## EE123 Discussion Section 4

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## Plan

- Recap of the DFT
- Practice problems for the DFT
- (If there's time) Gauss and the history of FFT and sparse FFT


## The Discrete Fourier Transform

$$
\begin{gathered}
X[k]=\sum_{n=0}^{N-1} x[n] W_{N}^{k n} \\
x[n]=\frac{1}{N} \sum_{n=0}^{N-1} X[k] W_{N}^{-k n}
\end{gathered}
$$

Equivalent interpretations of the DFT:

- Sampling the DTFT at $\omega=\frac{2 \pi}{N} k$
- The DTFT of periodically extended signal
- Sampling the z-transform at $z=e^{j \frac{2 \pi}{N} k}$


## Question 1

$$
x[n]=\{-3,5,4,-1,-9,-6,-8,2\}
$$

- a) Evaluate $\sum_{k=0}^{7}(-1)^{k} X[k]$
- b) Evaluate $\sum_{k=0}^{7}|X[k]|^{2}$


## Solution 1

- part a)

$$
\begin{aligned}
x[4] & =\frac{1}{8} \sum_{k=0}^{7}(-1)^{k} X[k] \\
\Rightarrow \sum_{k=0}^{7}(-1)^{k} X[k] & =8 x[4]=-72
\end{aligned}
$$

## Solution 1

- part a)

$$
\begin{aligned}
x[4] & =\frac{1}{8} \sum_{k=0}^{7}(-1)^{k} X[k] \\
\Rightarrow \sum_{k=0}^{7}(-1)^{k} X[k] & =8 x[4]=-72
\end{aligned}
$$

- part b)

By Parseval's Theorem:

$$
\sum_{k=0}^{7}|X[k]|^{2}=8 \sum_{n=0}^{7}|x[n]|^{2}=1888
$$

## Question 2

$X[k]$ is the 9 -point DFT of $x[n]$

- Given:

$$
X[0]=-j, X[2]=1-j, X[3]=2-j, X[5]=3-j, X[8]=4-j
$$

Determine the remaining 4 samples with the following assumptions:

- $x[n]$ is real
- $x[n]$ is symmetric
- $x[n]$ is conjugate symmetric


## Solution 2

- $x[n]$ is real, then $X[k]$ is conjugate symmetric, so
$X[1]=X^{*}[9-1]=4+j, X[4]=3+j, X[6]=2+j, X[7]=1+j$
BUT we also need $X[0]=X^{*}[0]$. Clearly $X[0]$ is not real, so this first condition cannot be met.
- $x[n]$ is symmetric, then $X[k]$ is symmetric, so
$X[1]=X[9-1]=4-j, X[4]=3-j, X[6]=2-j, X[7]=1-j$
- $x[n]$ is conjugate symmetric, then $X[k]$ is real, so it's not possible


## Question 3

Let $x[n]$ be some length $-N$ sequence. Let $X[k]=\operatorname{DFT}\{x[n]\}$

- Express $x_{2}[n]=\operatorname{DFT}\{X[k]\}$ in terms of $\mathrm{x}[\mathrm{n}]$
- Hint: Try to match $\operatorname{DFT}\{X[k]\}$ with the equation

$$
x[n]=\frac{1}{N} \sum_{k=0}^{N-1} x[k] W_{N}^{-k n}
$$

## Solution 3

$$
\begin{aligned}
x_{2}[n] & =D F T\{X[k]\} \\
x_{2}[n] & =\sum_{k=0}^{N-1} X[k] W_{N}^{k n} \\
x_{2}[n] & =\sum_{k=0}^{N-1}(X[-k])_{N} W_{N}^{-k n} \\
x_{2}[n] & =8 I D F T\left\{(X[-k])_{N}\right\} \\
x_{2}[n] & =8(x[-n])_{N}
\end{aligned}
$$

## Question 4

Let $x[n]$ be some length $-N$ sequence. Let $X[k]=\operatorname{DFT}\{x[n]\}$

- Express $x_{3}[k]=\operatorname{DFT}\{\operatorname{DFT}\{X[k]\}\}$ in terms of $X[k]$
- Express $x_{4}[n]=\operatorname{DFT}\{\operatorname{DFT}\{\operatorname{DFT}\{X[k]\}\}\}$ in terms of $x[\mathrm{n}]$


## Solution 4

- Using the DFT properties:

$$
\begin{aligned}
& x_{2}[n]=8(x[-n])_{N} \\
& x_{3}[k]=D F T\left\{x_{2}[n]\right\}=8(X[-k])_{N} \\
& x_{4}[n]=D F T\left\{x_{3}[n]\right\}=64 x[n]
\end{aligned}
$$

## Solution 4 continued

- Ignoring the scaling factors:

$$
\begin{aligned}
x_{0}[n] & =D F T^{0}\{x[n]\}=x[n] \\
x_{1}[k] & =D F T^{1}\{x[n]\}=X[k] \\
x_{2}[n] & =D F T^{2}\{x[n]\}=(x[-n])_{N} \\
x_{3}[k] & =D F T^{3}\{x[n]\}=(X[-k])_{N} \\
x_{4}[n] & =D F T^{4}\{x[n]\}=x[n]
\end{aligned}
$$

- The usual DFT operator is four-periodic. The $n$th power of the Fourier transform can be generalized as the fractional Fourier transform, which is used in optics.


## Question 5: DCT

- The Discrete Cosine Transform (DCT) is a DFT-related transform that decomposes a finite signal in terms of a sum of cosine functions
- The DCT is often used in compression schemes, such as MP3, JPEG and MPEG.
- One of the reasons is its energy compactness


## Question 5: DCT

- Demo


## Question 5: DCT

Wanting to know more why DCT performs much better than DFT, you decide to look closer at the definition of DCT

Given that the definition of the DCT (Type 2) is

$$
X_{c}[k]=2 \sum_{n=0}^{N-1} x[n] \cos \left(\frac{\pi k(2 n+1)}{2 N}\right)
$$

Express $X_{c}[k]$ in terms of $X[k]$, the 2 N -point DFT of $x[n]$

## Solution 5: DCT

$$
\begin{aligned}
& X_{c}[k]=2 \sum_{n=0}^{N-1} x[n] \cos \left(\frac{\pi k(2 n+1)}{2 N}\right) \\
& X_{c}[k]=\sum_{n=0}^{N-1} x[n]\left(e^{-j \frac{\pi k(2 n+1)}{2 N}}+e^{j \frac{\pi k(2 n+1)}{2 N}}\right) \\
& X_{c}[k]=\sum_{n=0}^{N-1} x[n] e^{-j \frac{2 \pi k n}{2 N}} e^{-j \frac{\pi k}{2 N}}+\sum_{n=0}^{N-1} x[n] e^{j \frac{2 \pi k n}{2 N}} e^{j \frac{\pi k}{2 N}} \\
& X_{c}[k]=X[k] e^{-j \frac{\pi k}{2 N}}+X[-k] e^{\frac{\pi k}{2 N}} \\
& X_{c}[k]=X[k] e^{-j \frac{\pi k}{2 N}}+X^{*}[k] e^{j \frac{\pi k}{2 N}} \\
& X_{c}[k]=2 \operatorname{Real}\left(X[k] e^{-j \frac{\pi k}{2 N}}\right)
\end{aligned}
$$

## Solution 5: DCT

Key point is:

$$
\begin{aligned}
X_{c}[k] & =X[k] e^{-j \frac{\pi k}{2 N}}+X[-k] e^{j \frac{\pi k}{2 N}} \\
X_{c}[k] & =e^{-j \frac{\pi k}{2 N}}\left(X[k]+X[-k] e^{j \frac{2 \pi k}{2 N}}\right)
\end{aligned}
$$

$$
x_{c}[n]=\operatorname{Shift}_{\frac{1}{2}}\left\{x\left[((n))_{2 N}\right]+x\left[((-n-1))_{2 N}\right]\right\}
$$

## Solution 5: DCT

Periodicity assumed by DFT


Periodicity assumed by DCT


DCT symmetric extension is better because sharp transitions require many coefficients to represent

