

EE123

Digital Signal Processing

Lecture 21

Tomography

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Impulse lines and line-integrals

In 1D

$$\int_{-\infty}^{\infty} f(x)\delta(x)dx = f(0) \quad \text{A sample @ } x=0$$

In 2D

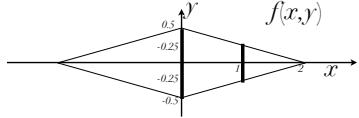
$$\int_{-\infty}^{\infty} f(x, y)\delta(x)dx = f(0, y) \quad \begin{matrix} \text{1D cross-section} \\ @x=0 \end{matrix}$$

Line Integral

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y)\delta(x)dy = \int_{-\infty}^{\infty} f(0, y)dy \quad \begin{matrix} \text{Integral of the 1D} \\ \text{cross-section @ } x=0 \end{matrix}$$

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Impulse lines and line-integrals



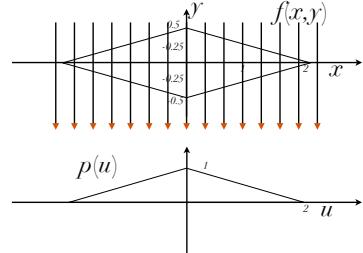
$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y)\delta(x)dx = f(0, y) = \Pi(y) \int_{-\infty}^{\infty} dy \begin{cases} 1 & -0.5 \leq y \leq 0.5 \\ 0 & \text{otherwise} \end{cases} = 1$$

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y)\delta(x-1)dx = f(1, y) = \Pi(2y) \int_{-\infty}^{\infty} dy \begin{cases} 1 & -0.25 \leq y \leq 0.25 \\ 0 & \text{otherwise} \end{cases} = 0.5$$

what about line integral with $\delta(x-u)$?

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Line Integral and Projection

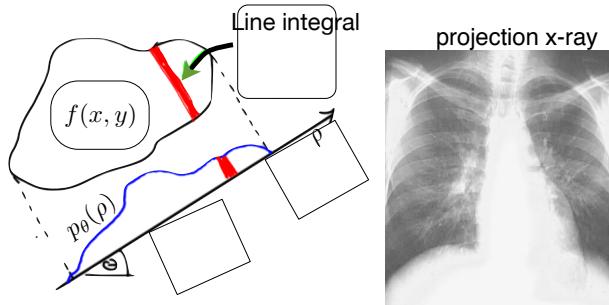


$$p(\underline{u}) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y)\delta(x-\underline{u})dxdy = \int_{-\infty}^{\infty} f(\underline{u}, y)dy$$

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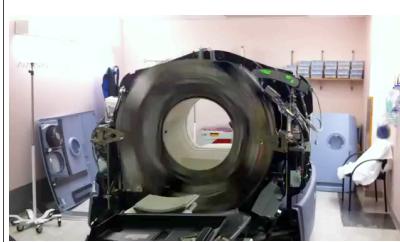
General Projections

$$p(\rho, \theta) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y)\delta(\rho - x \cos \theta - y \sin \theta)dxdy$$



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Many Projections - Tomography

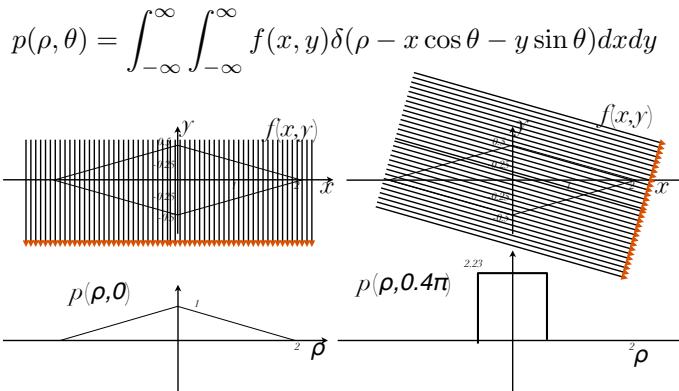


<http://www.youtube.com/watch?v=4gkOHM19aY&feature=related>



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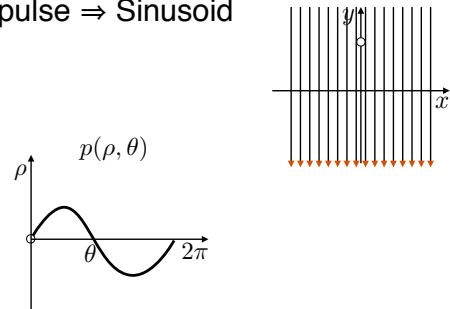
Radon Transform



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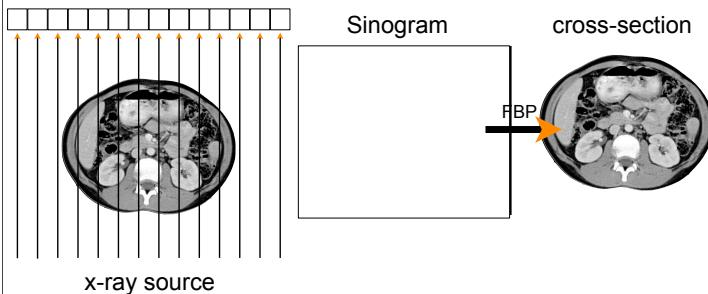
Radon Transform: Sinogram

- Also called Sinogram
- Impulse \Rightarrow Sinusoid



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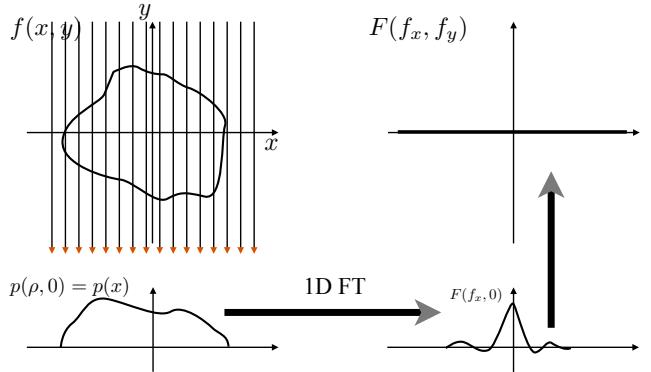
Computed Tomography



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Projection Slice Theorem (Bracewell)

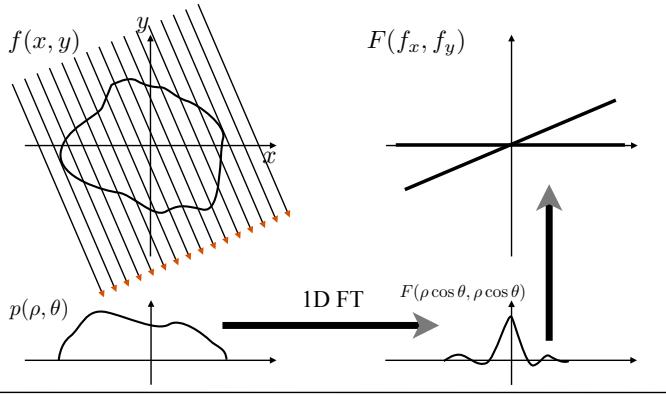
$$\mathcal{F}_{1D}\{p(\rho, \theta)\} = F(\rho \cos \theta, \rho \cos \theta)$$



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Projection Slice Theorem (Bracewell)

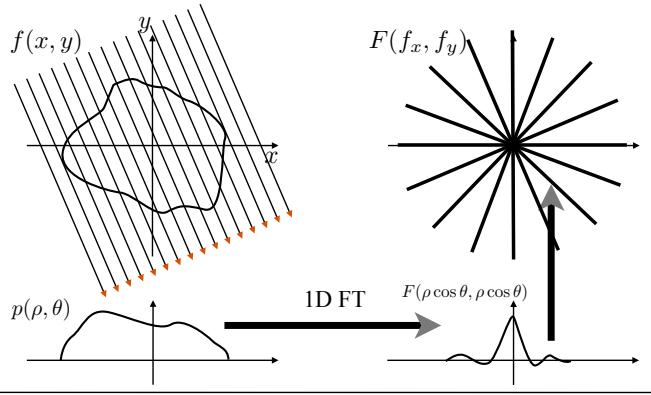
$$\mathcal{F}_{1D}\{p(\rho, \theta)\} = F(\rho \cos \theta, \rho \cos \theta)$$



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Projection Slice Theorem (Bracewell)

$$\mathcal{F}_{1D}\{p(\rho, \theta)\} = F(\rho \cos \theta, \rho \cos \theta)$$



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Projection Slice Theorem (Bracewell)

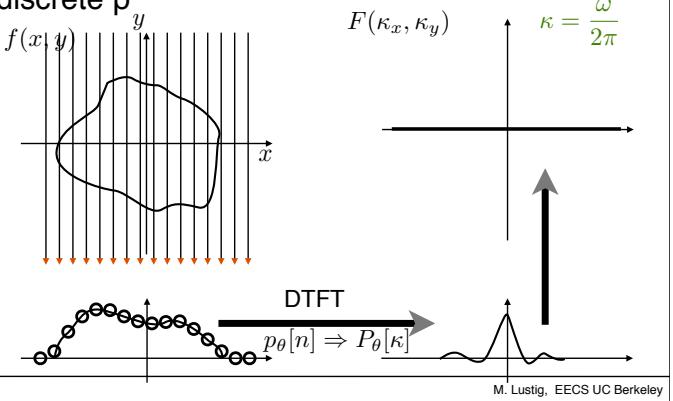
Proof (for $\Theta=0$)

$$\begin{aligned} p(x) &= \int_{-\infty}^{\infty} m(x, y) dy \\ M(k_x, 0) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} m(x, y) e^{-i2\pi(k_x x + k_y y)} dx dy = \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} m(x, y) e^{-i2\pi k_x x} dx dy = \\ &= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} m(x, y) dy \right] e^{-i2\pi k_x x} dx = \\ &= \int_{-\infty}^{\infty} p(x) e^{-i2\pi k_x x} dx = \mathcal{F}\{p(x)\} \end{aligned}$$

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Partly Discrete Reconstruction

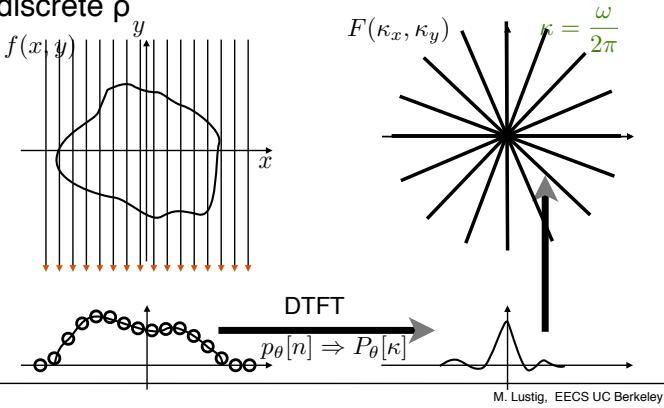
- Let's assume continuous angle Θ , discrete ρ



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Partly Discrete Reconstruction

- Let's assume continuous angle Θ , discrete ρ



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Reconstruction From Polar Coordinates

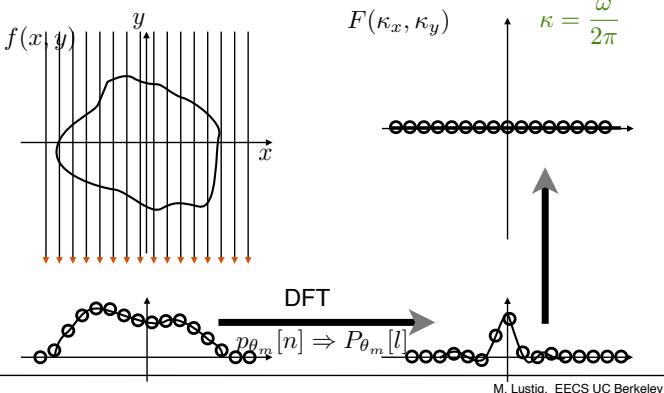
$$\begin{aligned} f[n, m] &= \int_{-0.5}^{0.5} \int_{-0.5}^{0.5} F(\kappa_x, \kappa_y) e^{2\pi j(\kappa_x n + \kappa_y m)} d\kappa_x d\kappa_y \\ &= \int_0^\pi \int_{-0.5}^{0.5} F(\rho, \theta) e^{2\pi j(\rho \cos(\theta)n + \rho \sin(\theta)m)} |\rho| d\rho d\theta \end{aligned}$$

- Polar frequency data must be multiplied by $|\rho|$
- Also called a rho filter

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Discrete Reconstruction

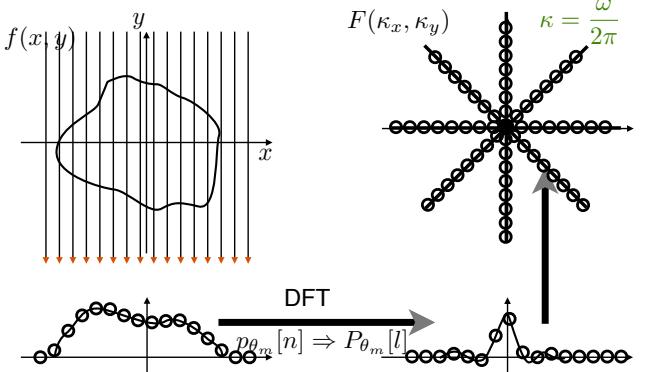
- Let's assume discrete angle Θ_m , discrete ρ



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Discrete Reconstruction

- Let's assume discrete angle Θ_m , discrete ρ



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Filtered Back Projection

- Replace integrals with sums. Sum over radius and angle
- Define a (filtered) backprojection:

$$C_{\theta_m}[n_x, n_y] = \sum_{l=-N/2}^{(N/2)-1} F[l, \theta_m] e^{2\pi j(l/N \cos(\theta_m)n_x + l/N \sin(\theta_m)n_y)} |l/N|$$

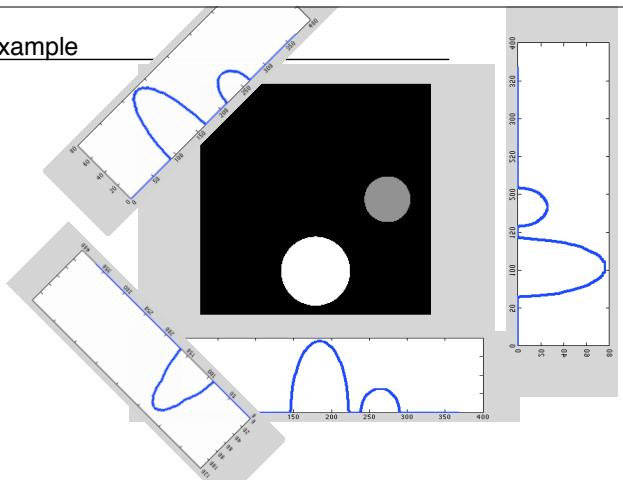
$\nwarrow \rho$

So,

$$f[n_x, n_y] = \sum_m C_{\theta_m}[n_x, n_y]$$

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Example

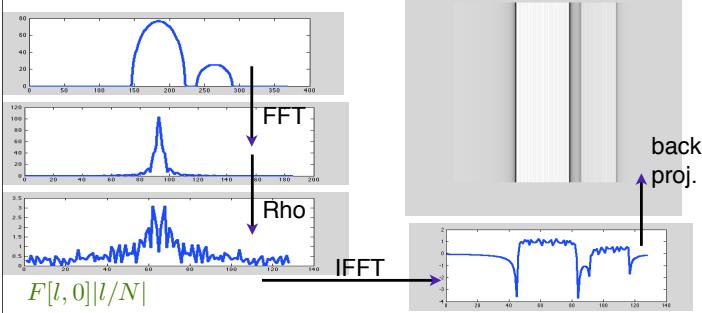


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Example Convolution Back Projection

- For $\Theta=0$

$$C_0[n_x, n_y] = \sum_{l=-N/2}^{(N/2)-1} F[l, 0] |l/N| e^{2\pi j(l/N n_x)}$$

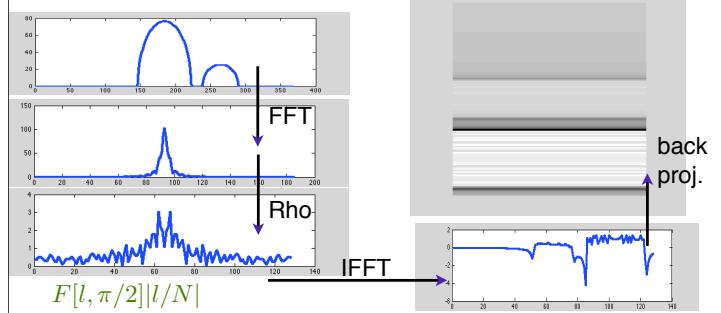


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Example Convolution Back Projection

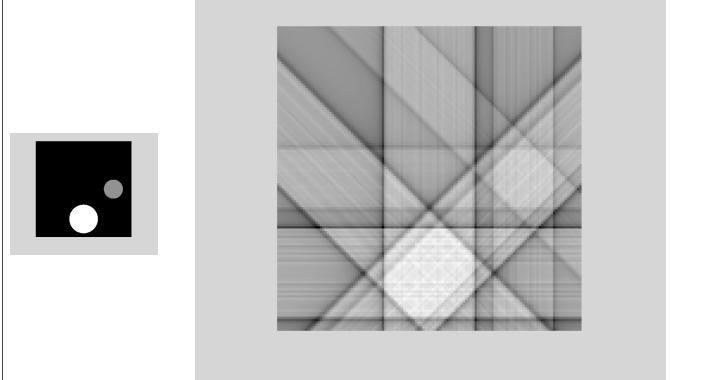
- For $\Theta=\pi/2$

$$C_{\pi/2}[n_x, n_y] = \sum_{l=-N/2}^{(N/2)-1} F[l, \pi/2] |l/N| e^{2\pi j(l/N n_y)}$$



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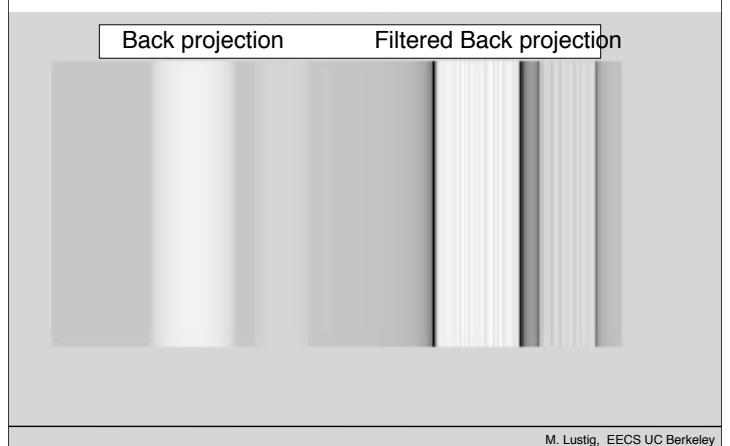
Convolution Back Projection



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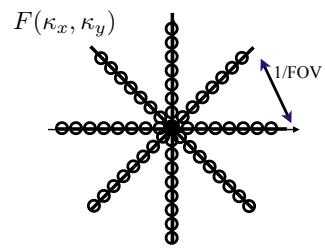
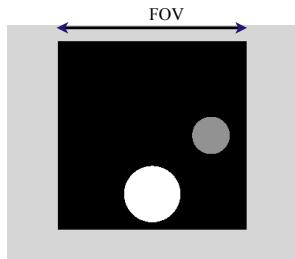
Filtered Back Projection

Back projection Filtered Back projection



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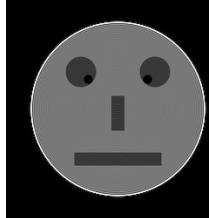
How Many Projections?



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How Many Projections?

256 Proj.



128 Proj



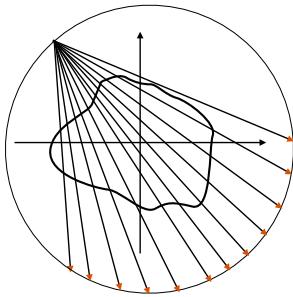
64 Proj



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Fan Beam CT

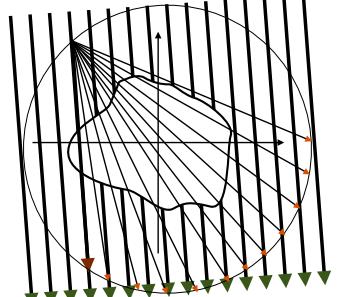
- Single Source
- Many detectors
- How to reconstruct?



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Fan Beam CT

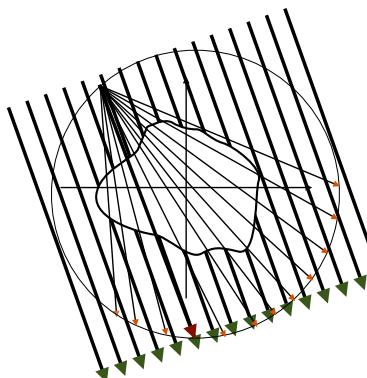
- Single Source
- Many detectors
- How to reconstruct?
- Re-binning!



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Fan Beam CT

- Single Source
- Many detectors
- How to reconstruct?
- Re-binning!



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