EE123 Discussion Section 3

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Plan

- Recap of Z-transform
- Practice problems
- Distribute SDRs

Z-transform

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

- Simpler notations (z instead of $e^{j\omega}$)
- Generalizes the Fourier transform to more signals (e.g. the unit step, exploding exponentials)
- Region of convergence, Zeros and Poles

TABLE 3.2 SOME z-TRANSFORM PROPERTIES

Section Reference	Sequence	Transform	ROC
	$x[n]$	$X(z)$	R_x
	$x_1[n]$	$X_1(z)$	R_{x_1}
	$x_2[n]$	$X_2(z)$	R_{x_2}
3.4.1	$ax_1[n] + bx_2[n]$	$aX_1(z) + bX_2(z)$	Contains $R_{x_1} \cap R_{x_2}$
3.4.2	$x[n - n_0]$	$z^{-n_0}X(z)$	R_x, except for the possible addition or deletion of the origin or ∞
3.4.3	$z_0^n x[n]$	$X(z/z_0)$	$\|z_0\|R_x$
3.4.4	$nx[n]$	$-z\dfrac{dX(z)}{dz}$	R_x, except for the possible addition or deletion of the origin or ∞
3.4.5	$x^*[n]$	$X^*(z^*)$	R_x
	$\mathcal{R}e\{x[n]\}$	$\frac{1}{2}[X(z) + X^*(z^*)]$	Contains R_x
	$\mathcal{I}m\{x[n]\}$	$\frac{1}{2j}[X(z) - X^*(z^*)]$	Contains R_x
3.4.6	$x^*[-n]$	$X^*(1/z^*)$	$1/R_x$
3.4.7	$x_1[n] * x_2[n]$	$X_1(z)X_2(z)$	Contains $R_{x_1} \cap R_{x_2}$
3.4.8	Initial-value theorem:		
	$x[n] = 0, \quad n < 0$	$\lim_{z\to\infty} X(z) = x[0]$	

TABLE 3.1 SOME COMMON z-TRANSFORM PAIRS

Sequence	Transform	ROC
1. $\delta[n]$	1	All z
2. $u[n]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
3. $-u[-n-1]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert < 1$
4. $\delta[n-m]$	z^{-m}	All z except 0 (if $m > 0$) or ∞ (if $m < 0$)
5. $a^n u[n]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
6. $-a^n u[-n-1]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert < \lvert a\rvert$
7. $na^n u[n]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert > \lvert a\rvert$
8. $-na^n u[-n-1]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert < \lvert a\rvert$
9. $[\cos\omega_0 n]u[n]$	$\frac{1-[\cos\omega_0]z^{-1}}{1-[2\cos\omega_0]z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
10. $[\sin\omega_0 n]u[n]$	$\frac{[\sin\omega_0]z^{-1}}{1-[2\cos\omega_0]z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
11. $[r^n\cos\omega_0 n]u[n]$	$\frac{1-[r\cos\omega_0]z^{-1}}{1-[2r\cos\omega_0]z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$
12. $[r^n\sin\omega_0 n]u[n]$	$\frac{[r\sin\omega_0]z^{-1}}{1-[2r\cos\omega_0]z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$
13. $\begin{cases} a^n, & 0 \le n \le N-1, \\ 0, & \text{otherwise} \end{cases}$	$\frac{1-a^Nz^{-N}}{1-az^{-1}}$	$\lvert z\rvert > 0$

Z-transform Inversion in this class

Three methods to invert z-transform:

- Inspection / Properties of z-transform

$$a^n u[n] \Leftrightarrow \frac{1}{1 - az^{-1}} \quad \{z : |z| > a\}$$

- Partial fraction expansion
- Long division / Power series

Question 1

- Find the inverse z-transform of the following, assuming right-sided:

$$X(z) = \frac{1}{1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}}$$

Solution 1

- Example: Partial Fraction Expansion

$$\begin{aligned} X(z) &= \frac{1}{1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}} \\ &= \frac{1}{\left(1 - \frac{1}{4}z^{-1}\right)\left(1 - \frac{1}{2}z^{-1}\right)} \\ &= \frac{A_1}{1 - \frac{1}{4}z^{-1}} + \frac{A_2}{1 - \frac{1}{2}z^{-1}} \end{aligned}$$

Solution 1

- Example: Partial Fraction Expansion

$$
\begin{aligned}
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&= \frac{1}{\left(1 - \frac{1}{4}z^{-1}\right)\left(1 - \frac{1}{2}z^{-1}\right)} \\
&= \frac{A_1}{1 - \frac{1}{4}z^{-1}} + \frac{A_2}{1 - \frac{1}{2}z^{-1}}
\end{aligned}
$$

Find A_1 and A_2

$$
\begin{aligned}
A_1 &= (1 - \frac{1}{4}z^{-1})X(z)\Big|_{z=\frac{1}{4}} = -1 \\
A_2 &= (1 - \frac{1}{2}z^{-1})X(z)\Big|_{z=\frac{1}{2}} = 2
\end{aligned}
$$

Solution 1

Partial fraction expansion:

$$X(z) = \frac{-1}{1 - \frac{1}{4}z^{-1}} + \frac{2}{1 - \frac{1}{2}z^{-1}}$$

From the tables:

Solution 1

Partial fraction expansion:

$$X(z) = \frac{-1}{1 - \frac{1}{4}z^{-1}} + \frac{2}{1 - \frac{1}{2}z^{-1}}$$

From the tables:

$$x[n] = \left[-(\frac{1}{4})^n + 2(\frac{1}{2})^n\right] u[n]$$ because right sided

Question 2

- Find the inverse z-transform of the following system:

$$X(z) = \frac{1 + 2z^{-1} + z^{-2}}{1 - \frac{3}{2}z^{-1} + \frac{1}{2}z^{-2}} \quad \text{ROC} = \{z : |z| > 1\}$$

- Hint: If $M < N$,

$$X(z) = \sum_{k=1}^{N} \frac{A_k}{1 - d_k z^{-1}}$$

If $M \geq N$,

$$X(z) = \sum_{r=0}^{M-N} B_r z^{-r} + \sum_{k=1}^{N} \frac{A_k}{1 - d_k z^{-1}}$$

Solution 2

- Find the impulse response of the following system:

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$$X(z) = \frac{1 + 2z^{-1} + z^{-2}}{1 - \frac{3}{2}z^{-1} + \frac{1}{2}z^{-2}} \quad M = N = 2$$

$$= B_0 + \frac{A_1}{1 - \frac{1}{2}z^{-1}} + \frac{A_2}{1 - z^{-1}}$$

Matching coefficients: $A_1 = -9$, $A_2 = 8$, $B_0 = 2$

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Matching coefficients: $A_1 = -9$, $A_2 = 8$, $B_0 = 2$

$$\text{ROC} = \{z : |z| > 1\}$$

$$\Rightarrow x[n] = 2\delta[n] - 9(\frac{1}{2})^n u[n] + 8u[n]$$

Question 3

- What is the inverse z-transform of

$$X(z) = \log(1 + az^{-1}), \qquad |z| > |a|$$

- Hint: use the property $nx[n] \Leftrightarrow -z\frac{dX(z)}{dz}$

Solution 3

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- Use the shift property

$$nx[n] = a(-a)^{n-1}u[n-1]$$

Solution 3

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- Use the differentiation property
$$nx[n] \Leftrightarrow -z\frac{dX(z)}{dz} = \frac{-az^{-1}}{1 + az^{-1}}$$
- Use the shift property
$$nx[n] = a(-a)^{n-1}u[n-1]$$
- $x[n] = (-1)^{n+1}\frac{a^n}{n}u[n-1]$ (This is actually the power series of X(z))

Question 4

- What is the ROC and inverse z-transform of

$$X(z) = z^2(1 - \frac{1}{2}z^{-1})(1 + z^{-1})(1 - z^{-1})$$

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- Expanding the terms

$$X(z) = z^2 - \frac{1}{2}z - 1 + \frac{1}{2}z^{-1}$$

Solution 4

$$X(z) = z^2(1 - \frac{1}{2}z^{-1})(1 + z^{-1})(1 - z^{-1})$$

- Expanding the terms

$$X(z) = z^2 - \frac{1}{2}z - 1 + \frac{1}{2}z^{-1}$$

- By inspection

$$X(z) = \delta[n+2] - \frac{1}{2}\delta[n+1] - \delta[n] + \frac{1}{2}\delta[n-1]$$

Question 5

- Let $x[n]$ be causal, i.e., $x[n] = 0$ for $n < 0$. And assume $X(z)$ is rational.
- Can $\lim_{z \to \infty} X(z)$ be infinite? That is, can there be a pole at $z = \infty$?

Solution 5

- The answer is no as long as $x[0] \neq \infty$ because of the initial value theorem:
- Proof:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

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- Since $x[n] = 0$:

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- Proof:

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- Since $x[n] = 0$:

$$X(z) = \sum_{n=0}^{\infty} x[n] z^{-n}$$

$$\lim_{z \to \infty} X(z) = x[0]$$

Question 6

Which of the followings could be causal and stable? Find the ROC whenever it can be causal and stable

- $\frac{(1-z^{-1})^2}{(1-\frac{1}{4}z^{-1})}$
- $\frac{(z-\frac{1}{4})^6}{(z-\frac{1}{2})^5}$
- $\frac{1-3z^{-2}}{(1-2z^{-1})}$
- $\frac{1}{(1-2z^{-1})} - \frac{4z^{-2}}{(1-2z^{-1})}$

Solution 6

- $\frac{(1-z^{-1})^2}{(1-\frac{1}{4}z^{-1})}$ Could be. ROC $= \{z : |z| > \frac{1}{4}\}$
- $\frac{(z-\frac{1}{4})^6}{(z-\frac{1}{2})^5}$ Could not be causal, because $\lim_{z\to\infty} X(z) = \infty$
- $\frac{1-3z^{-2}}{(1-2z^{-1})}$ Could not be causal and stable, because the outermost pole is outside the unit circle.
- $\frac{1}{(1-2z^{-1})} - \frac{4z^{-2}}{(1-2z^{-1})} = 1 + 2z^{-1}$ Always causal and stable