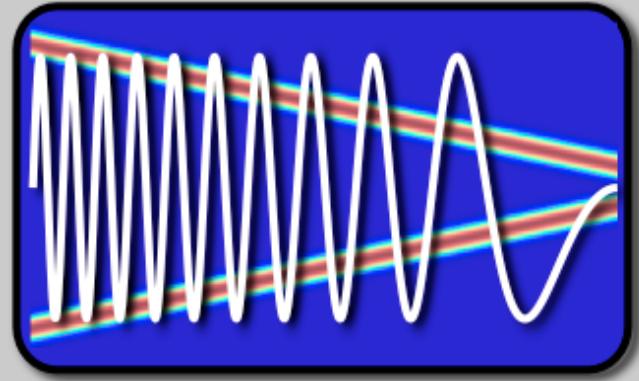


EE123



# Digital Signal Processing

## Lecture 21 Tomography

## Impulse lines and line-integrals

In 1D

$$\int_{-\infty}^{\infty} f(x)\delta(x)dx = f(0)$$

A sample @x=0

In 2D

$$\int_{-\infty}^{\infty} f(x, y)\delta(x)dx = f(0, y)$$

1D cross-section  
@x=0

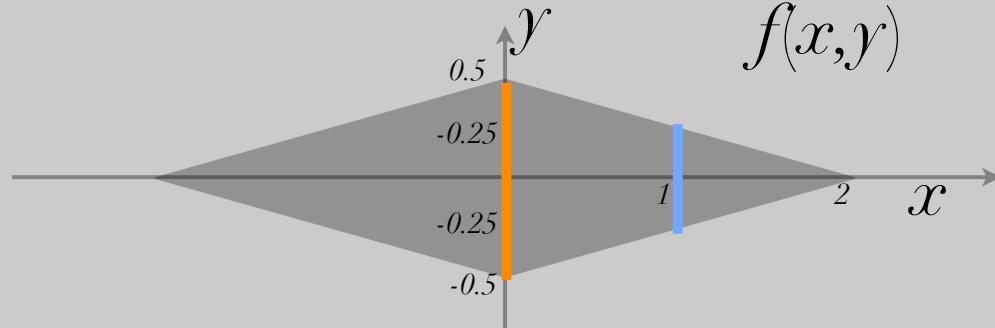
Line Integral

dx

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y)\delta(x)dy = \int_{-\infty}^{\infty} f(0, y)dy$$

Integral of the 1D  
cross-section @x=0

## Impulse lines and line-integrals



$$\int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} f(x,y) \delta(x) dx = f(0,y) = \square(y) \int_{-\infty}^{\infty} dy$$

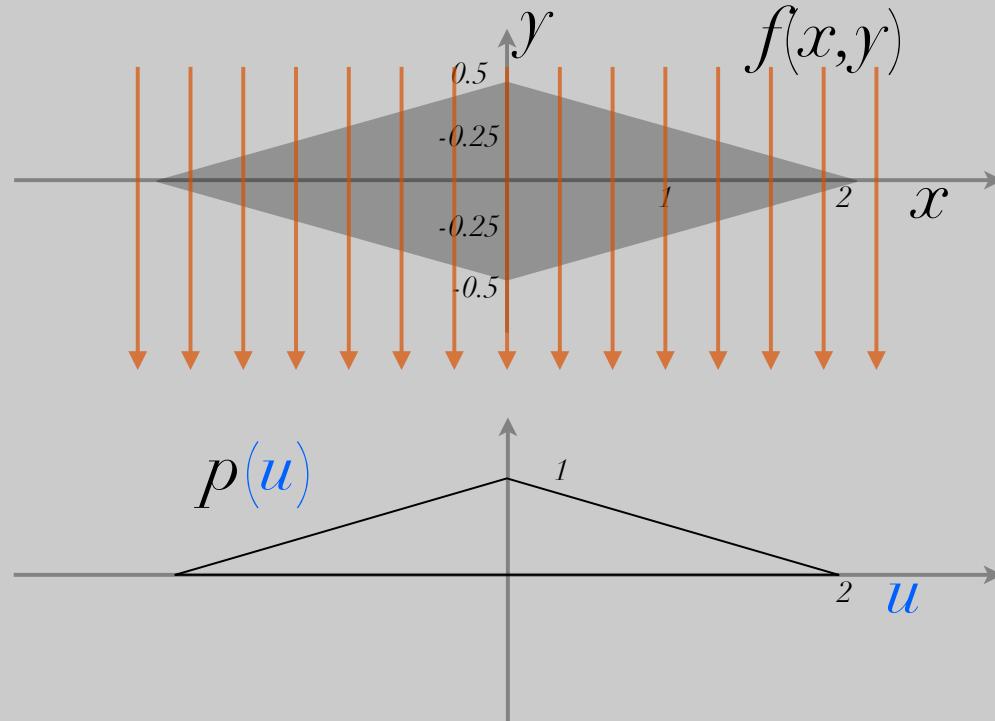
$= 1$

$$\int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} f(x,y) \delta(x-1) dx = f(1,y) = \square(2y) \int_{-\infty}^{\infty} dy$$

$= 0.5$

what about line integral with  $\delta(x-u)$ ?

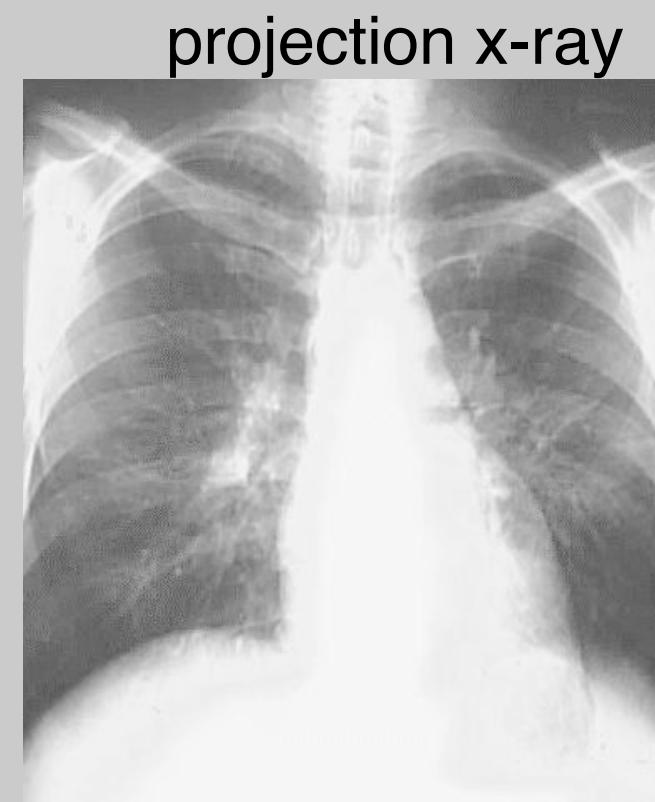
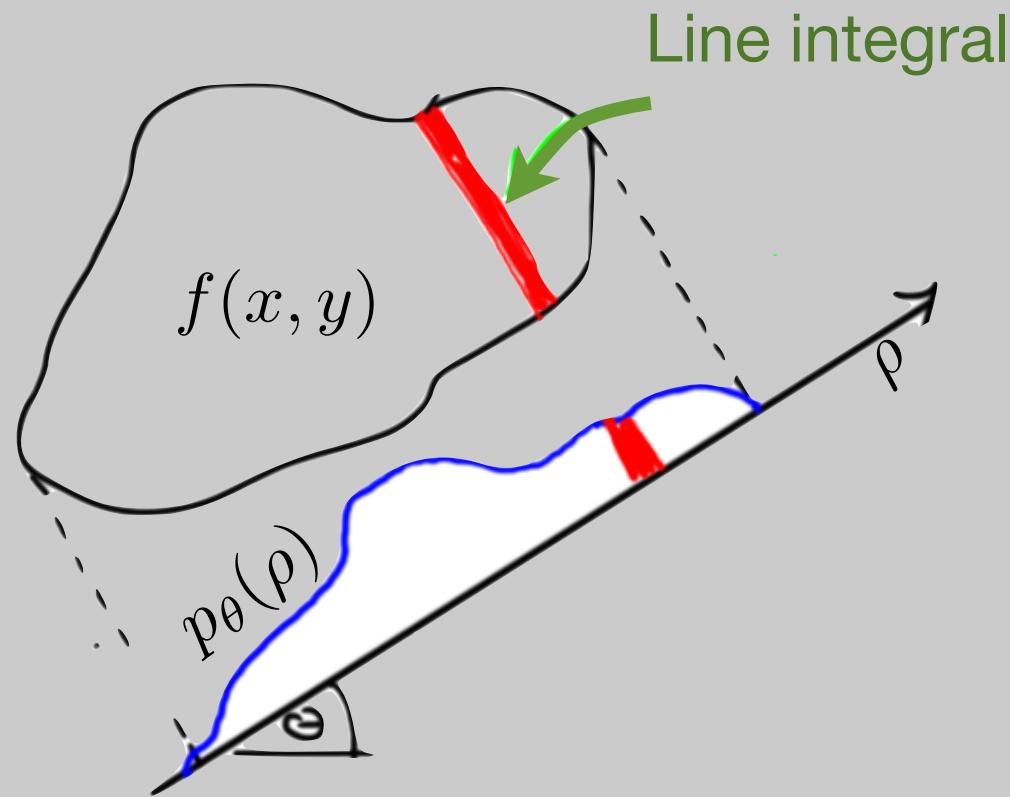
# Line Integral and Projection



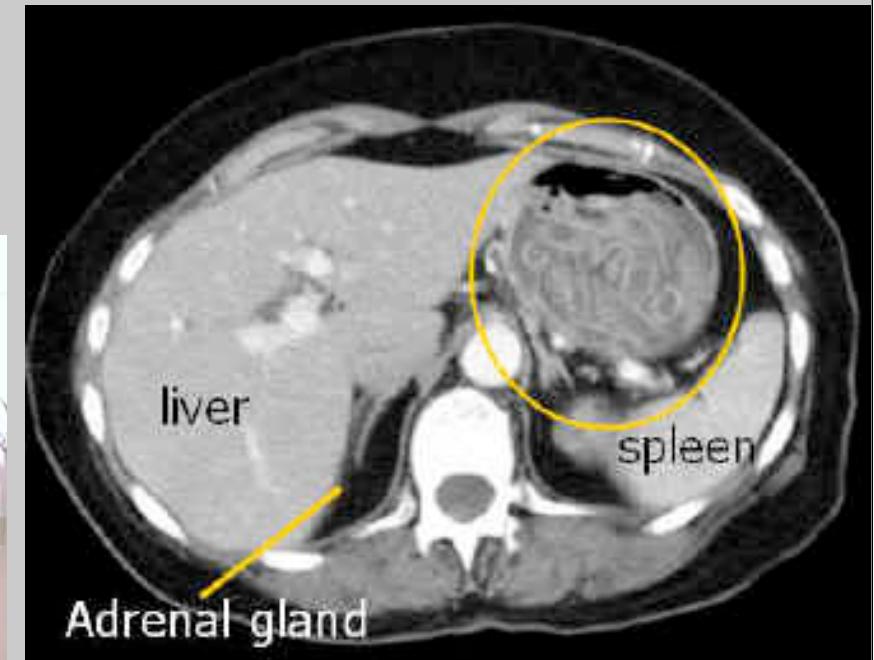
$$p(\textcolor{blue}{u}) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \delta(x - \textcolor{blue}{u}) dx dy = \int_{-\infty}^{\infty} f(\textcolor{blue}{u}, y) dy$$

# General Projections

$$p(\rho, \theta) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \delta(\rho - x \cos \theta - y \sin \theta) dx dy$$



# Many Projections - Tomography

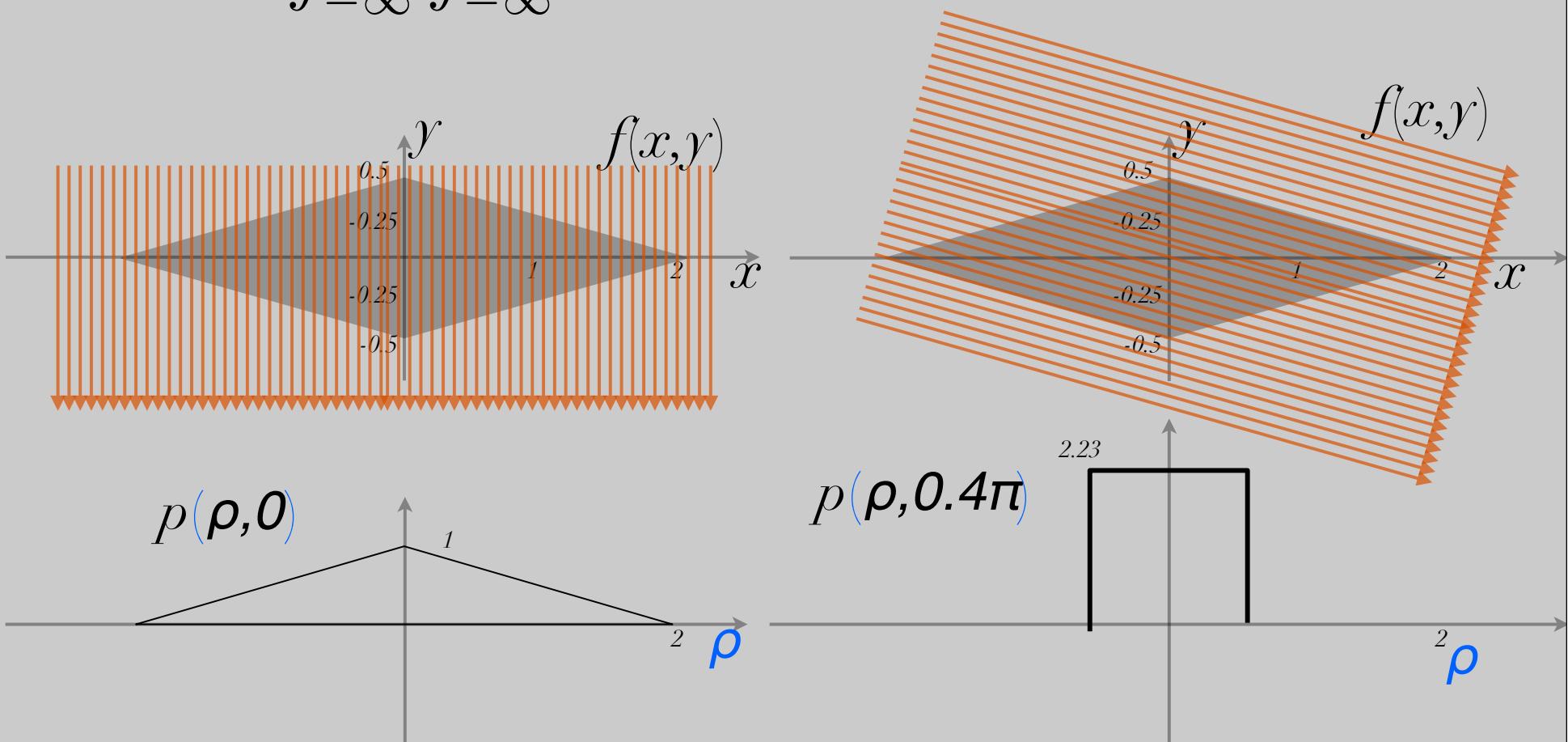


<http://www.youtube.com/watch?v=4gklQHM19aY&feature=related>



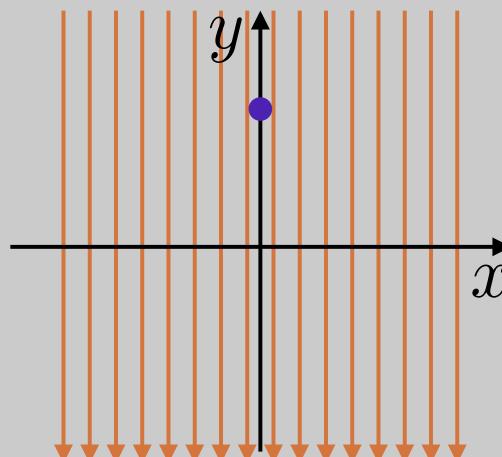
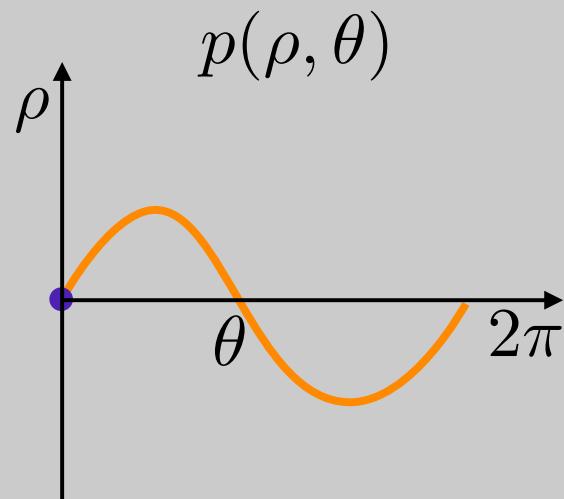
# Radon Transform

$$p(\rho, \theta) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \delta(\rho - x \cos \theta - y \sin \theta) dx dy$$

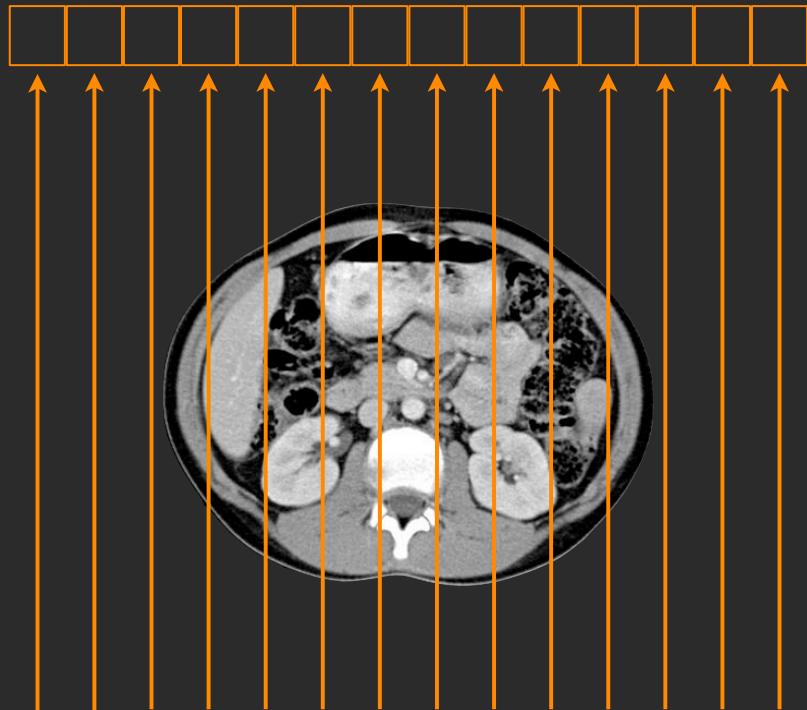


## Radon Transform: Sinogram

- Also called Sinogram
- Impulse  $\Rightarrow$  Sinusoid

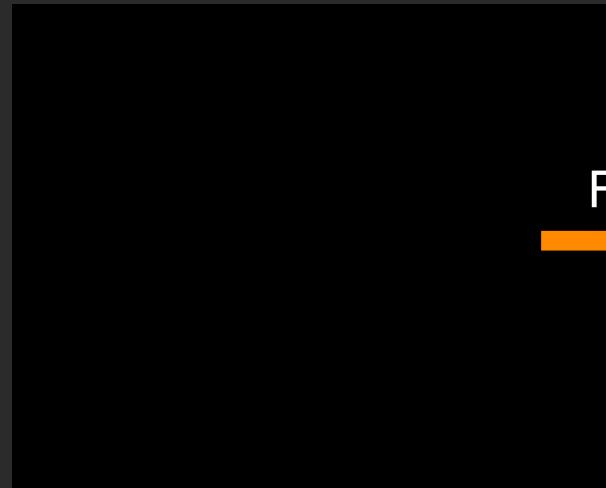


# Computed Tomography

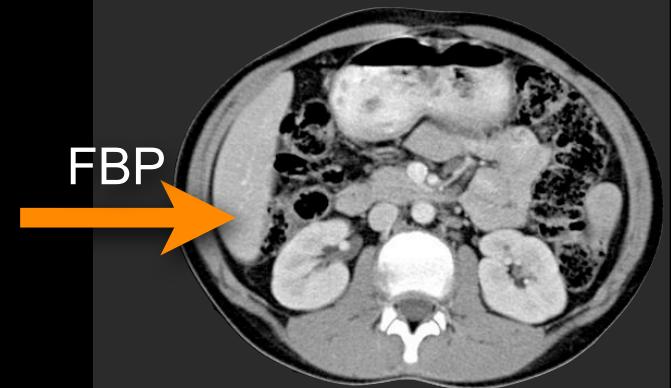


x-ray source

Sinogram



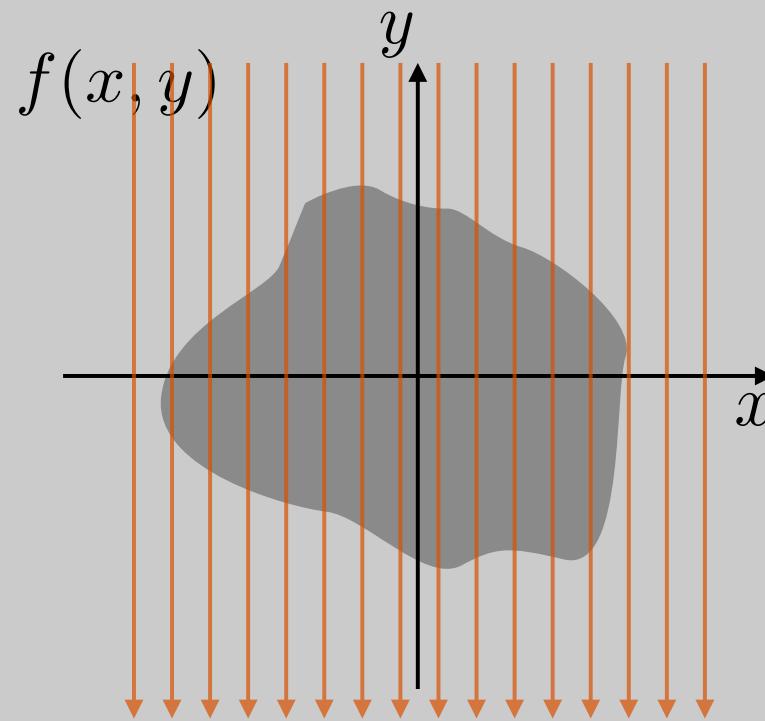
cross-section



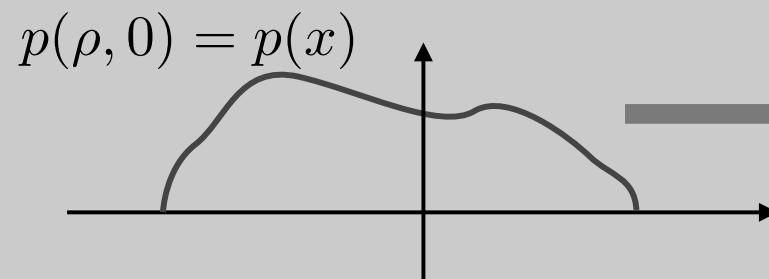
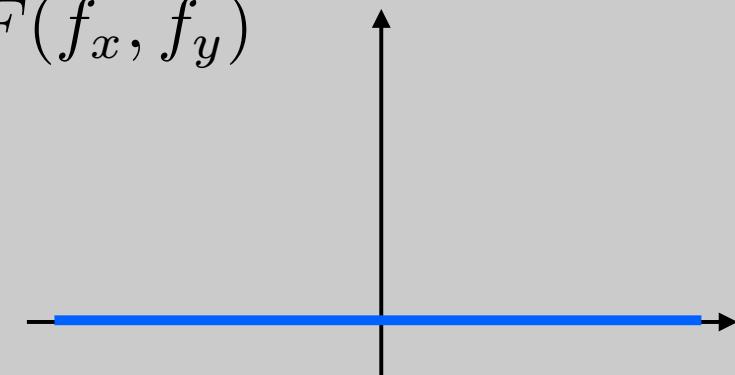
## Projection Slice Theorem (Bracewell)

sine

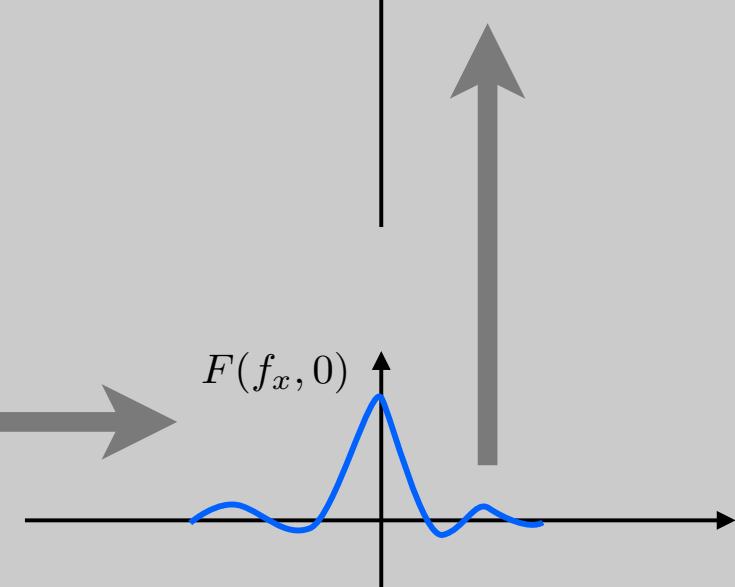
$$\mathcal{F}_{1D}\{p(\rho, \theta)\} = F(\rho \cos \theta, \rho \cos \theta)$$



$$F(f_x, f_y)$$



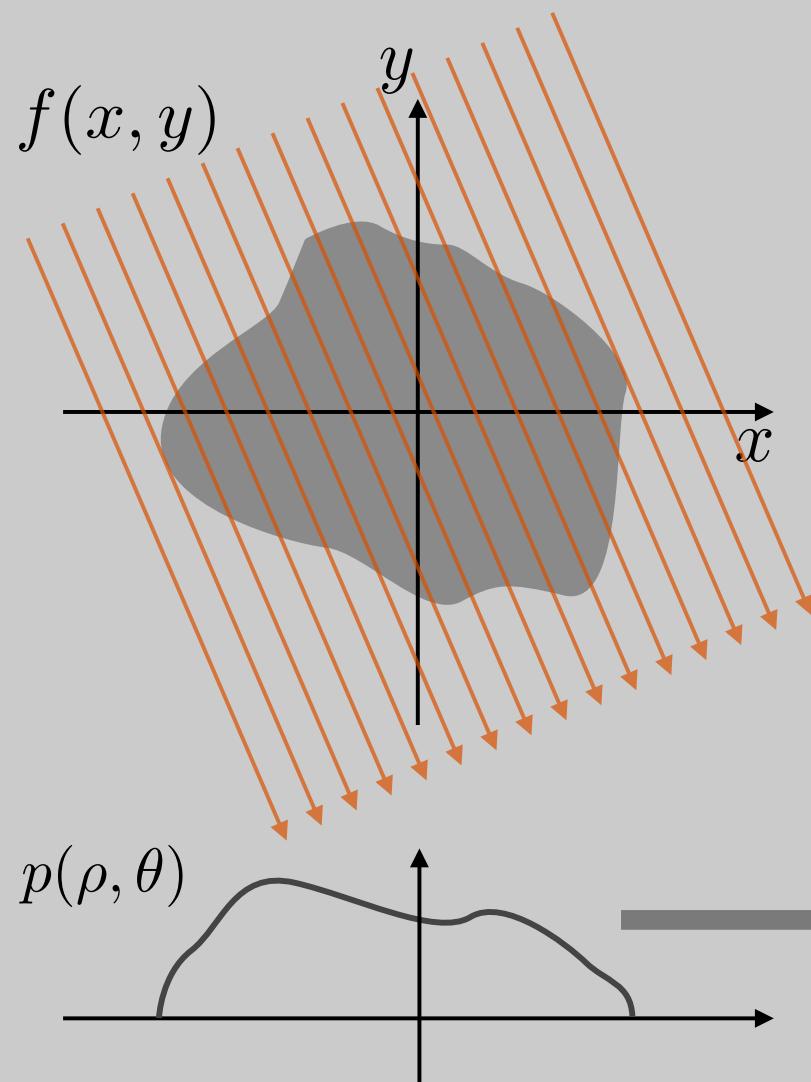
1D FT



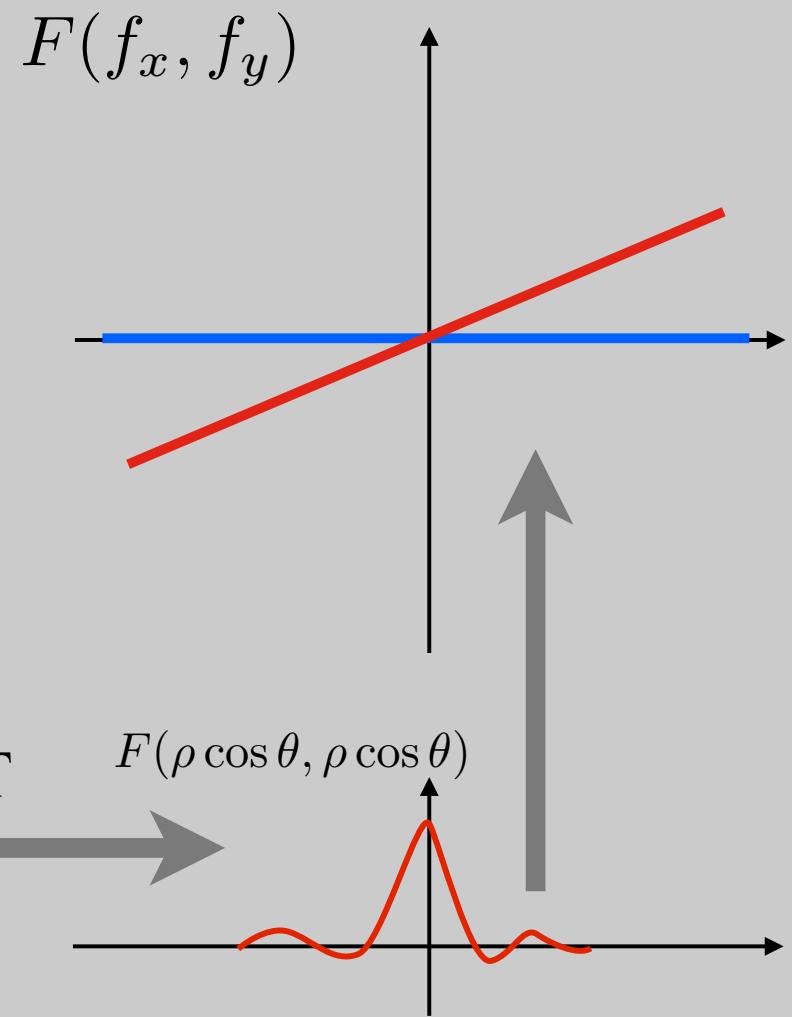
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$$\mathcal{F}_{1D}\{p(\rho, \theta)\} = F(\rho \cos \theta, \rho \cos \theta)$$



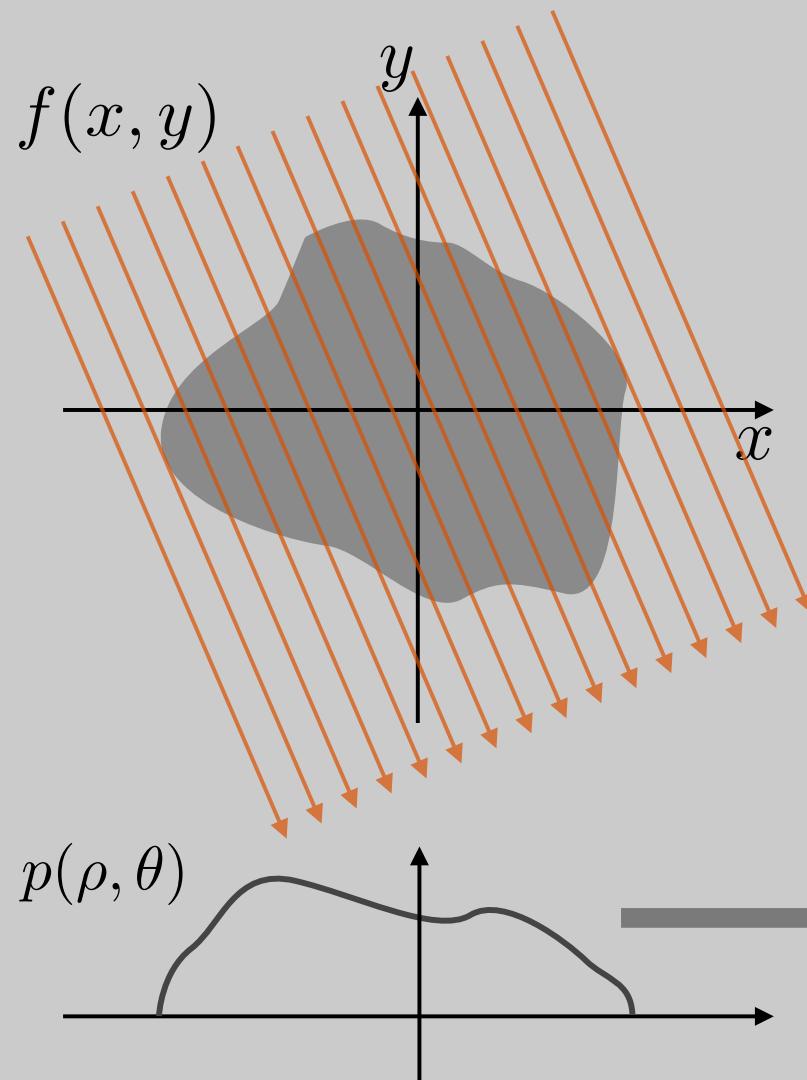
1D FT



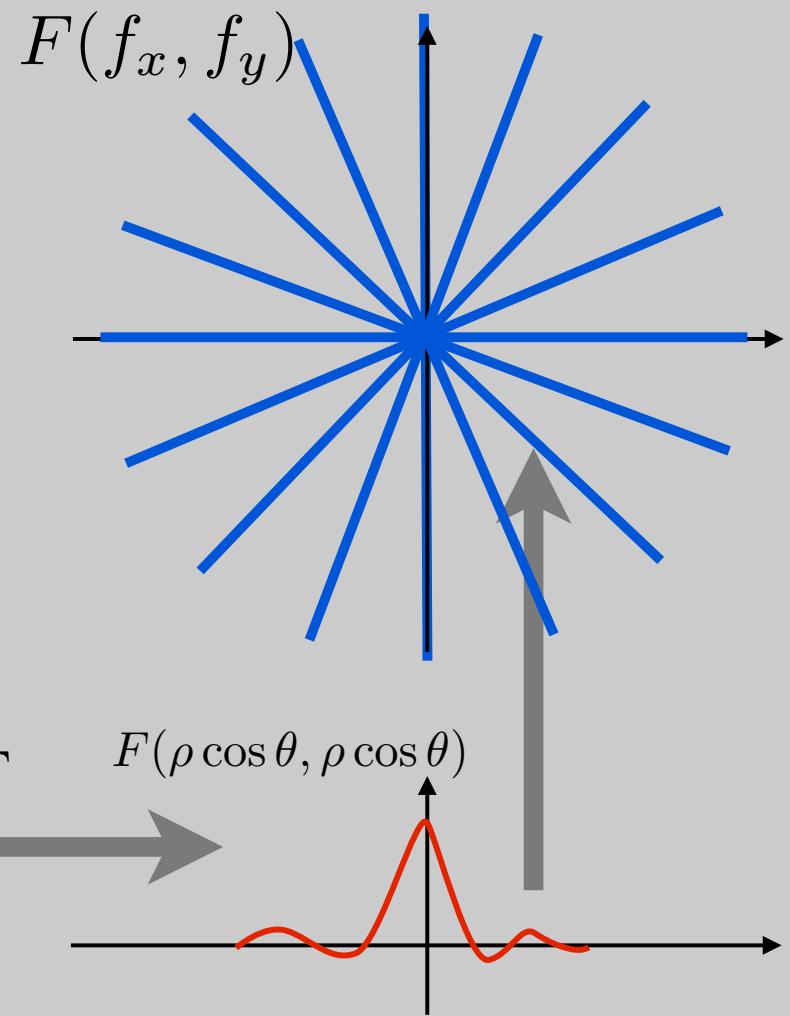
## Projection Slice Theorem (Bracewell)

sine

$$\mathcal{F}_{1D}\{p(\rho, \theta)\} = F(\rho \cos \theta, \rho \cos \theta)$$



1D FT



## Projection Slice Theorem (Bracewell)

Proof (for  $\Theta=0$ )

$$p(x) = \int_{-\infty}^{\infty} m(x, y) dy$$

$$M(k_x, 0) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} m(x, y) e^{-i2\pi(k_x x + k_y y)} dx dy =$$

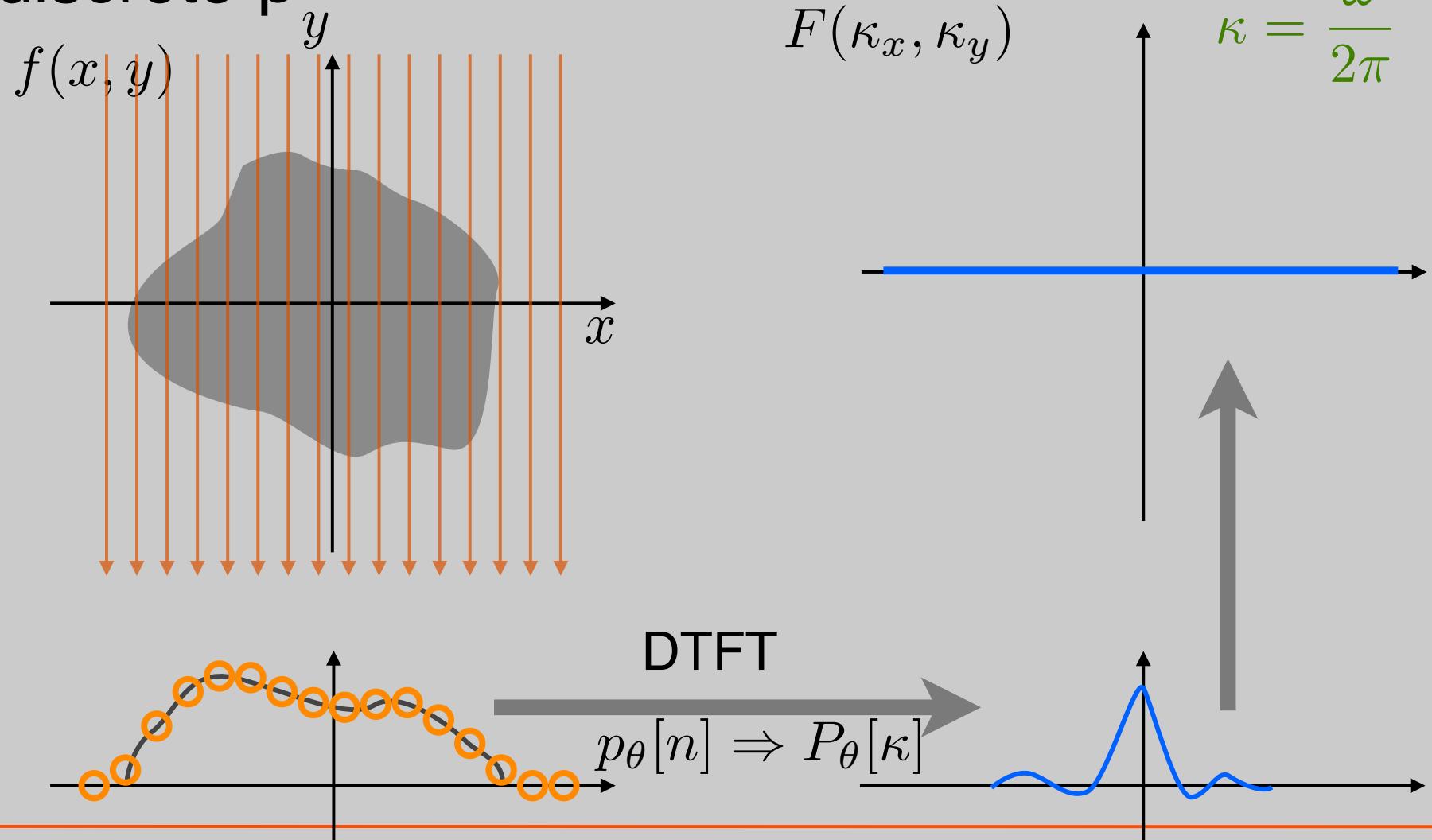
$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} m(x, y) e^{-i2\pi k_x x} dx dy =$$

$$= \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} m(x, y) dy \right] e^{-i2\pi k_x x} dx =$$

$$= \int_{-\infty}^{\infty} p(x) e^{-i2\pi k_x x} dx = \tilde{f} \{ p(x) \}$$

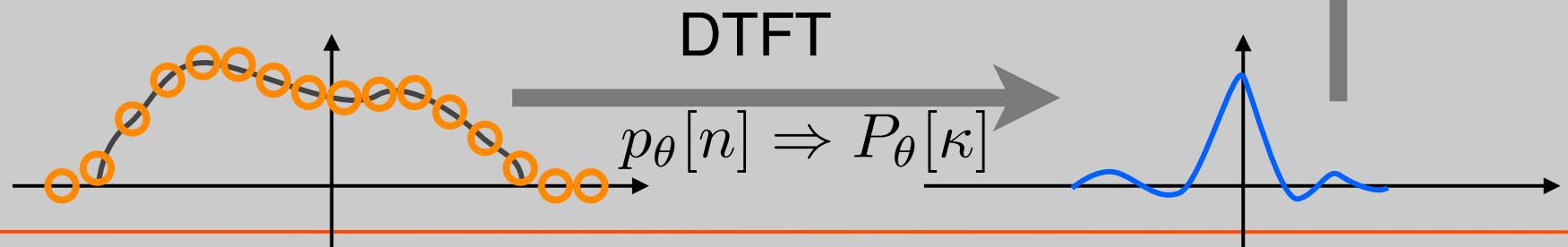
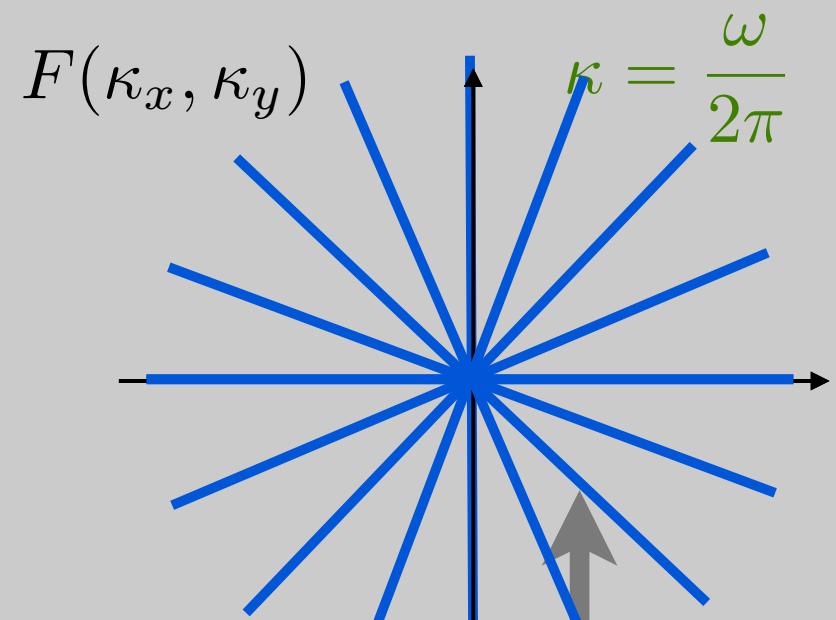
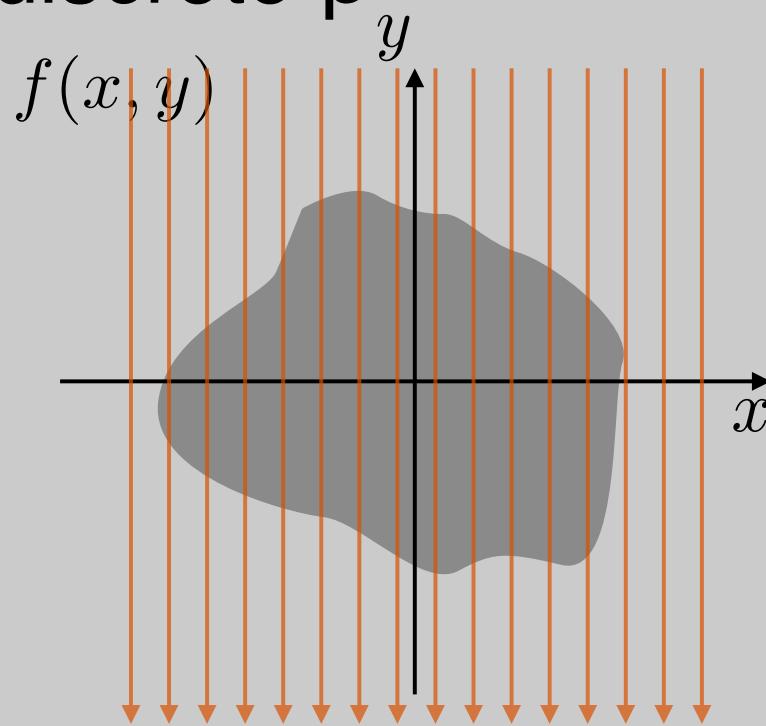
## Partly Discrete Reconstruction

- Let's assume continuous angle  $\Theta$ , discrete  $\rho$



## Partly Discrete Reconstruction

- Let's assume continuous angle  $\Theta$ , discrete  $\rho$



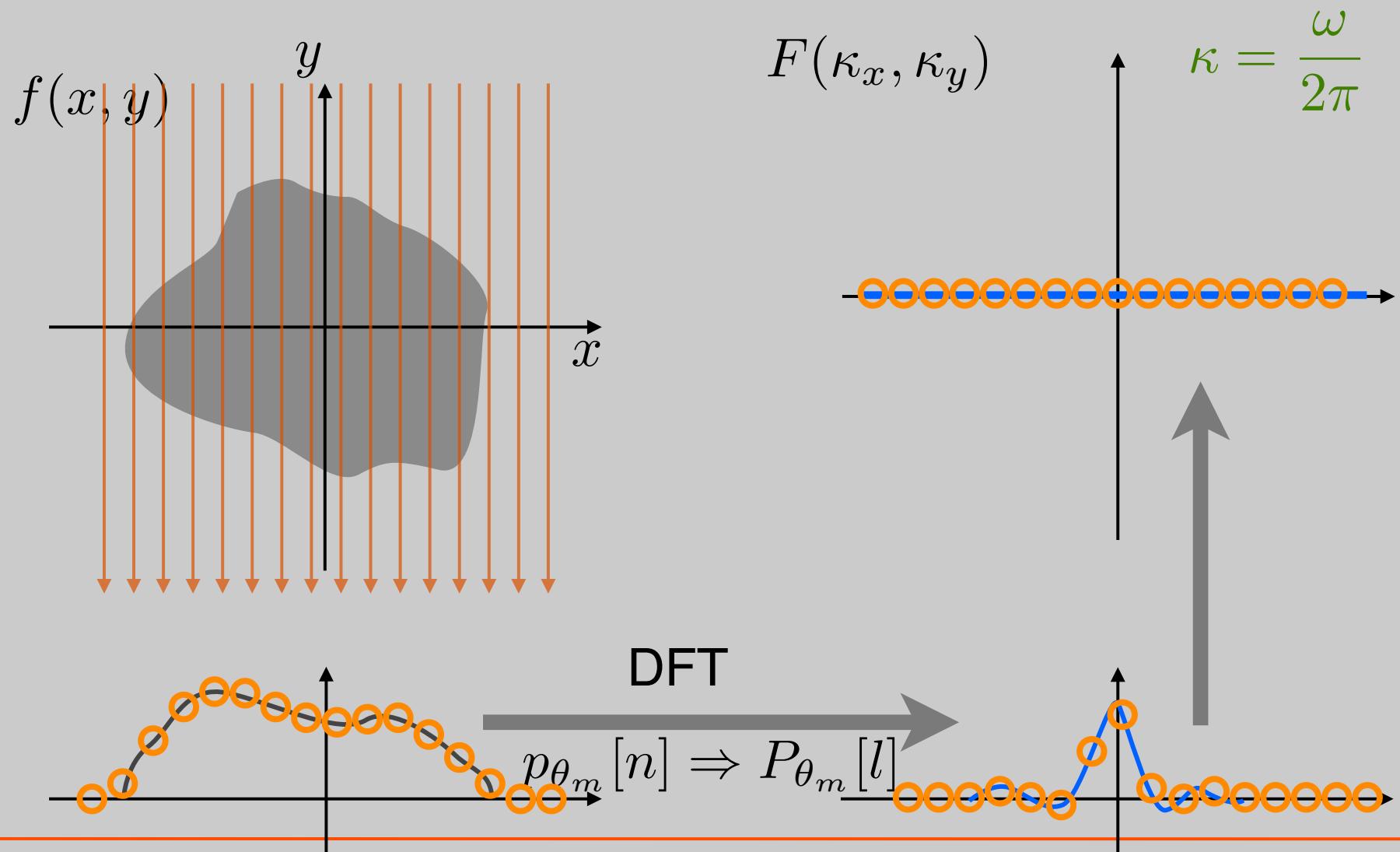
## Reconstruction From Polar Coordinates

$$\begin{aligned}f[n, m] &= \int_{-0.5}^{0.5} \int_{-0.5}^{0.5} F(\kappa_x, \kappa_y) e^{2\pi j(\kappa_x n + \kappa_y m)} d\kappa_x d\kappa_y \\&= \int_0^\pi \int_{-0.5}^{0.5} F(\rho, \theta) e^{2\pi j(\rho \cos(\theta)n + \rho \sin(\theta)m)} |\rho| d\rho d\theta\end{aligned}$$

- Polar frequency data must be multiplied by  $|\rho|$
- Also called a rho filter

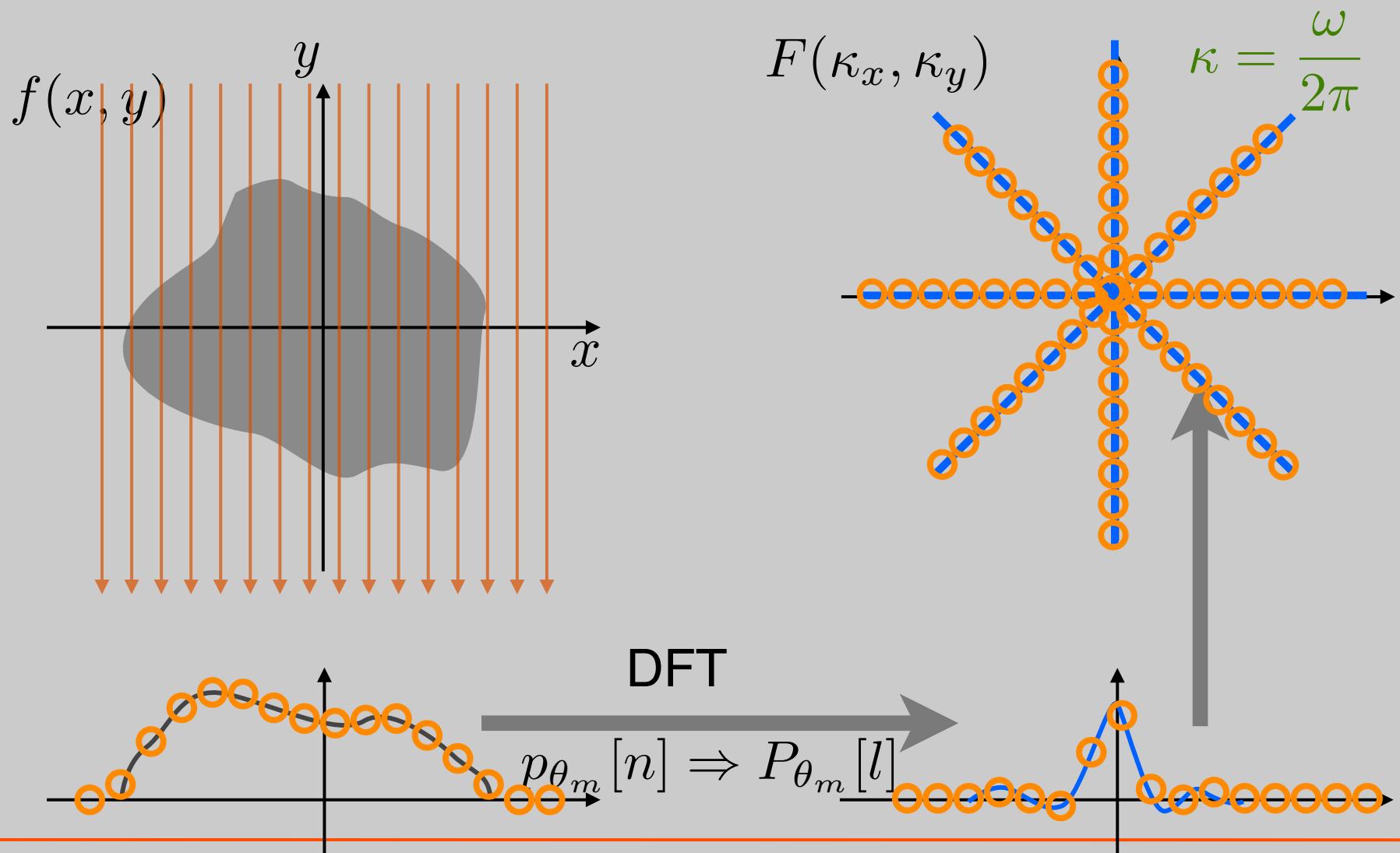
## Discrete Reconstruction

- Let's assume discrete angle  $\Theta_m$ , discrete  $\rho$



## Discrete Reconstruction

- Let's assume discrete angle  $\Theta_m$ , discrete  $\rho$



## Filtered Back Projection

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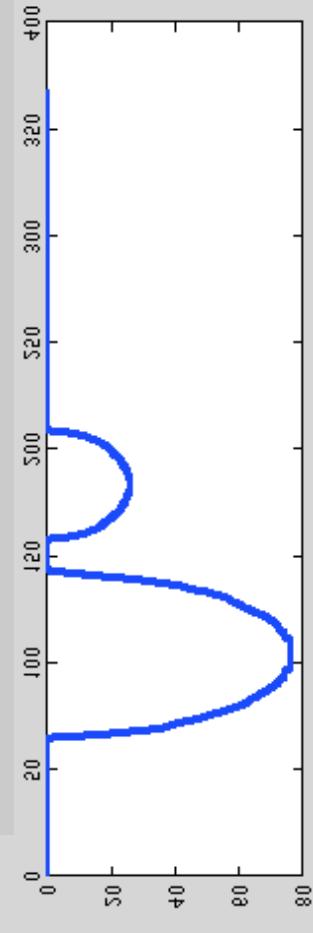
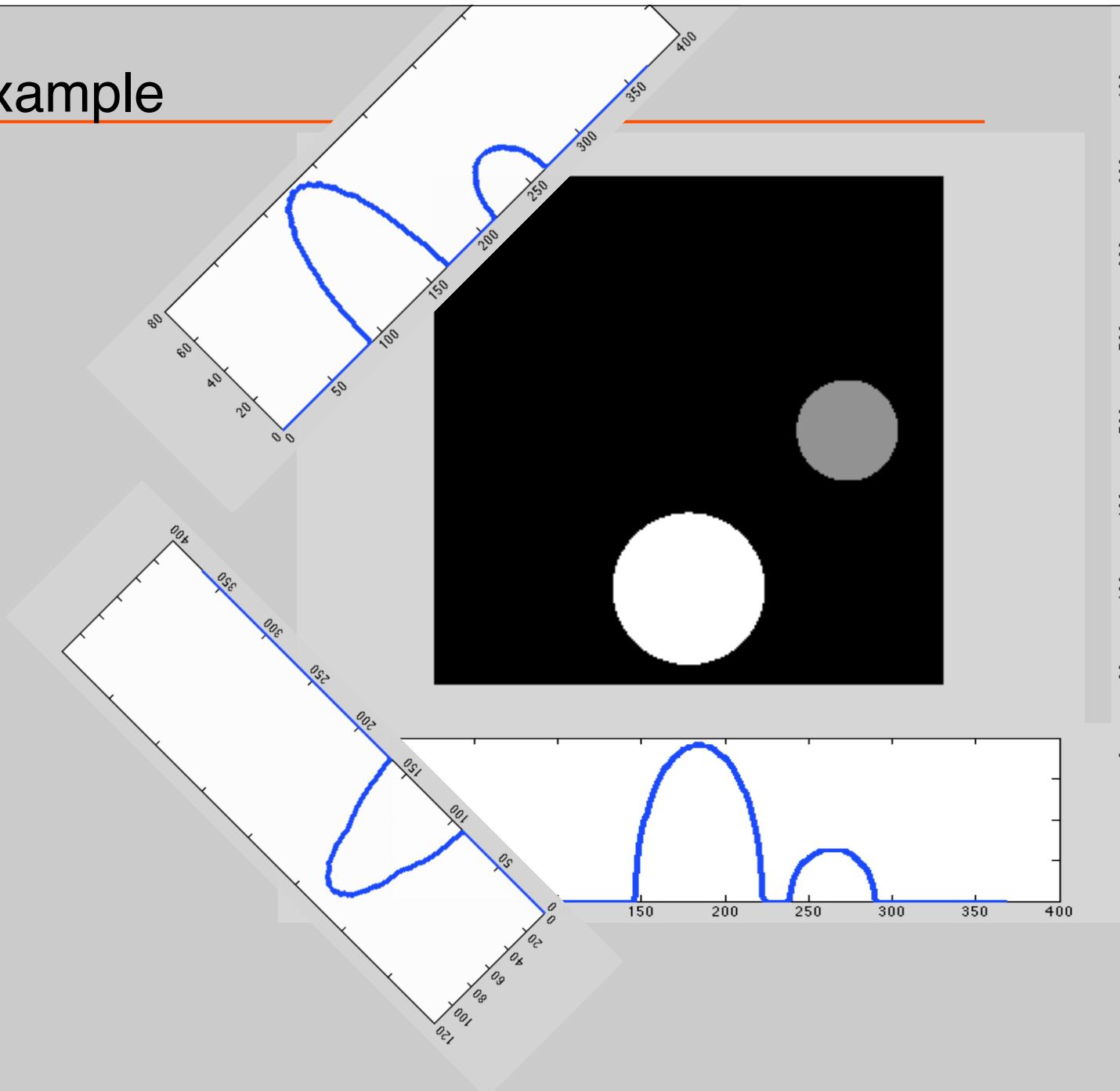
- Replace integrals with sums. Sum over radius and angle
- Define a (filtered) backprojection:

$$C_{\theta_m}[n_x, n_y] = \sum_{l=-N/2}^{(N/2)-1} F[l, \theta_m] e^{2\pi j(l/N \cos(\theta_m)n_x + l/N \sin(\theta_m)n_y)} |l/N| \ll \rho$$

So,

$$f[n_x, n_y] = \sum_m C_{\theta_m}[n_x, n_y]$$

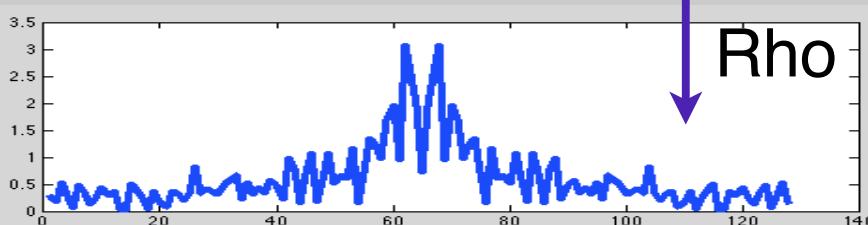
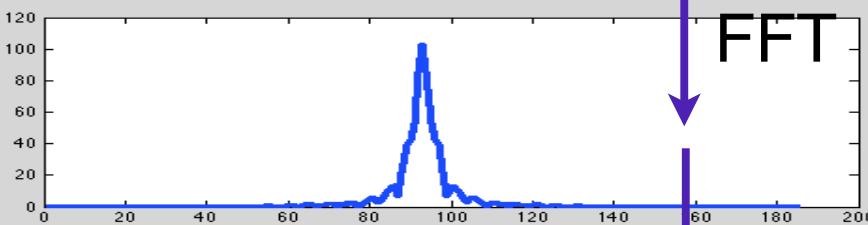
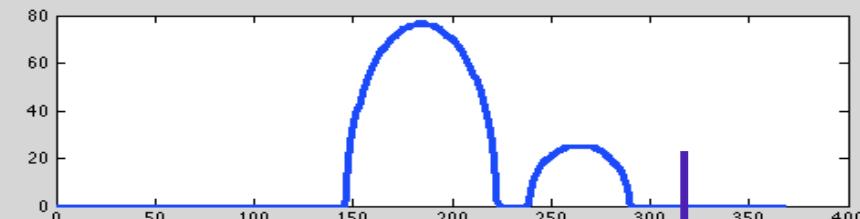
## Example



## Example Convolution Back Projection

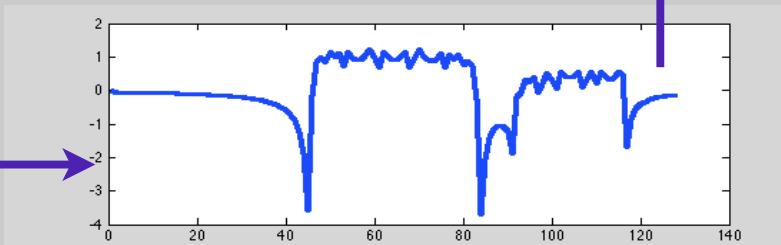
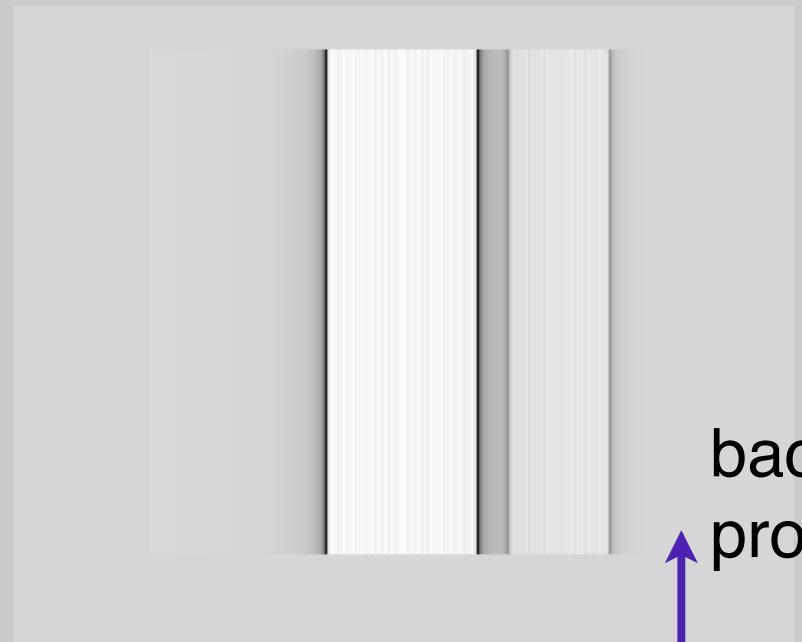
- For  $\Theta=0$

$$C_0[n_x, n_y] = \sum_{l=-N/2}^{(N/2)-1} F[l, 0] |l/N| e^{2\pi j(l/N n_x)}$$



$F[l, 0] |l/N|$

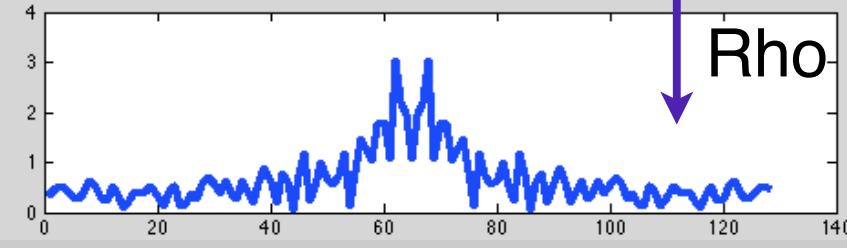
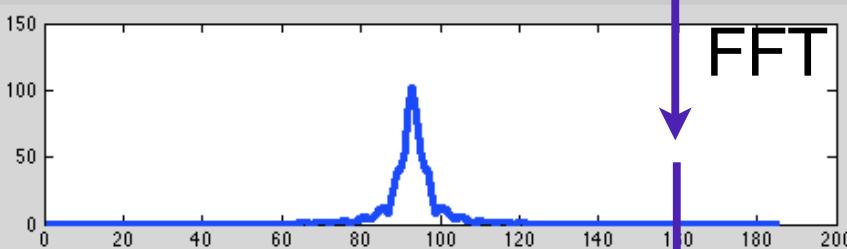
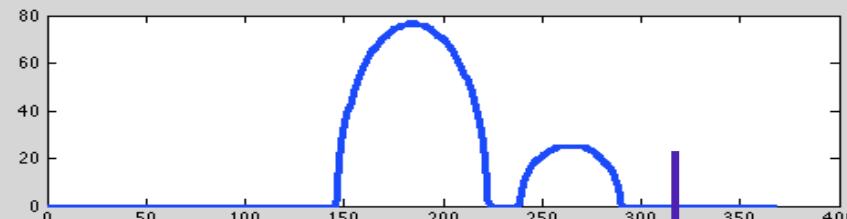
IFFT



back  
proj.

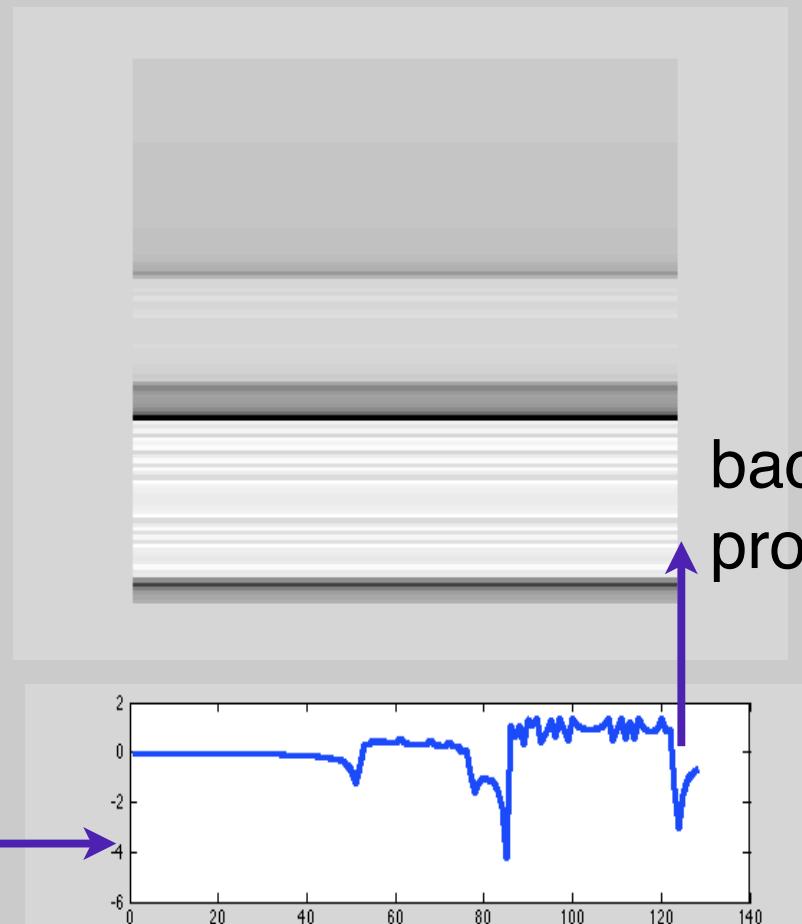
## Example Convolution Back Projection

- For  $\Theta=\pi/2$   $C_{\pi/2}[n_x, n_y] = \sum_{l=-N/2}^{(N/2)-1} F[l, \pi/2] |l/N| e^{2\pi j(l/N)n_y}$



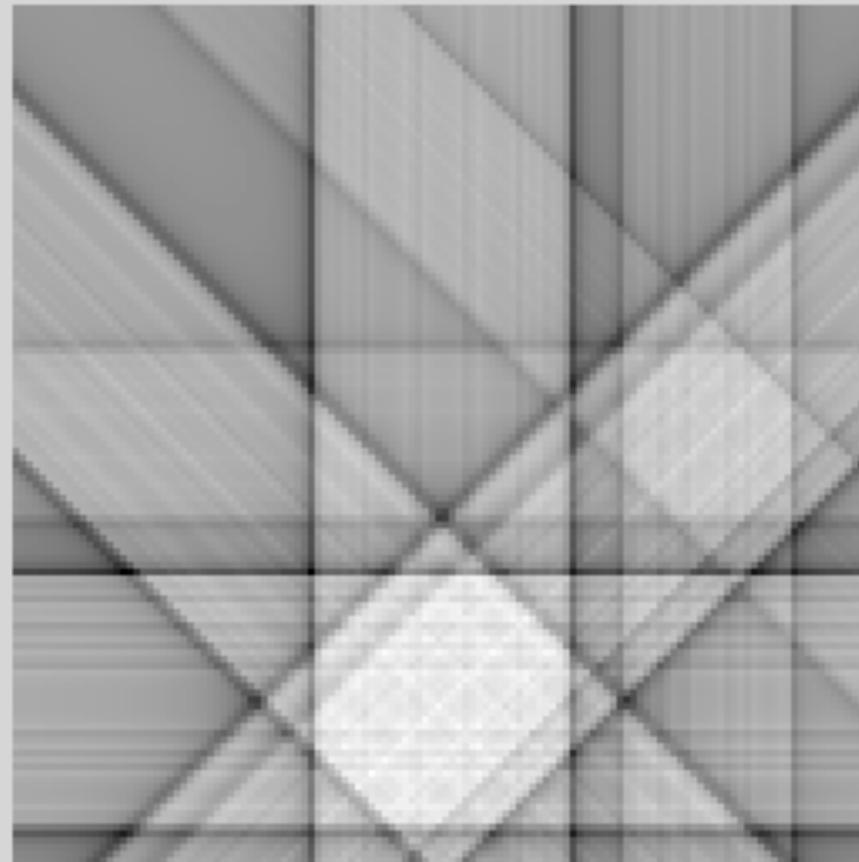
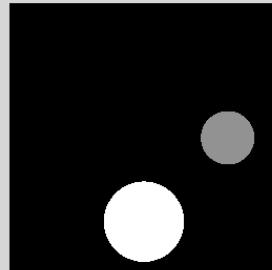
$F[l, \pi/2] |l/N|$

IFFT



# Convolution Back Projection

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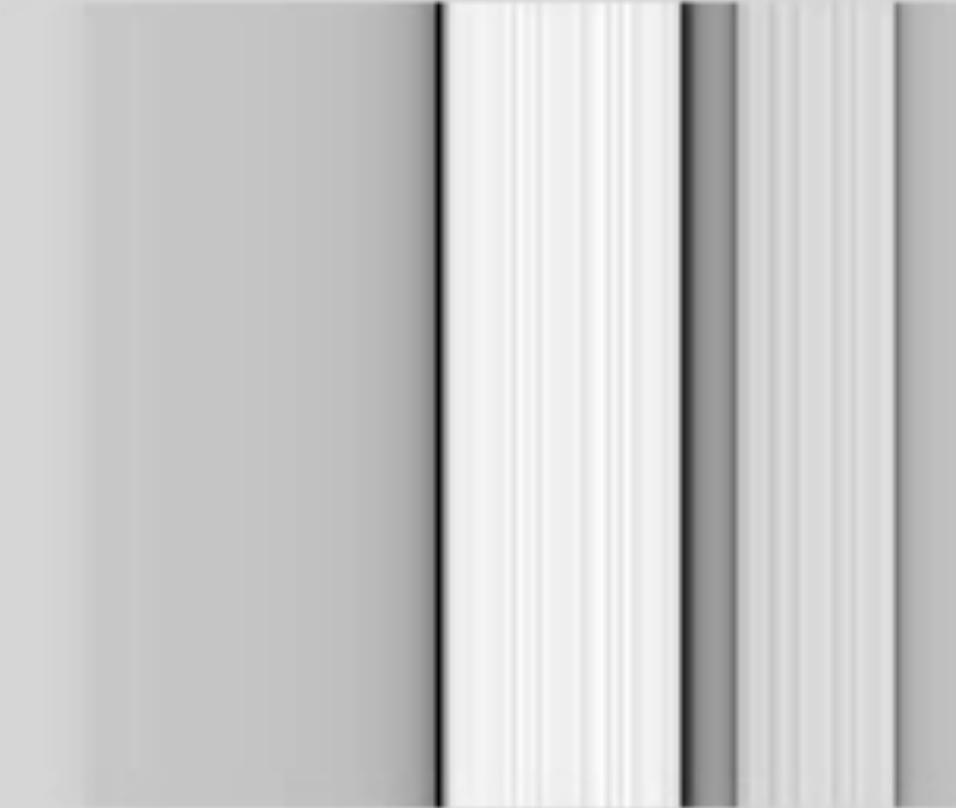


# Filtered Back Projection

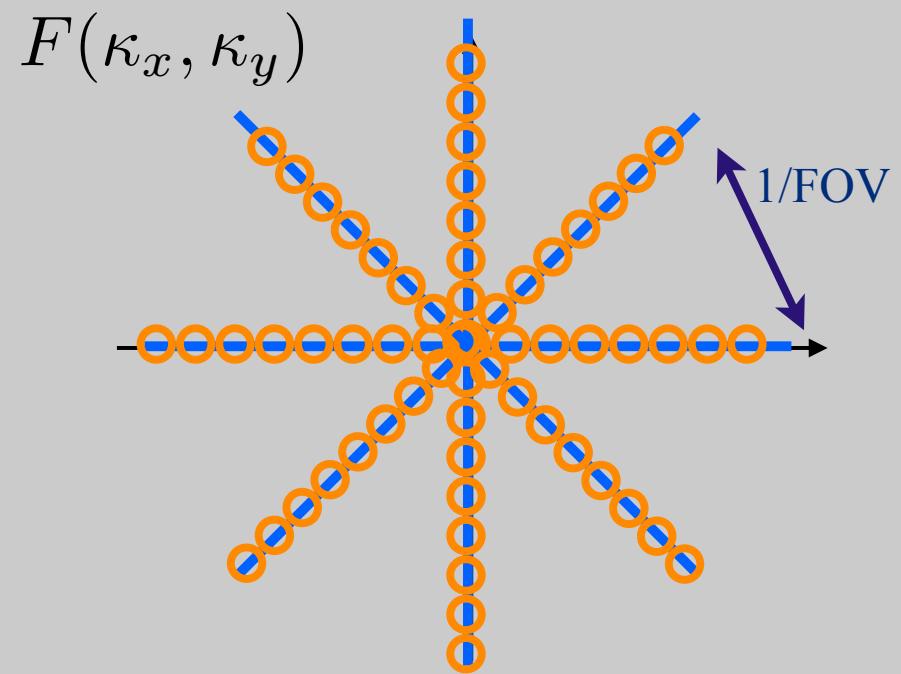
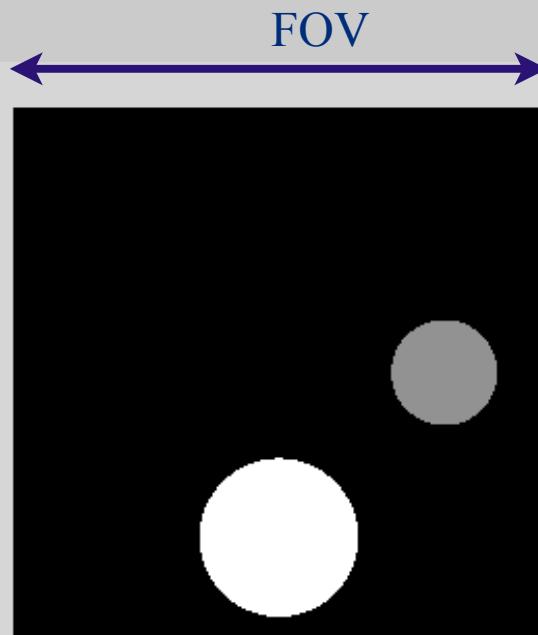
Back projection



Filtered Back projection

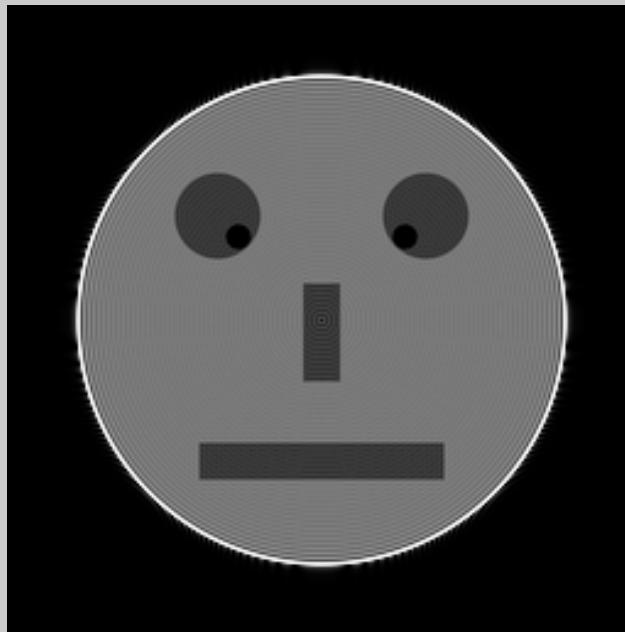


# How Many Projections?

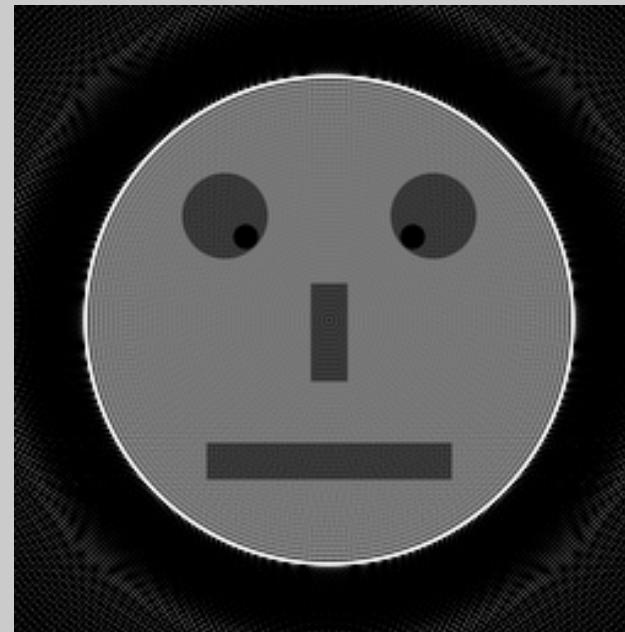


# How Many Projections?

256 Proj.



128 Proj

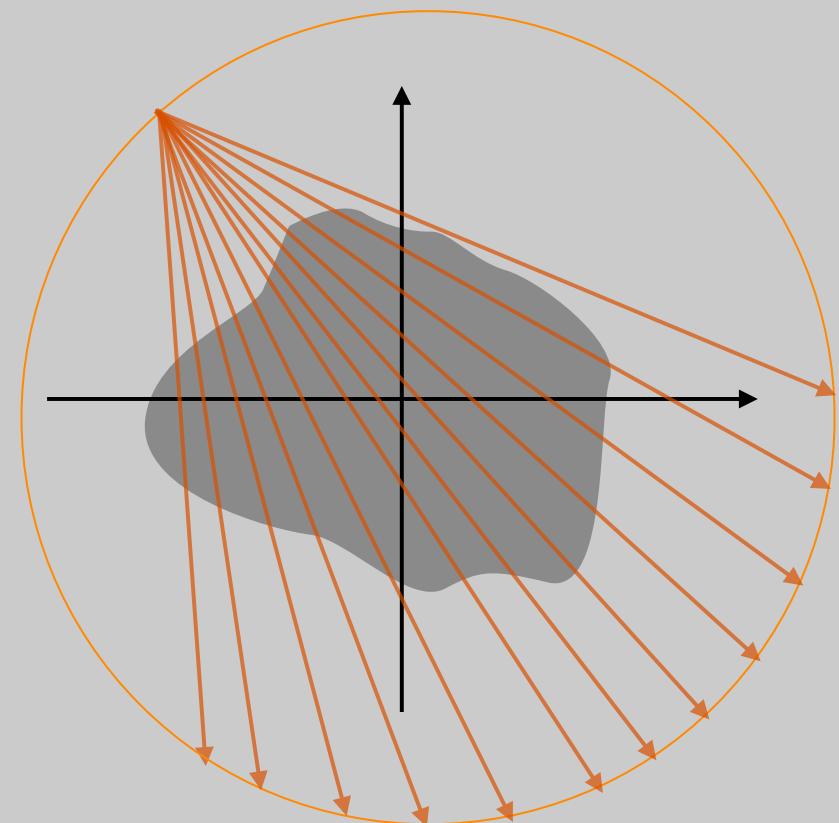


64 Proj



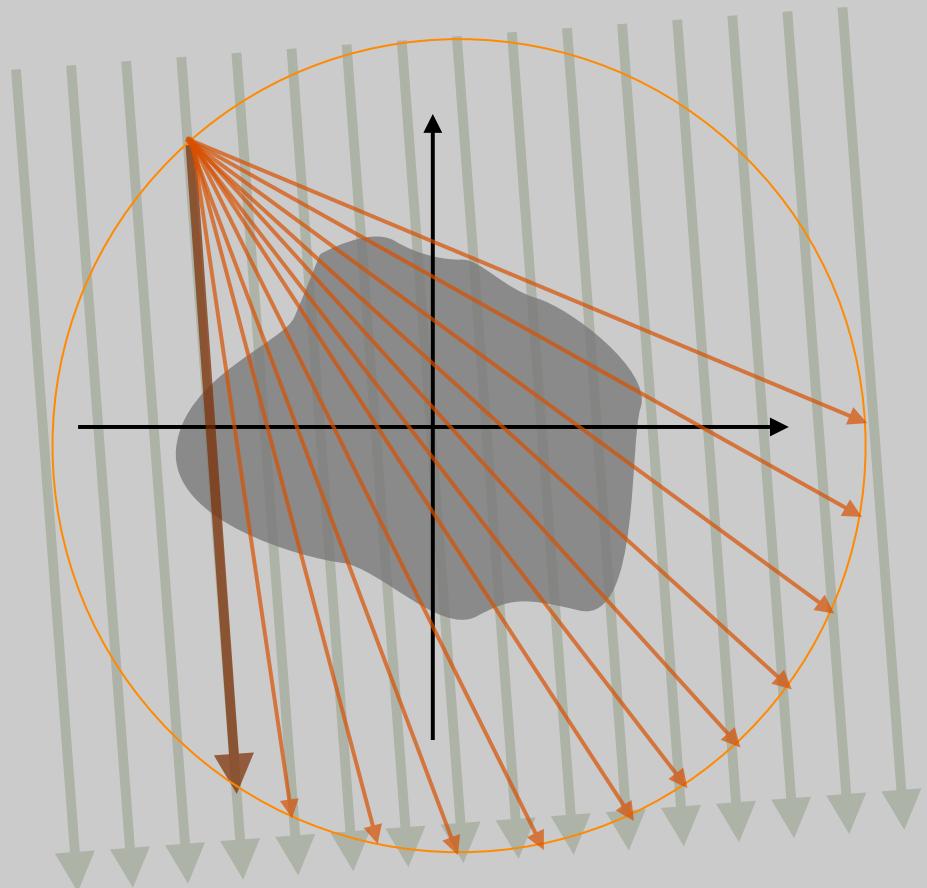
# Fan Beam CT

- Single Source
- Many detectors
- How to reconstruct?



# Fan Beam CT

- Single Source
- Many detectors
- How to reconstruct?
- Re-binning!



# Fan Beam CT

- Single Source
- Many detectors
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- Re-binning!

