EE123
Digital Signal Processing

Lecture 28
Phase Response
All-Pass and Minimum Phase

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* Beautiful handwritten figures by Prof. Murat Arcak

GEOMAGNETIC STORM SPARKS AURORAS: Unsetted solar wind conditions + the possible arrival of a weak CME ignited a G2-class geomagnetic storm during the early hours of April 10th. Northern Lights spilled over the Canadian border into the USA as far south as Idaho, Montana and Colorado. The storm is subsiding now, but NOAA forecasters estimate a 50% chance that it could flare up again before the end of the day

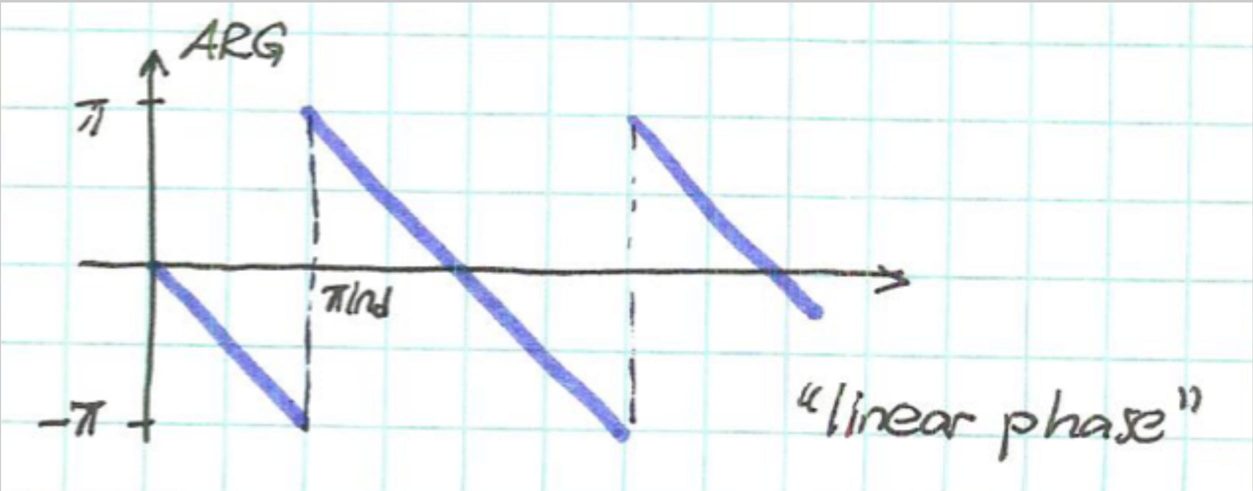
Phase response

Example: $H(e^{j\omega}) = e^{-j\omega n_d} \quad \leftrightarrow \quad h[n] = \delta[n - n_d]$

$$|H(e^{j\omega})| = 1$$

$$\arg[H(e^{j\omega})] = -\omega n_d$$

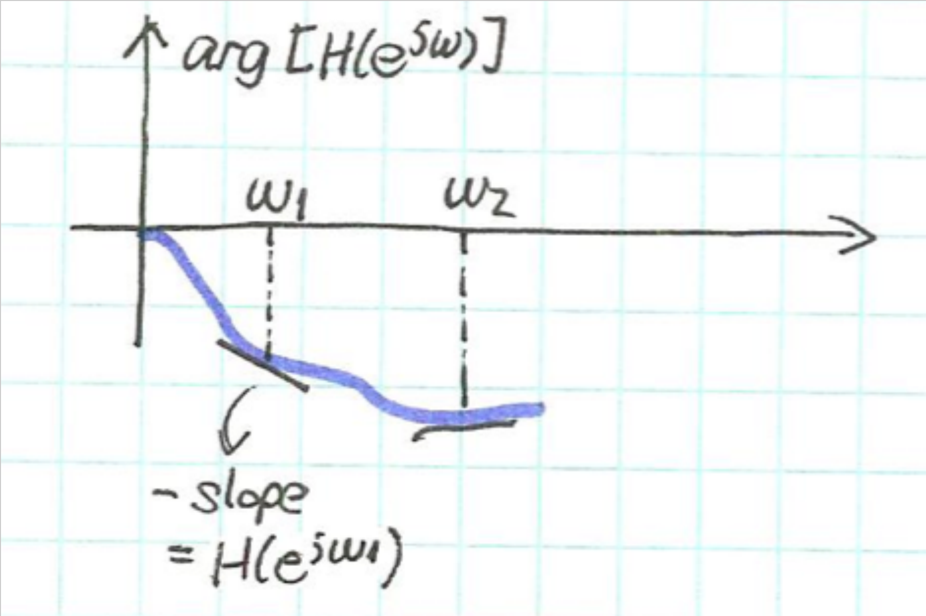
ARG is the wrapped phase
arg is the unwrapped phase



Group delay

To characterize general phase response, look at the group delay:

$$\mathrm{grd}[H(e^{j\omega})] = -\frac{d}{d\omega}\{\arg[H(e^{j\omega})]\}$$

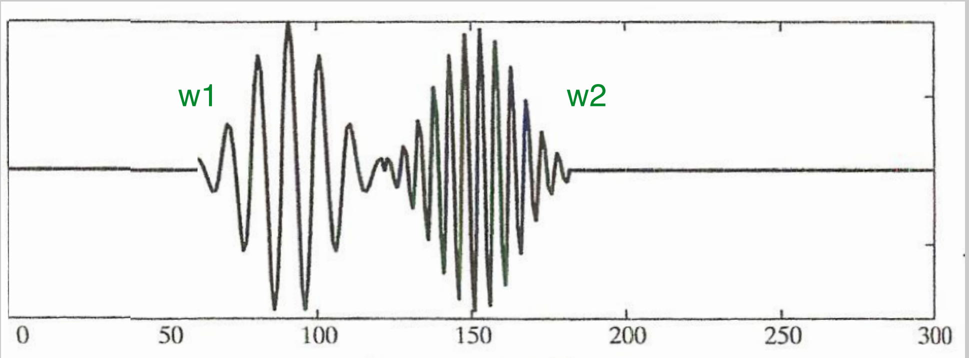


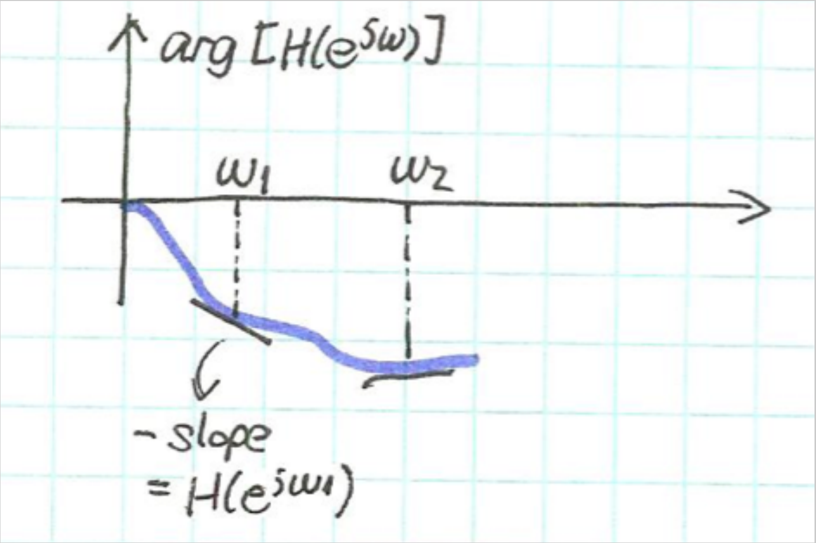
For linear phase system, the group delay is n_d

Group delay

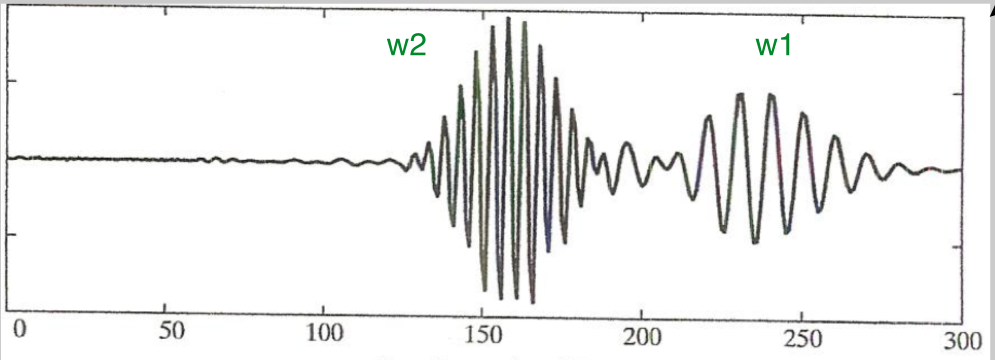
$$\text{grd}[H(e^{j\omega})] = -\frac{d}{d\omega}\{\arg[H(e^{j\omega})]\}$$

Input





Output



For narrowband signals, phase response looks like a linear phase

Group delay math

$$H(z) = \frac{b_0}{a_0} \frac{\prod_{k=1}^{M} (1 - c_k z^{-1})}{\prod_{k=1}^{N} (1 - d_k z^{-1})}$$

arg of products is sum of args

$$\arg[H(e^{j\omega})] = -\sum_{k=1}^{N} \arg[1 - d_k e^{-j\omega}] + \sum_{k=1}^{M} \arg[1 - c_k e^{-j\omega}]$$

$$\operatorname{grd}[H(e^{j\omega})] = -\sum_{k=1}^{N} \operatorname{grd}[1 - d_k e^{-j\omega}] + \sum_{k=1}^{M} \operatorname{grd}[1 - c_k e^{-j\omega}]$$

Group delay math

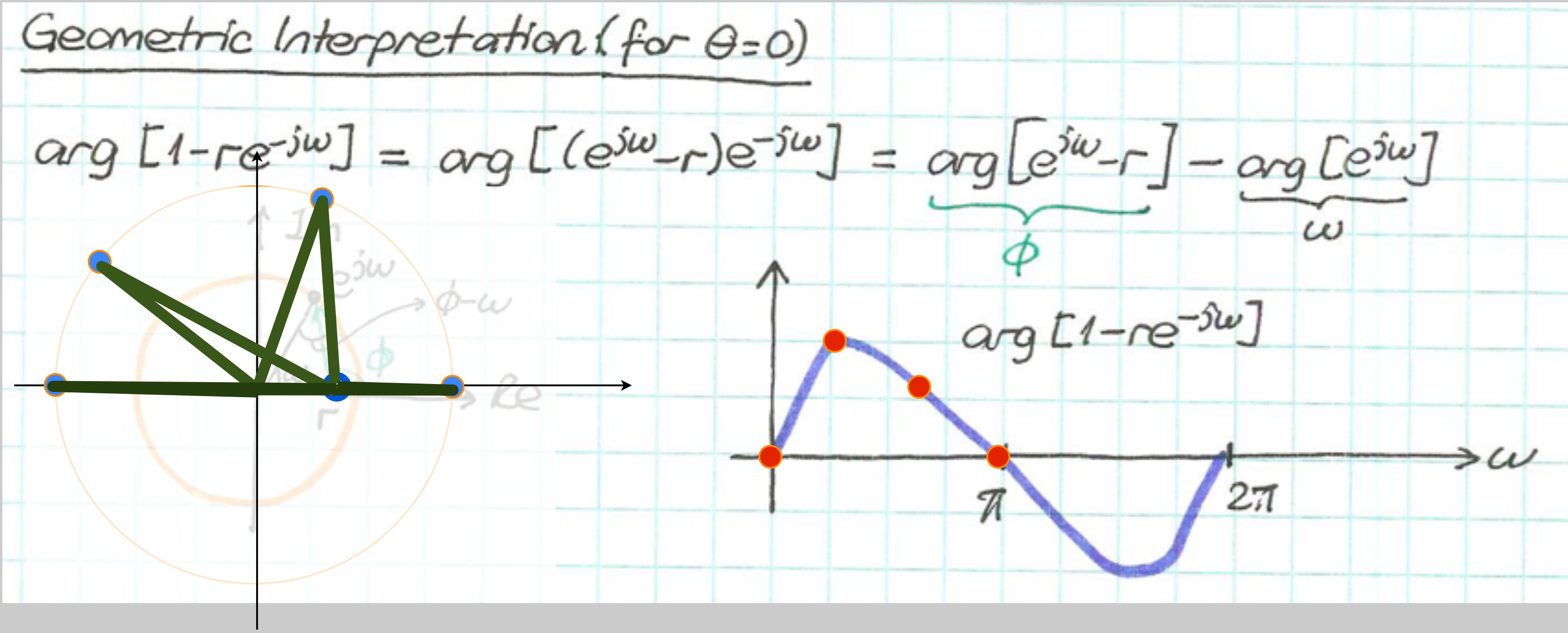
$$\text{grd}[H(e^{j\omega})] = -\sum_{k=1}^{N} \text{grd}[1-d_k e^{-j\omega}] + \sum_{k=1}^{M} \text{grd}[1-c_k e^{-j\omega}]$$

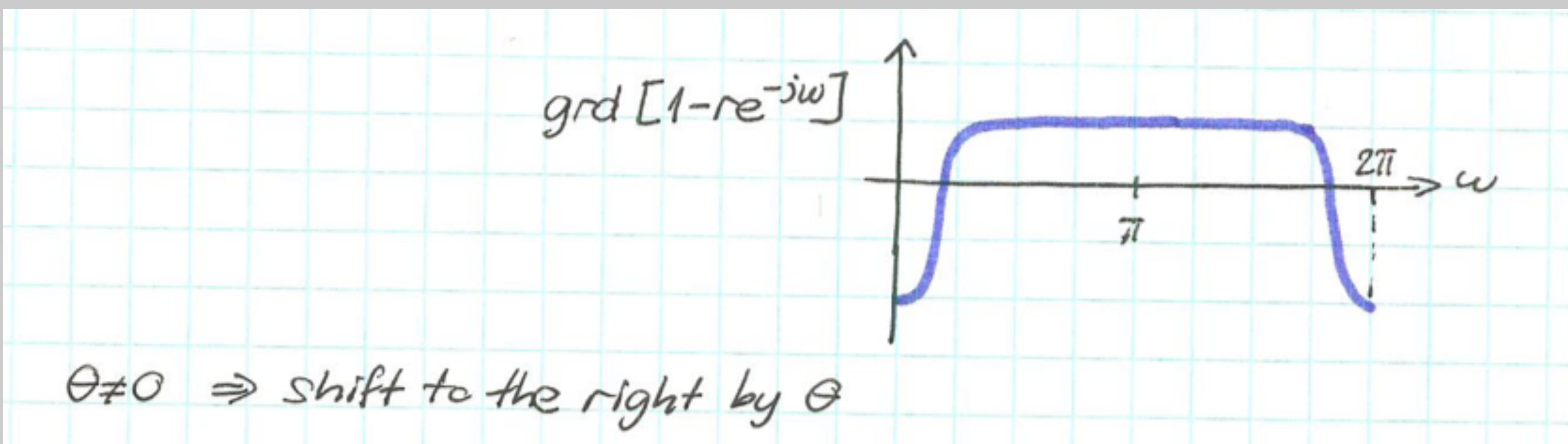
Look at each factor:

$$\arg[1-\underbrace{re^{j\theta}}_{c_k \text{ or } d_k} e^{-j\omega}] = \tan^{-1}\left(\frac{r\sin(\omega-\theta)}{1-r\cos(\omega-\theta)}\right)$$

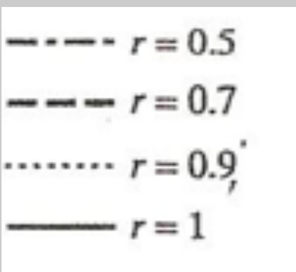
$$\text{grd}[1-re^{j\theta}e^{-j\omega}] = \frac{r^2-r\cos(\omega-\theta)}{|1-re^{j\theta}e^{-j\omega}|^2}$$

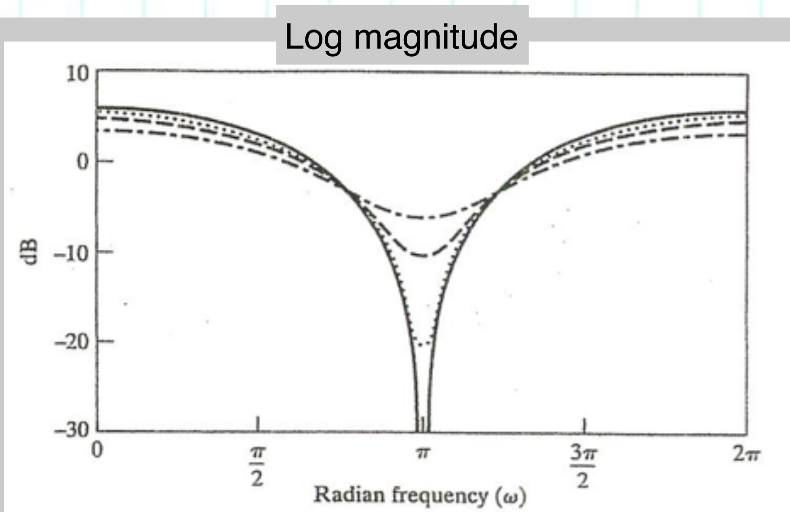
Look at a zero lying on the real axis

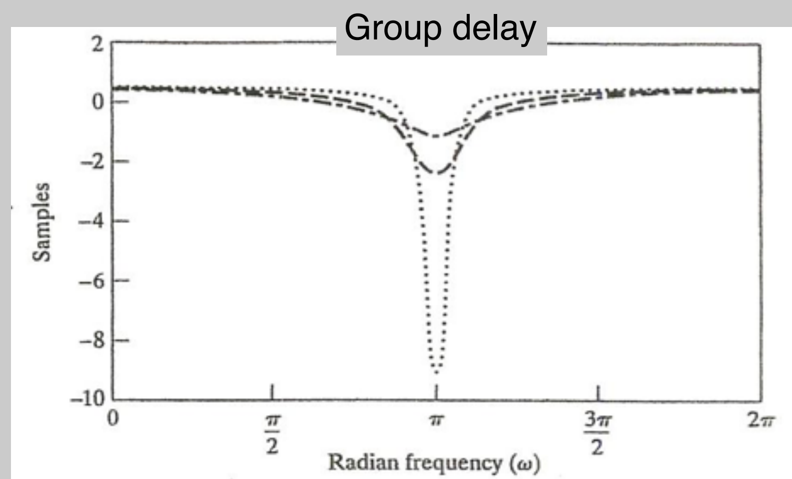


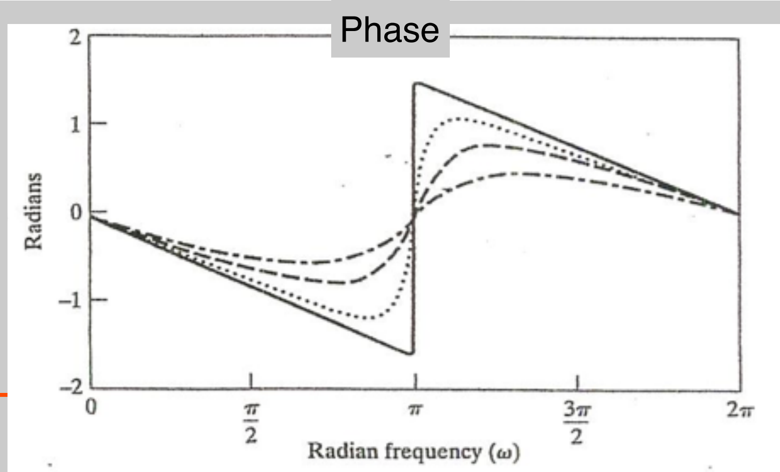


* Poles increase magnitude, but introduce phase lag and group delay.
* Zeros do the opposite.
* These effects are more marked when $r \to 1$.





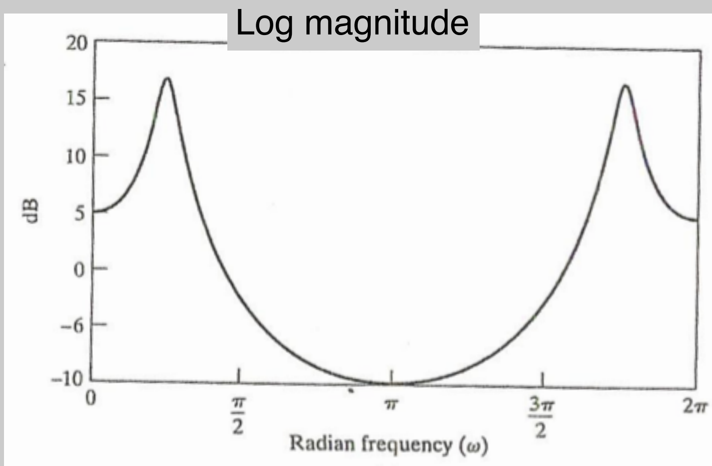


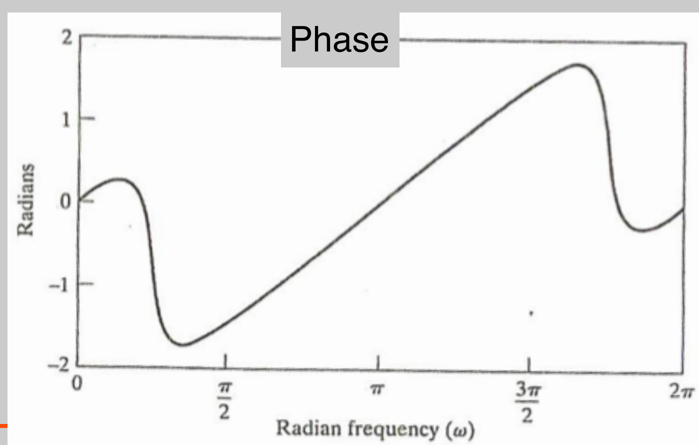


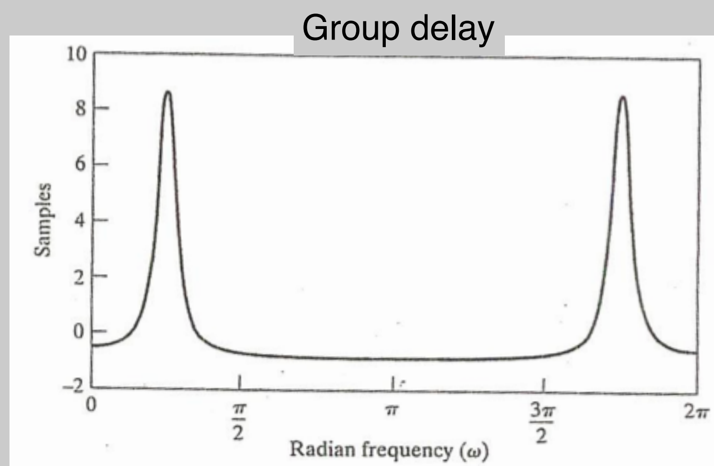
2nd order IIR example

Example: 2nd order IIR with complex poles

$$H(z) = \frac{1}{(1-re^{j\theta}z^{-1})(1-re^{-j\theta}z^{-1})}$$

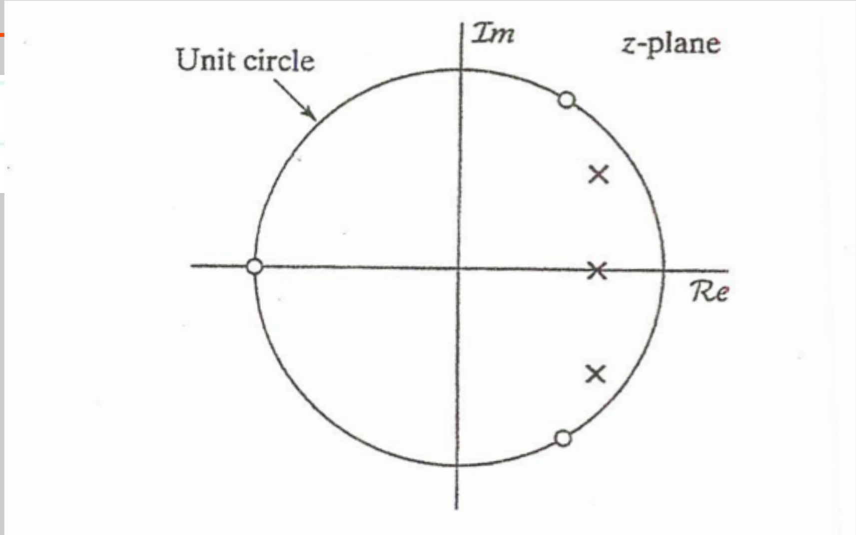




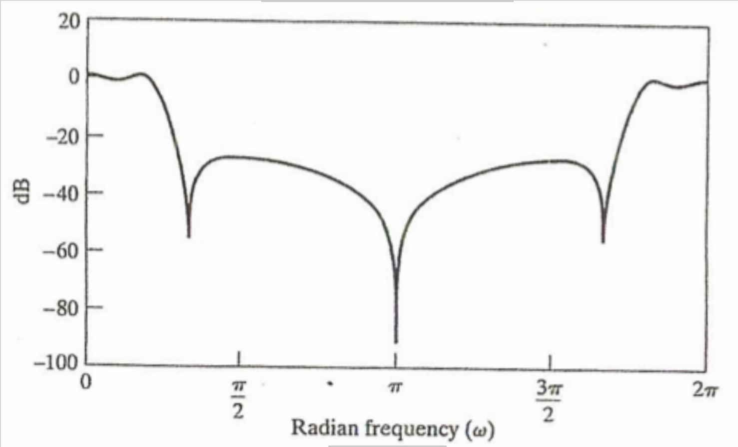


3nd order IIR example

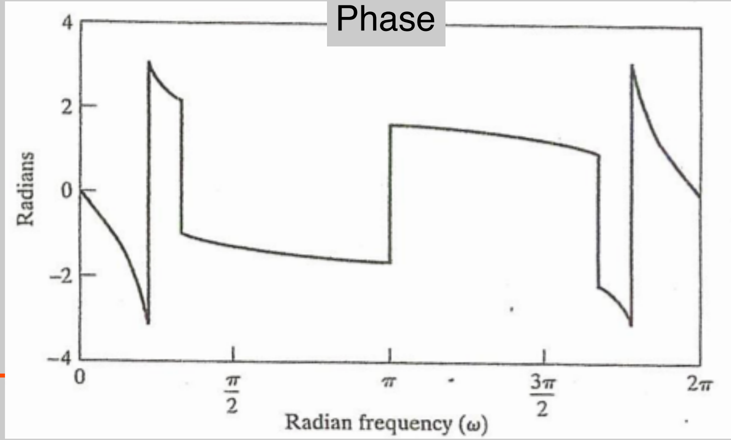
Example: 3rd order IIR



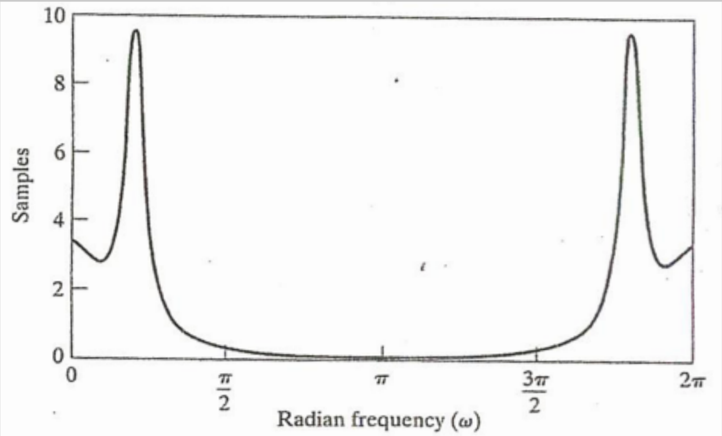
Log magnitude



Phase



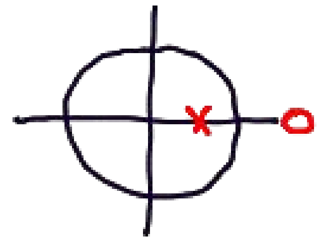
Group delay



All-Pass Systems

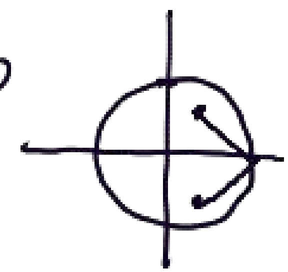
what is the magnitude response of

$$H(z) = \frac{z^{-1} - a^*}{1 - a z^{-1}}$$



$$|H(e^{j\omega})| = \frac{|e^{-j\omega} - a^*|}{|1 - a e^{-j\omega}|} = \frac{|e^{-j\omega}(1 - a^* e^{j\omega})|}{|1 - a e^{-j\omega}|} =$$

$$= \frac{|1 - a^* e^{j\omega}|}{|1 - (a^* e^{j\omega})^*|} = 1 \quad \forall \omega$$



A generall all-pass system: ③

$$H_{ap}(z) = \prod_{k=1}^{M_r} \frac{z^{-1} - d_k}{1 - d_k z^{-1}} \cdot \prod_{k=1}^{M_c} \frac{z^{-1} - e_k^*}{1 - e_k z^{-1}} \cdot \frac{z^{-1} - e_k}{1 - e_k^* z^{-1}}$$

Real

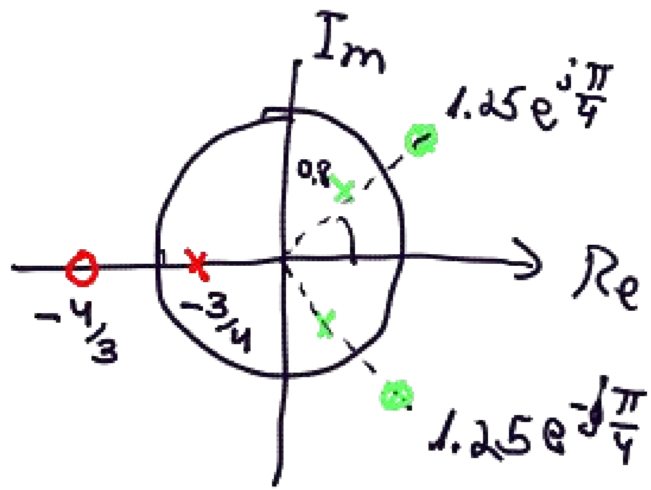
complex

d_k: real Poles

e_k: complex poles paired w/ conjugate e_k^*

$$|H_{ap}(e^{j\omega})| \equiv 1$$

Example



phase response of an all-pass:

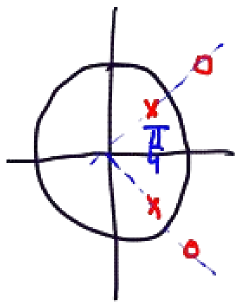
(4)

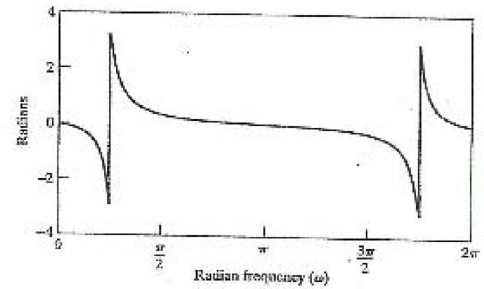
$$\arg\left[\frac{e^{-j\omega} - \overbrace{r e^{-j\theta}}^{a^*}}{1 - \underbrace{r e^{j\theta}}_{a} e^{-j\omega}}\right] = \arg\left[\frac{e^{-j\omega}\left(1 - r e^{-j\theta} e^{-j\omega}\right)}{1 - r e^{j\theta} e^{-j\omega}}\right] =$$

$$= \underbrace{\arg\left[e^{-j\omega}\right]}_{-\omega} - 2\arg\left[1 - r e^{j\theta} e^{-j\omega}\right]$$

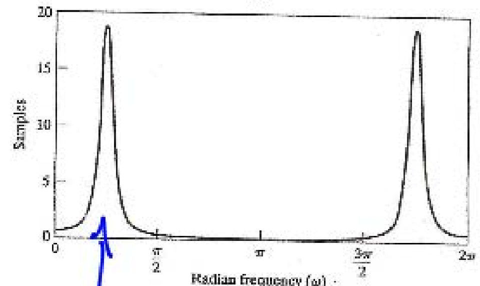
$$\operatorname{grd}\left[\frac{e^{-j\omega} - r e^{-j\theta}}{1 - r e^{j\theta} e^{-j\omega}}\right] = 1 - 2\operatorname{grd}\left[1 - r e^{j\theta} e^{-j\omega}\right]$$

Example: < figure 5.20 > ⑤





(b)



(c)

can be used to compensate phase distortion.

Claim: for a stable ap system $H_{ap}(z)$: ⑥

(i) $\text{grd}\left[H_{ap}(e^{j\omega})\right] > 0$

(ii) $\arg\left[H_{ap}(e^{j\omega})\right] \leq 0$

Delay positive → causal
phase negative → phase lag.

proof in book.