EE123 Spring 2015
Discussion Section 6

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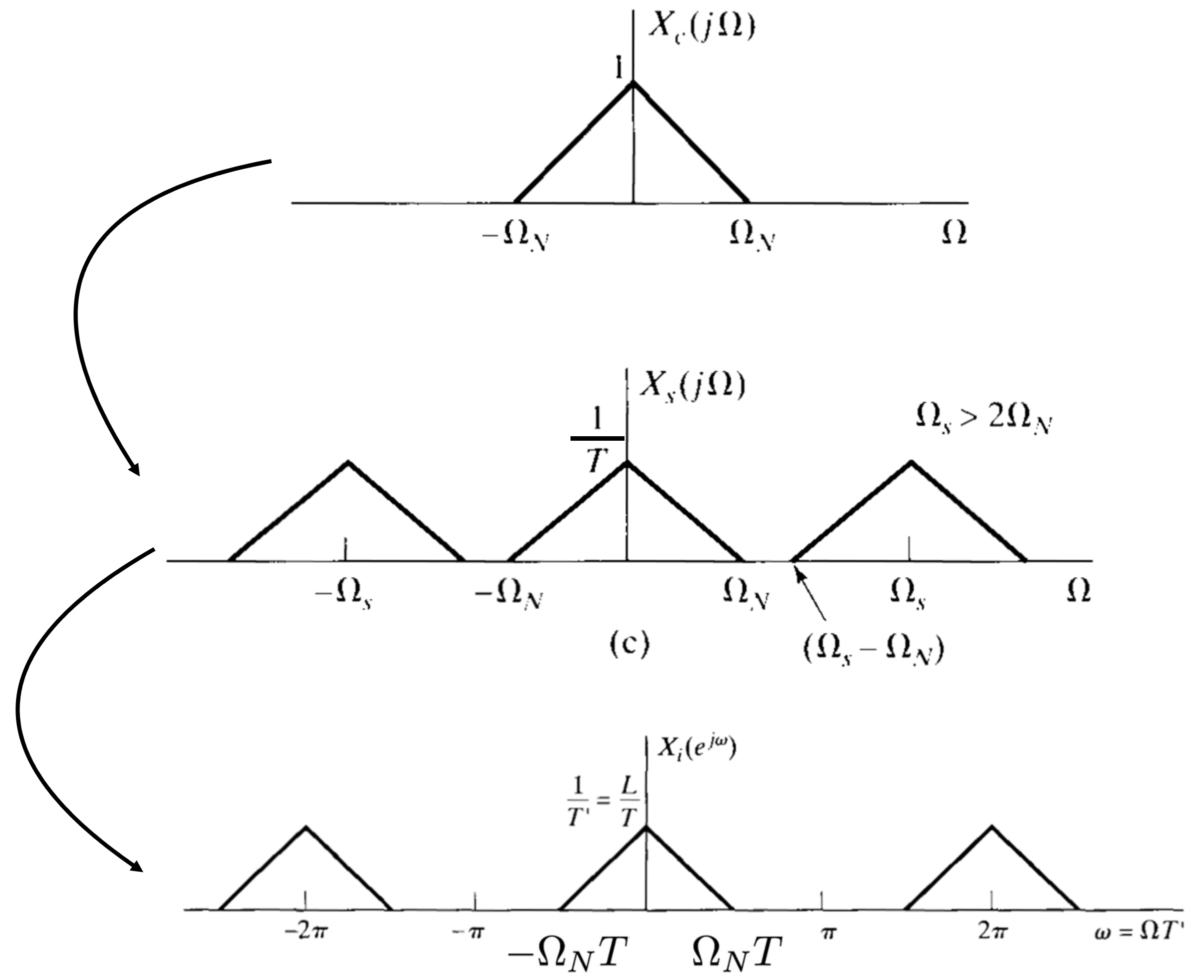
Plan

- Sampling
- Non-uniform FFT

Two steps to sample continuous signals



Multiply by impulse trains

$\Omega_s \Rightarrow 2\pi$

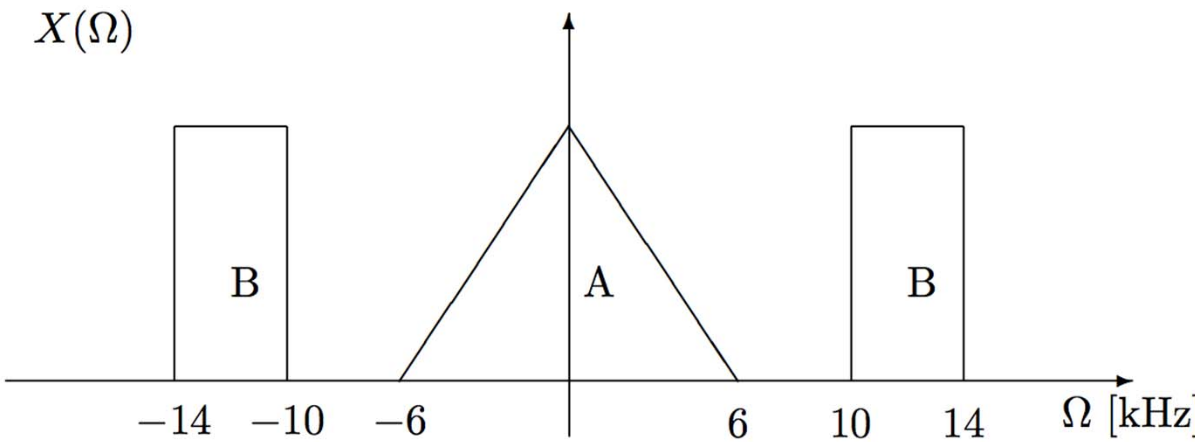
Set $\omega = \Omega T$

Problem 1

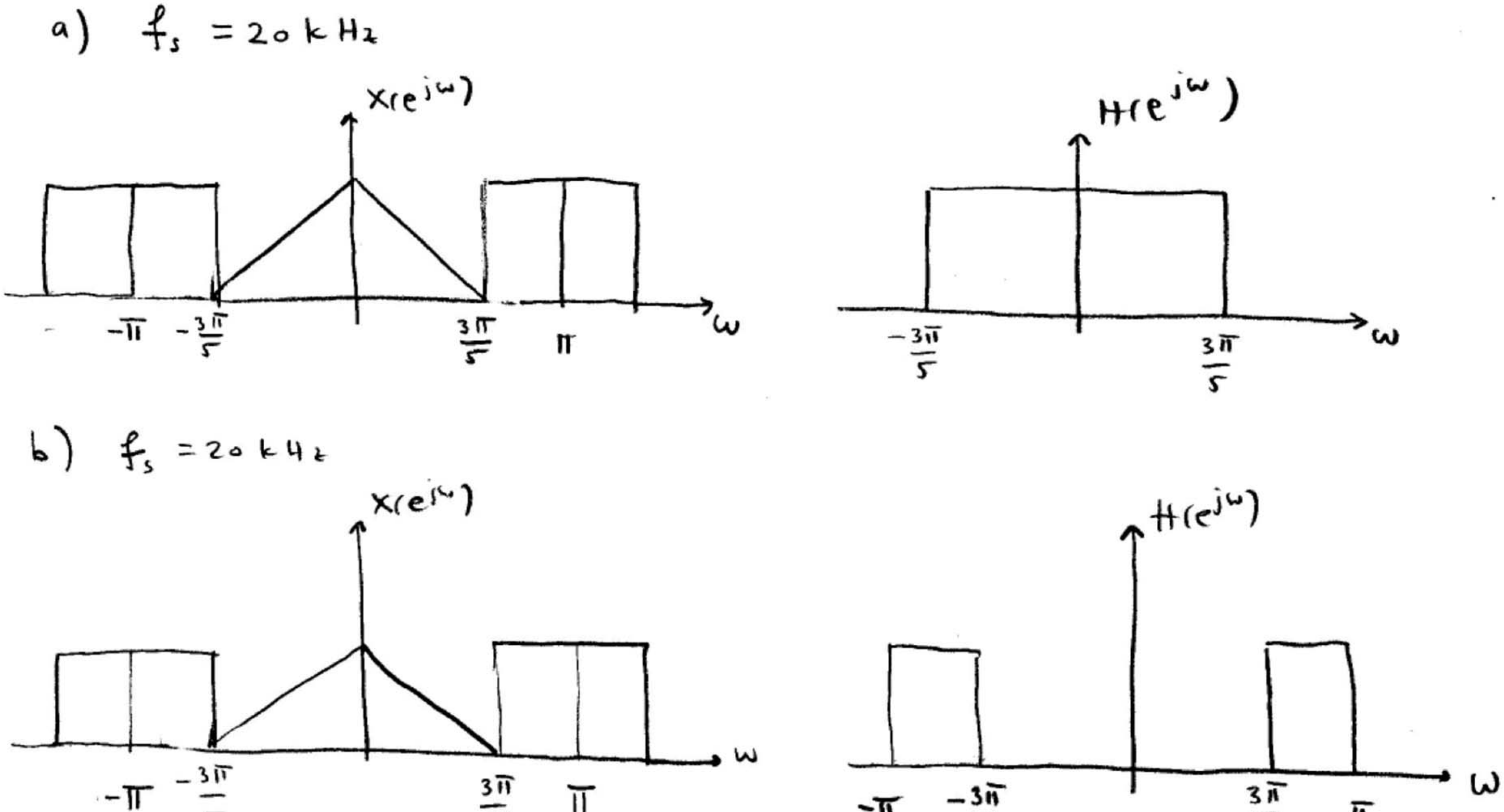
4. An analog signal, whose spectrum is shown below, is to be processed with a digital filter using ideal C/D and D/C converters (with no analog anti-aliasing filters).

a) (8 points) What is the minimum sufficient sampling rate to extract portion A of the signal? Sketch the magnitude of the digital filter that would be used at this sampling rate.

b) (7 points) Repeat for portion B of the signal.



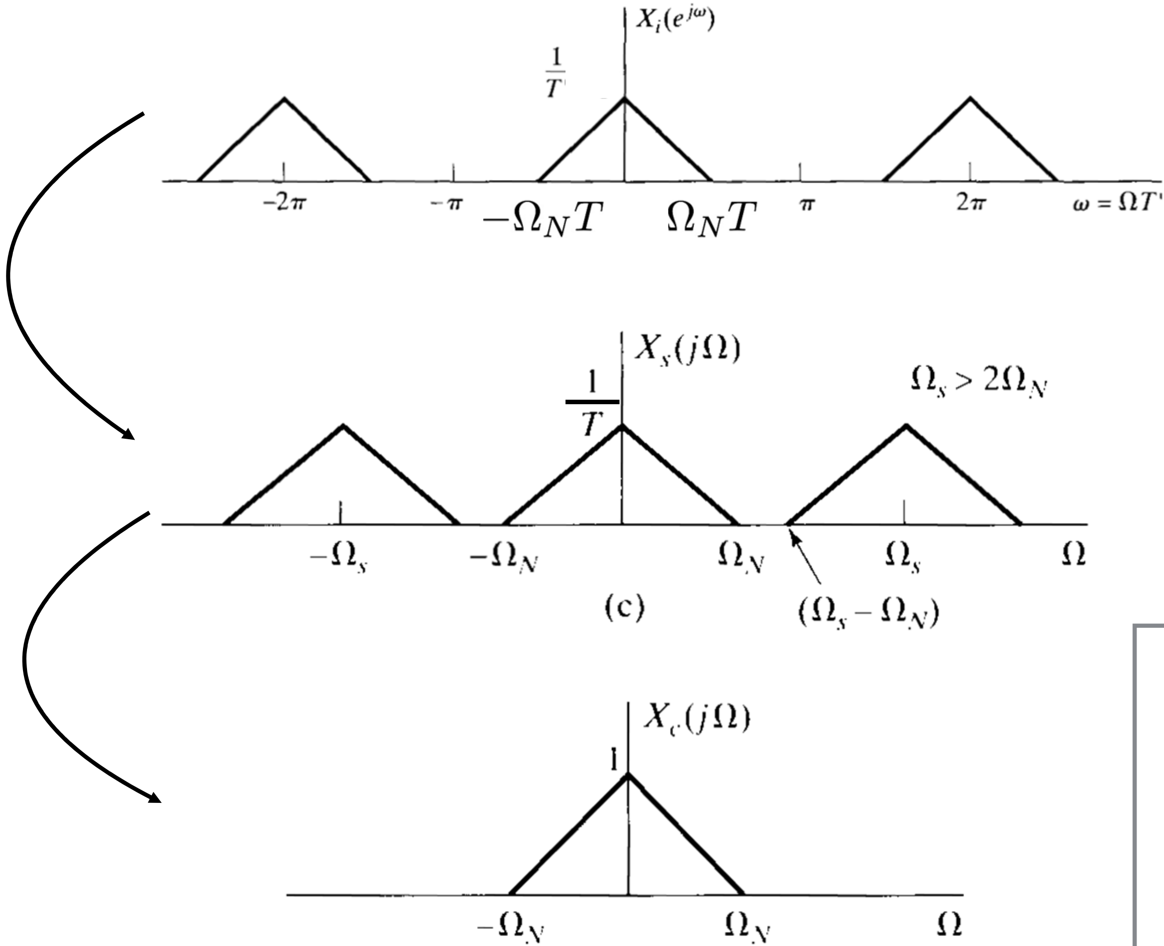
Solution 1



a) f_s = 20 kHz
X(e^jω)
H(e^jω)
−π
−3π/5
3π/5
π
ω
b) f_s = 20 kHz
X(e^jω)
H(e^jω)
−π
−3π/5
3π/5
π
ω

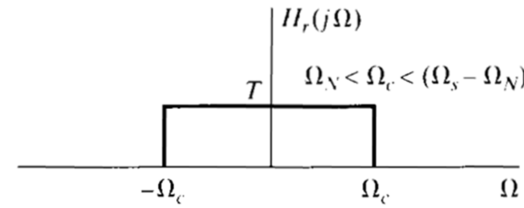
Two steps to reconstruct continuous signal

$2\pi \Rightarrow \Omega_s$



Set $\Omega = \omega/T$

Multiply by



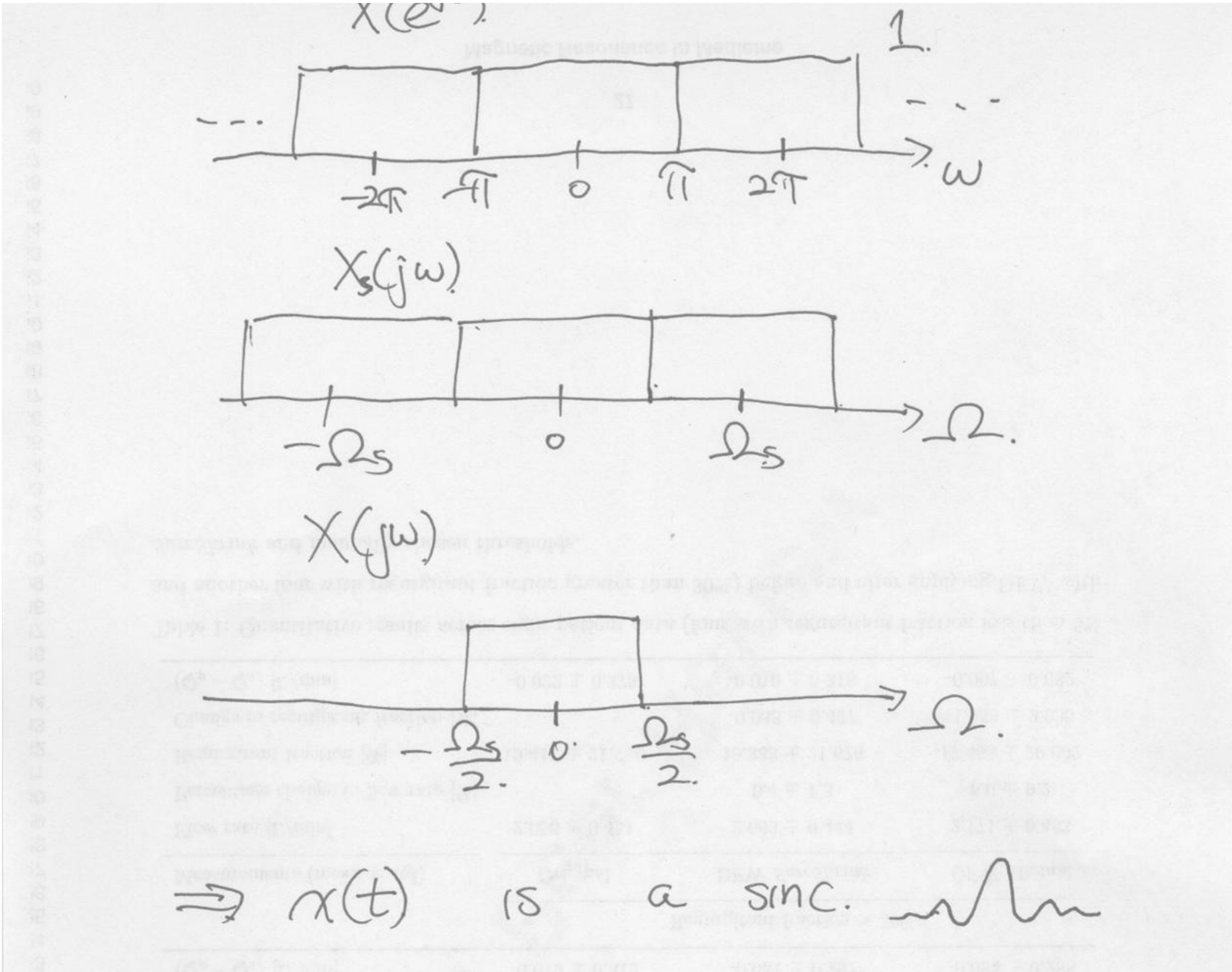
Question 2

- Suppose you start with a discrete time signal,

$$x[n] = \delta[n]$$

- What is the reconstructed continuous-time signal when you use the ideal reconstruction filter with sampling period T?

Solution 2



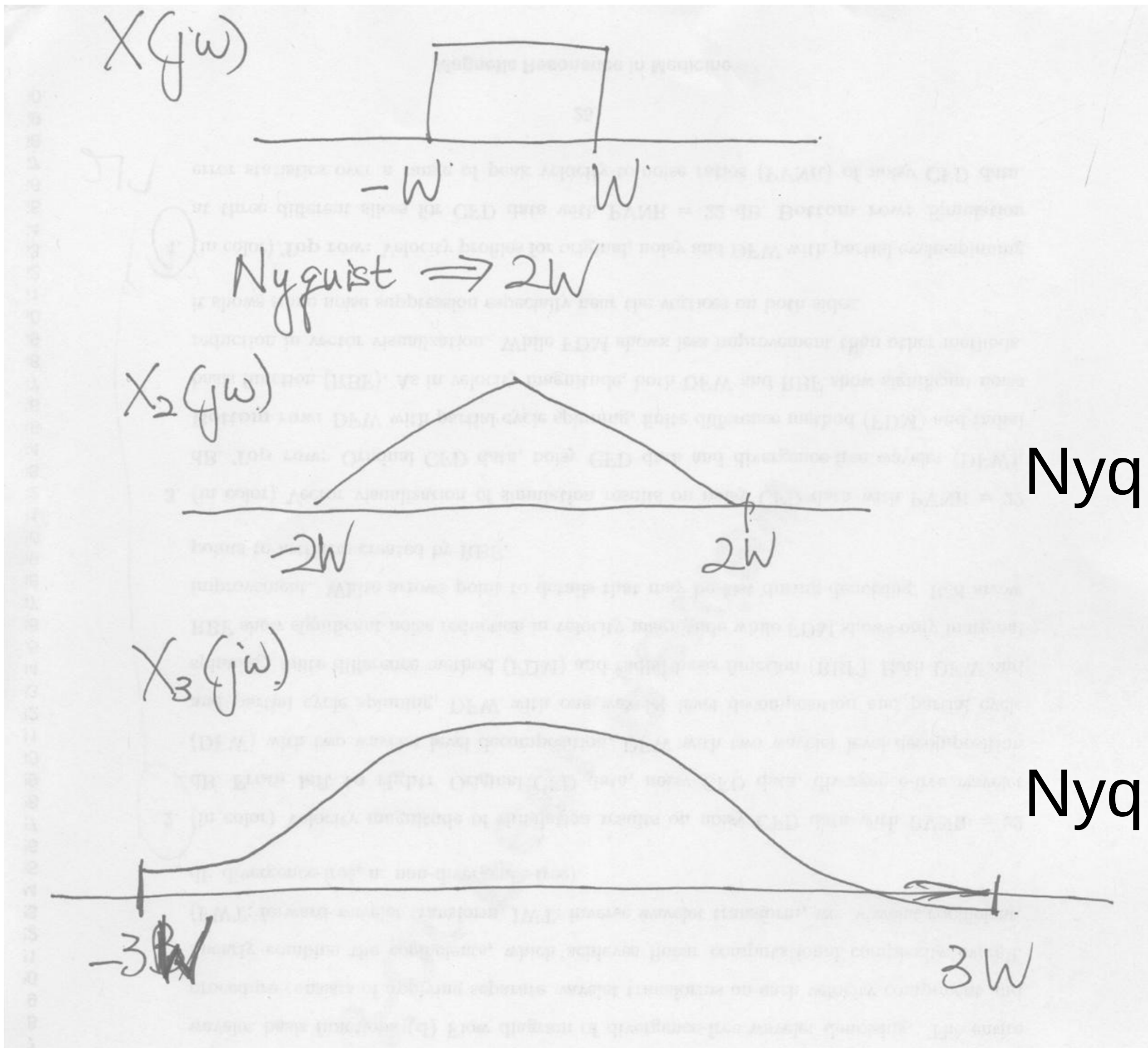
Question 3

- For any signal x(t) bandlimited from -W to W.

Suppose you sample $x^2(t)$ and wish to reconstruct $x^2(t)$. At what rate must $x^2(t)$ be sampled?

Suppose you sample $x^3(t)$ and wish to reconstruct $x^3(t)$. At what rate must $x^3(t)$ be sampled?

Solution 3

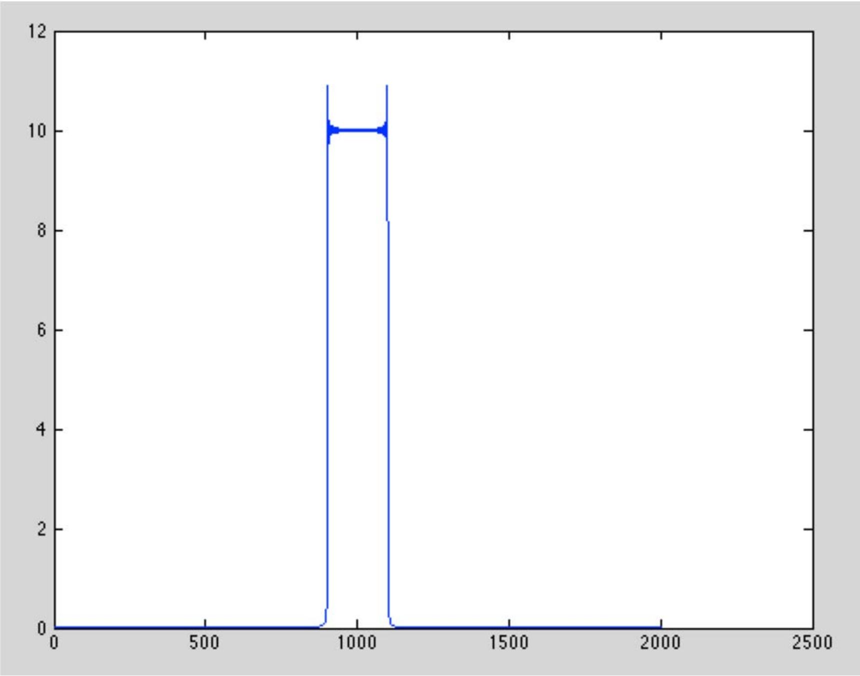


Nyquist = 4W

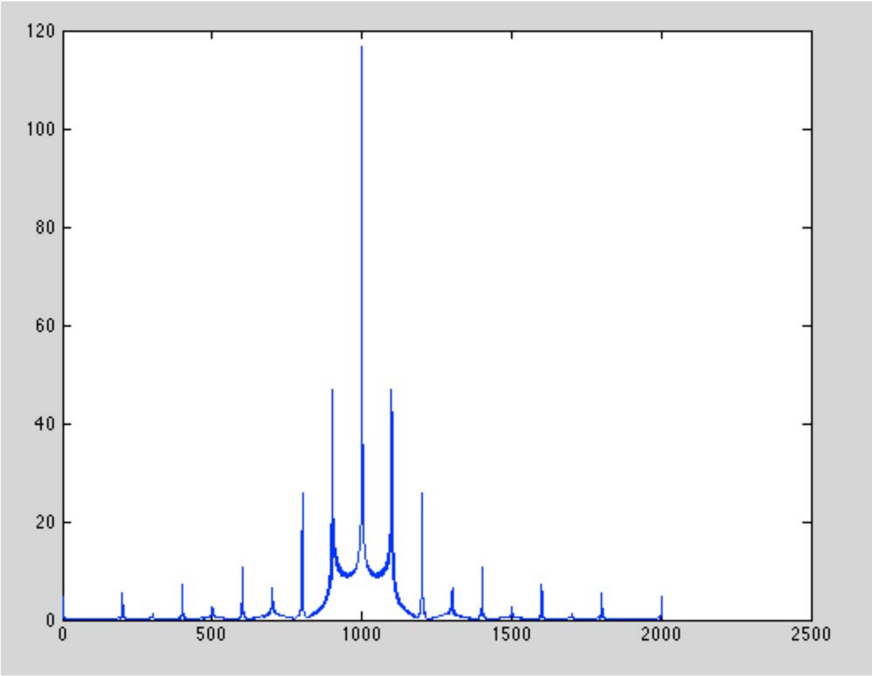
Nyquist = 6W

Does not go the other way

DFT(sinc)



DFT(sqrt(sinc))



Derivative Sampling

Dilemma:
1) Dora’s boss wants her to measure Hetch Hetchy reservoir water levels daily.
2) Dora just wants to go exploring.

	t=1	t=2	t=3	t=4	t=5	t=6
Want	$f(t)$	$f(t)$	$f(t)$	$f(t)$	$f(t)$	$f(t)$
Get	$f(t)$		$f(t)$		$f(t)$	
	$\frac{d}{dt}f(t)$		$\frac{d}{dt}f(t)$		$\frac{d}{dt}f(t)$	

$$f[n] = f(nT)$$

and

$$\dot{f}[n] = \frac{d}{dt}f(nT)$$

where $f(t)$ is the continuous water level in the lake and $T = 2$ [days]. We will show in this question that it is possible to reconstruct the continuous signal, even though the sampling is at half the Nyquist rate. However, there are some costs associated with this. We will solve the problem in the frequency domain.

(a) $f(t)$ is band limited. What is its highest frequency Ω_N?

Solutions:

Since sampling period required to meet the Nyquist rate is T=1 day, the highest frequency, Ω_n is $\pi/T = \pi$ radians per day.

(b) Show that if $f(t)$ is band limited by Ω_N so is $\frac{d}{dt}f(t)$.

Solutions:

Denote the fourier transform of $f(t)$ by $F(j\Omega)$. Then the fourier transform of $\frac{d}{dt}f(t)$, denoted by $\dot{F}(j\Omega)$, is equal to $j\Omega F(j\Omega)$. For $\Omega > \Omega_N$, $\dot{F}(j\Omega) = j\Omega F(j\Omega) = j\Omega 0 = 0$.

We will use the impulsive representations $f_s(t)$ and $\dot{f}_s(t)$ of $f[n]$ and $\dot{f}[n]$. Since we sampled at a rate of 2 [days], the Fourier transforms $F_s(j\Omega)$ and $\dot{F}_s(j\Omega)$ are periodic with a period of $\frac{2\pi}{2} = \pi$ [1/days]. We would like to express $F(j\Omega)$ as a function $F_s(j\Omega)$ and $\dot{F}_s(j\Omega)$. Since $F(j\Omega)$ is band limited we need only to focus on the interval $-\pi < \Omega < \pi$.

(c) For the interval $0 \leq \Omega \leq \pi$ express $F_s(j\Omega)$ as a function of $F(j\Omega)$.

Solutions:

Note: It's okay to work in units of days, in which case T=1.

$$F_s(j\Omega) = \frac{1}{2T} \sum_{r=-\infty}^{\infty} F\left(j\left(\Omega - \frac{2\pi r}{2T}\right)\right)$$

Splitting the sum into odd and even terms using the change of variable $r = 2k + i$ for $i = 0, 1$:

$$F_s(j\Omega) = \frac{1}{2T} \sum_{k=-\infty}^{\infty} \left[F\left(j\left(\Omega - \frac{2\pi k}{T}\right)\right) + F\left(j\left(\Omega - \frac{\pi}{T} - \frac{2\pi k}{T}\right)\right)\right]$$

If we restrict to $0 < \Omega < \frac{\pi}{T}$, we only get terms for $k = 0$.

$$F_s(j\Omega) = \frac{1}{2T}\left[F(j\Omega) + F\left(j\left(\Omega - \frac{\pi}{T}\right)\right)\right]$$

(d) For the interval $0 \leq \Omega \leq \pi$ express $\dot{F}_s(j\Omega)$ as a function of $F(j\Omega)$.

Solutions:
Since $\dot{F}(j\Omega) = j\Omega F(j\Omega)$, we have

$$\dot{F}_s(j\Omega) = \frac{1}{2T} \sum_{r=-\infty}^{\infty} j\left(\Omega - \frac{2\pi r}{2T}\right) F\left(j\left(\Omega - \frac{2\pi r}{2T}\right)\right)$$

Using the same change of variable as in the previous part

$$\dot{F}_s(j\Omega) = \frac{1}{2T} \sum_{k=-\infty}^{\infty} \left[j\left(\Omega - \frac{2\pi k}{T}\right) F\left(j\left(\Omega - \frac{2\pi k}{T}\right)\right) + j\left(\Omega - \frac{\pi}{T} - \frac{2\pi k}{T}\right) F\left(j\left(\Omega - \frac{\pi}{T} - \frac{2\pi k}{T}\right)\right)\right]$$

Again restricting to $0 < \Omega < \frac{\pi}{T}$, we only get terms for $k = 0$.

$$\dot{F}_s(j\Omega) = \frac{1}{2T}\left[j\Omega F(j\Omega) + j\left(\Omega - \frac{\pi}{T}\right) F\left(j\left(\Omega - \frac{\pi}{T}\right)\right)\right]$$

(e) Use the above equations to find $F(j\Omega)$ for $0 \leq \Omega \leq \pi$

From the previous 2 parts, we can write the matrix equation:

$$\begin{bmatrix} F_s(j\Omega) \\ \dot{F}_s(j\Omega) \end{bmatrix} = \frac{1}{2T} \begin{bmatrix} 1 & 1 \\ j\Omega & j(\Omega - \frac{\pi}{T}) \end{bmatrix} \begin{bmatrix} F(j\Omega) \\ F(j\Omega - \frac{\pi}{T}) \end{bmatrix}$$

Solving:

$$\begin{bmatrix} F(j\Omega) \\ F(j(\Omega - \frac{\pi}{T})) \end{bmatrix} = 2T \begin{bmatrix} 1 & 1 \\ j\Omega & j(\Omega - \frac{\pi}{T}) \end{bmatrix}^{-1} \begin{bmatrix} F_s(j\Omega) \\ \dot{F}_s(j\Omega) \end{bmatrix}$$

$$F(j\Omega) = \frac{2jT}{\pi/T}\left(j(\Omega - \frac{\pi}{T})F_s(j\Omega) - \dot{F}_s(j\Omega)\right)$$

(f) Repeat c-e and find $F(j\Omega)$ for $-\pi \leq \Omega \leq 0$

Solutions:

The solution for $-\frac{\pi}{T} < \Omega < 0$ can be obtained from the second row of the preceding matrix equation, letting $\Omega^+ = \Omega + \frac{\pi}{T}$, since when $\Omega \in [-\frac{\pi}{T}, 0]$, we will have $\Omega^+ = \in [0, \frac{\pi}{T}]$. This yields

$$\begin{aligned} F(j\Omega) &= \frac{2jT}{\pi/T}\left(-j(\Omega^+)F_s(j\Omega^+) + \dot{F}_s(j\Omega^+)\right) \\ &= \frac{2jT}{\pi/T}\left(-j(\Omega + \frac{\pi}{T})F_s(j\Omega + \frac{\pi}{T}) + \dot{F}_s(j\Omega + \frac{\pi}{T})\right) \end{aligned}$$

Alternatively, the analysis can also be repeated in an identical fashion, with the two contributing terms being those at Ω and $\Omega + \frac{\pi}{T}$. This yields the marix equation

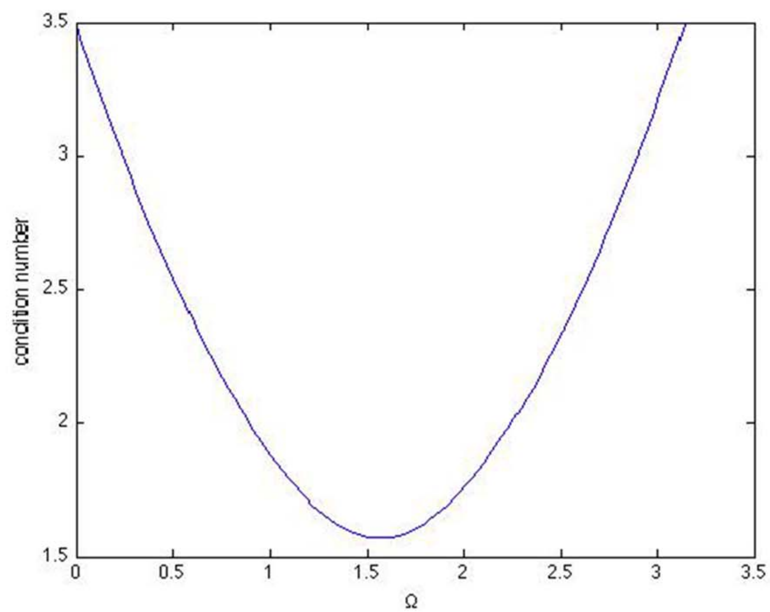
$$\begin{bmatrix} F_s(j\Omega) \\ \dot{F}_s(j\Omega) \end{bmatrix} = \frac{1}{2T}\begin{bmatrix} 1 & 1 \\ j\Omega & j(\Omega + \frac{\pi}{T}) \end{bmatrix}\begin{bmatrix} F(j\Omega) \\ F(j\Omega + \frac{\pi}{T}) \end{bmatrix}$$

and solution:

$$\begin{bmatrix} F(j\Omega) \\ F(j(\Omega + \frac{\pi}{T})) \end{bmatrix} = 2T\begin{bmatrix} 1 & 1 \\ j\Omega & j(\Omega + \frac{\pi}{T}) \end{bmatrix}^{-1}\begin{bmatrix} F_s(j\Omega) \\ \dot{F}_s(j\Omega) \end{bmatrix}$$

$$F(j\Omega) = \frac{2jT}{\pi/T}\left(j(\Omega + \frac{\pi}{T})F_s(j\Omega) - \dot{F}_s(j\Omega)\right)$$

(g) In order to recover $F(j\Omega)$ you had to invert a set of linear equations. Express the set of equations in c+d in matrix form. Plot the condition number (use the `cond` function in Matlab) for $0 \leq \Omega \leq \pi$. The condition number is the ratio between the highst and lowest eigen-values and indicates how much noise amplification occurs when inverting the matrix. From the plot, what can you conclude on the cost of derivative sampling?



Matrix inversion causes a noise amplification. The matrix inversion problem becomes increasingly ill-conditioned for Ω close to 0 or π/T, where noise is amplified by as much as a factor of 3.5. For example, given a signal to noise ratio of 20 in the measurements (which is pretty good), after the reconstruction we will have a signal to noise ratio of 5.7, which is much worse! To gain back SNR by averaging samples we will need the equivalent of $3.5^2 = 12.25$ averages.