? \qquad \rightarrow \boxed{H(z^{\frac{1}{L}})} \rightarrow \boxed{\uparrow L} \rightarrow$$

- A: Yes, if H(z) is rational
  No, otherwise

Example:


↑2
1+z^-1
H(z^1/2) = 1+z^-1/2 not rational
this can't be written as output of expander
(every other value is not zero)
↑2
1+z^-2
≡
1+z^-1
↑2
every other value is zero