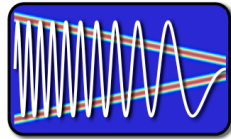


EE123



# Digital Signal Processing

## Lecture 26 Optimal Filter Design

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### Announcements

- Compressed Sensing HW, optional
- New lab posted -- tests your audio interfaces
- Projects: Start thinking about it.
  - It's should 1 week of work.
  - Groups of 2 minimum, 3 possible depending on the scope
  - There will be a default project based on lab 6

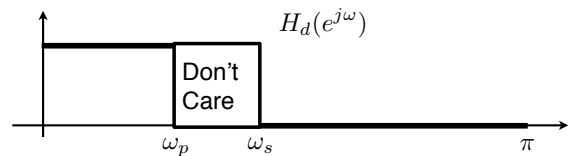
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### Optimal Filter Design

- Window method
  - Design Filters heuristically using windowed sinc functions
- Optimal design
  - Design a filter  $h[n]$  with  $H(e^{j\omega})$
  - Approximate  $H_d(e^{j\omega})$  with some optimality criteria - or satisfies specs.

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### Optimality



- Least Squares:

$$\text{minimize} \int_{\omega \in \text{care}} |H(e^{j\omega}) - H_d(e^{j\omega})|^2 d\omega$$

Variation: weighted least-squares

$$\text{minimize} \int_{-\pi}^{\pi} W(\omega) |H(e^{j\omega}) - H_d(e^{j\omega})|^2 d\omega$$

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### Optimality

- Chebychev Design (min-max)

$$\text{minimize}_{\omega \in \text{care}} \max |H(e^{j\omega}) - H_d(e^{j\omega})|$$

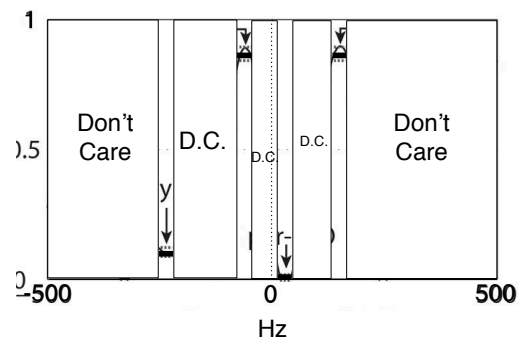
- Parks-McClellan algorithm - equi-ripple
- Also known as Remez exchange algorithms (signal.remez)

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### Example of Complex Filter

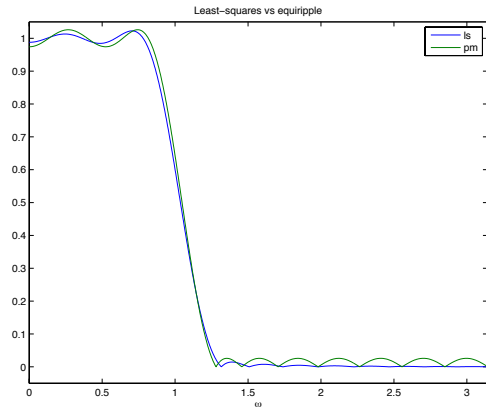
Larson et. al, "Multiband Excitation Pulses for Hyperpolarized <sup>13</sup>C Dynamic Chemical Shift Imaging" JMR 2008;194(1):121-127

Need to design 11 taps filter with following frequency response:



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## Least-Squares v.s. Min-Max



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## Design Through Optimization

- Idea: Sample/discretize the frequency response

$$H(e^{j\omega}) \Rightarrow H(e^{j\omega_k})$$

- Sample points are fixed  $\omega_k = k \frac{\pi}{P}$

$$-\pi \leq \omega_1 < \dots < \omega_p \leq \pi$$

- M+1 is the filter order
- P >> M + 1 (rule of thumb P=15M)
- Yields a (good) approximation of the original problem

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## Example: Least Squares

- Target: Design M+1=2N+1 filter
- First design non-causal  $\tilde{H}(e^{j\omega})$  and hence  $\tilde{h}[n]$
- Then, shift to make causal

$$h[n] = \tilde{h}[n - M/2]$$

$$H(e^{j\omega}) = e^{-j\frac{M}{2}\omega} \tilde{H}(e^{j\omega})$$

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## Example: Least Squares

- Matrix formulation:

$$\tilde{h} = [\tilde{h}[-N], \tilde{h}[-N+1], \dots, \tilde{h}[N]]^T$$

$$b = [H_d(e^{j\omega_1}), \dots, H_d(e^{j\omega_P})]^T$$

$$A = \begin{bmatrix} e^{-j\omega_1(-N)} & \dots & e^{-j\omega_1(+N)} \\ \vdots & & \vdots \\ e^{-j\omega_P(-N)} & \dots & e^{-j\omega_P(+N)} \end{bmatrix}$$

$$\operatorname{argmin}_{\tilde{h}} \|A\tilde{h} - b\|^2$$

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## Least Squares

$$\operatorname{argmin}_{\tilde{h}} \|A\tilde{h} - b\|^2$$

Solution:

$$\tilde{h} = (A^*A)^{-1}A^*b$$

- Result will generally be non-symmetric and complex valued.
- However, if  $\tilde{H}(e^{j\omega})$  is real,  $\tilde{h}[n]$  should have symmetry!

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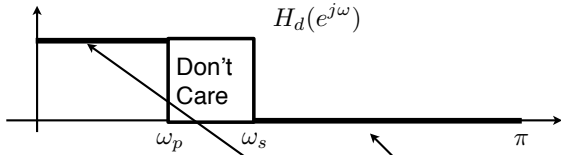
## Design of Linear-Phase L.P Filter

- Suppose:
  - $\tilde{H}(e^{j\omega})$  is real-symmetric
  - M is even (M+1 taps)
- Then:
  - $\tilde{h}[n]$  is real-symmetric around midpoint
- So:

$$\begin{aligned} \tilde{H}(e^{j\omega}) &= \tilde{h}[0] + \tilde{h}[1]e^{-j\omega} + \tilde{h}[-1]e^{+j\omega} \\ &\quad + \tilde{h}[2]e^{-j2\omega} + \tilde{h}[-2]e^{+j2\omega} \dots \\ &= \tilde{h}[0] + 2 \cos(\omega)\tilde{h}[1] + 2 \cos(2\omega)\tilde{h}[2] + \dots \end{aligned}$$

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### Least-Squares Linear-Phase Filter



Given  $M$ ,  $\omega_p$ ,  $\omega_s$  find the best LS filter:

$$A = \begin{bmatrix} \text{---} \\ \text{---} \end{bmatrix} \quad b = \begin{bmatrix} \text{---} \\ \text{---} \end{bmatrix}^T$$

### Least-Squares Linear-Phase Filter

Given  $M$ ,  $\omega_p$ ,  $\omega_s$  find the best LS filter:

$$A = \begin{bmatrix} \text{---} \\ \text{---} \end{bmatrix} \quad b = \begin{bmatrix} \text{---} \\ \text{---} \end{bmatrix}^T$$

$$\tilde{h}_+ = [\tilde{h}[0], \dots, \tilde{h}[\frac{M}{2}]]^T = (A^*A)^{-1}A^*b$$

$$\tilde{h} = \begin{cases} \tilde{h}_+[n] & n \geq 0 \\ \tilde{h}_+[-n] & n < 0 \end{cases}$$

$$h[n] = \tilde{h}[n - M/2]$$

### Extension:

- LS has no preference for pass band or stop band
- Use weighting of LS to change ratio

want to solve the discrete version of:

$$\text{minimize} \int_{-\pi}^{\pi} W(\omega) |H(e^{j\omega}) - H_d(e^{j\omega})|^2 d\omega$$

where  $W(\omega)$  is  $\delta_p$  in the pass band and  $\delta_s$  in stop band

Similarly:  $W(\omega)$  is 1 in the pass band and  $\delta_p/\delta_s$  in stop band

### Weighted Least-Squares

$$\text{argmin}_{\tilde{h}_+} (A\tilde{h}_+ - b)^* W^2 (A\tilde{h}_+ - b)$$

Solution:

$$\tilde{h}_+ = (A^* W^2 A)^{-1} A^* W^2 b$$

$$W = \begin{bmatrix} 1 & & & & & & & & 0 \\ & 1 & & & & & & & \\ & & \dots & & & & & & \\ & & & \delta_p & & & & & \\ & & & & \delta_s & & & & \\ & & & & & \dots & & & \\ 0 & & & & & & & \delta_p & \\ & & & & & & & & \delta_s \end{bmatrix}$$