Discrete Fourier Series

$x(t)$ real. continuous $\xrightarrow{\text{C.T.F.T.}}$ $X(\Omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\Omega t}\, dt$

Ω real. C.T.F.T

$x(n)$ int. discrete time signal (∞ or finite length) $\xrightarrow{\text{D.T.F.T}}$ $X(\omega) = \sum_n x(n) e^{-j\omega n}$.

ω real.

$x(n)$ int discrete time (∞ly long, finite long.) $\xrightarrow{\text{Z.T}}$ $X(z) = \sum_n x(n) z^{-n}$

z complex.

$\tilde{x}(n)$ int periodic. $\xrightarrow{\text{D.F. Series.}}$ $\tilde{X}(k) = \sum_{n \text{ over one period}} \tilde{x}(n) e^{-j\frac{2\pi nk}{N}}$

$x(n)$ int finite length. $\xrightarrow{\text{D.F.T.}}$ $X(k) = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi nk}{N}}$

k int

finite length

DFS = Discrete Fourier Series

Deals with $\tilde{x}(n$ periodic, discrete Time signal.

$$\tilde{x}(n) = \tilde{x}(n + kN)$$

k any integer; N = period.

Idea: Decompose $\tilde{x}(n)$ in Terms of exponentials periodic with period N.

$$e_k(n) = e^{j\frac{2\pi n k}{N}} \qquad k = 0, \ldots, N-1$$

$Re[e_k(n)]$

There are N, periodic exponentials with period N.

$$\tilde{x}(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\, e^{j\frac{2\pi n k}{N}}$$

weights

$e_k(n)$ is periodic with period N:

$$e_k(n) \overset{??}{=} e_{k+rN}(n)$$

arbitary int.

<u>proof</u>:

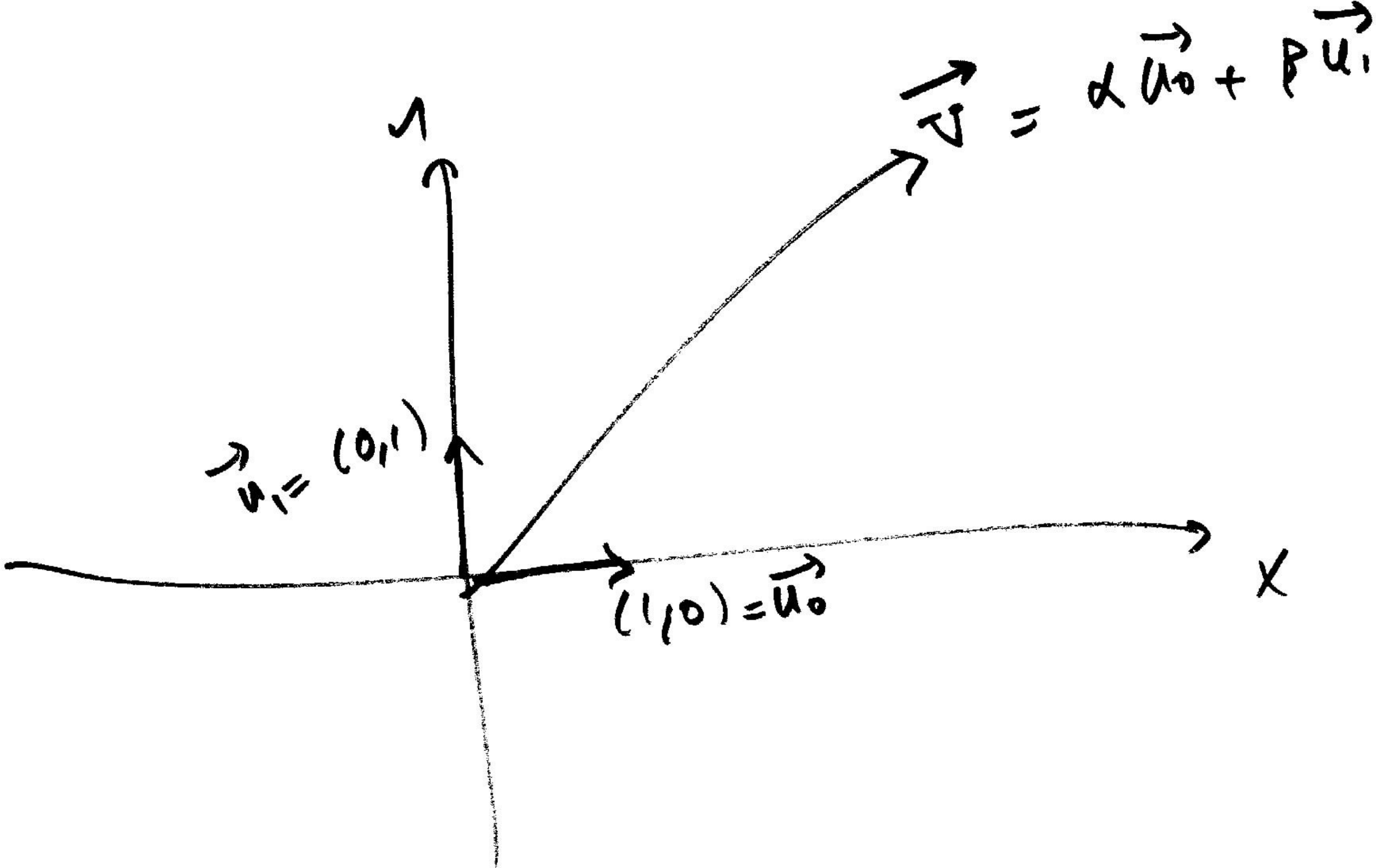
$$e^{j2\pi n\frac{k}{N}} \overset{?}{=} e^{j2\pi n\left(\frac{k+rN}{N}\right)}$$

$$= e^{j2\pi\frac{kn}{N}} e^{j2\pi n\frac{rN}{N}}$$

$$= e^{j2\pi\frac{kn}{N}}$$

$$e_0(n) = e_N(n) = e_{2N}(n) = e_{3N}(n) = \cdots$$

$$e_1(n) = e_{N+1}(n) = e_{2N+1}(n) = \cdots$$


$\vec{v} = \alpha\vec{u_0} + \beta\vec{u_1}$
y
$\vec{u_1} = (0,1)$
$(1,0) = \vec{u_0}$
x

$\underline{Q}$ How find "weights"? $\tilde{X}(k)$?

proposal:

$$\tilde{X}(k) \overset{??}{=} \sum_{n=0}^{N-1} \underbrace{\hat{x}(n)}\, e^{-j\frac{2\pi nk}{N}}$$

proof:

$$\tilde{X}(k) \overset{??}{=} \sum_{n=0}^{N-1} \left(\frac{1}{N} \sum_{l=0}^{N-1} \tilde{X}(l)\, e^{j\frac{2\pi nl}{N}} \right) e^{-j\frac{2\pi nk}{N}}$$

$$\overset{??}{=} \sum_{l=0}^{N-1} \tilde{X}(l) \underbrace{\left[\frac{1}{N} \sum_{n=0}^{N-1} e^{j\frac{2\pi(l-k)n}{N}} \right]}_{A}$$

$$A = \delta(l-k-rN)$$

what is A?

$$\checkmark \overset{??}{=} \sum_{l=0}^{N-1} \tilde{X}(l)\,\delta(l-k-rN) = \tilde{X}(k+rN) = \tilde{X}(k)$$

$r \to$ arb. int.

$$\tilde{X}(k) = \tilde{X}(k+N) = \tilde{X}(k+2N)$$

Case 1: If $l-k$ is an int. multiple of N.

$$l-k = rN$$

$$A = \frac{1}{N} \sum_{n=0}^{N-1} e^{j\frac{2\pi r N n}{N}} = 1$$

Case 2 $\quad l-k \neq rN \qquad$ $l-k$ is not an int. multiple of N.

$$A = \frac{1}{N} \sum_{n=0}^{N-1} e^{j\frac{2\pi(l-k)n}{N}}$$

Recall $\displaystyle\sum_{n=0}^{N-1} \alpha^n = \frac{1-\alpha^N}{1-\alpha}$

$$A = \frac{1}{N} \frac{1 - e^{j2\pi\frac{(l-k)N}{N}}}{1 - e^{j2\pi\frac{(l-k)}{N}}} = \phi$$

$$A = \delta(l-k-rN)$$

$\bar{X}(k) = X(k + rN) \qquad r = \text{arb. int.}$

$\Rightarrow$ $X(k)$ is a periodic sequence with period N.

$\Rightarrow$ From now on refer to $X(k)$ as $\tilde{X}(k)$

DFS pair.

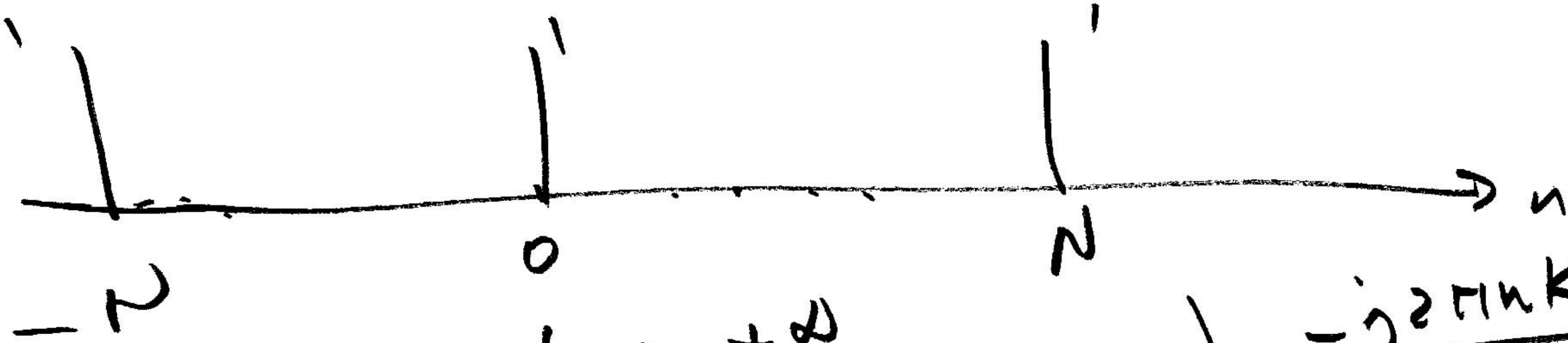
Analysis.
$$\tilde{X}(k) = \sum_{n=0}^{N-1} \tilde{x}(n)\, e^{-j\frac{2\pi n k}{N}}$$

synthesis
$$\tilde{x}(n) = \frac{1}{N}\sum_{k=0}^{N-1} \tilde{X}(k)\, e^{+j\frac{2\pi n k}{N}}$$

DFS:

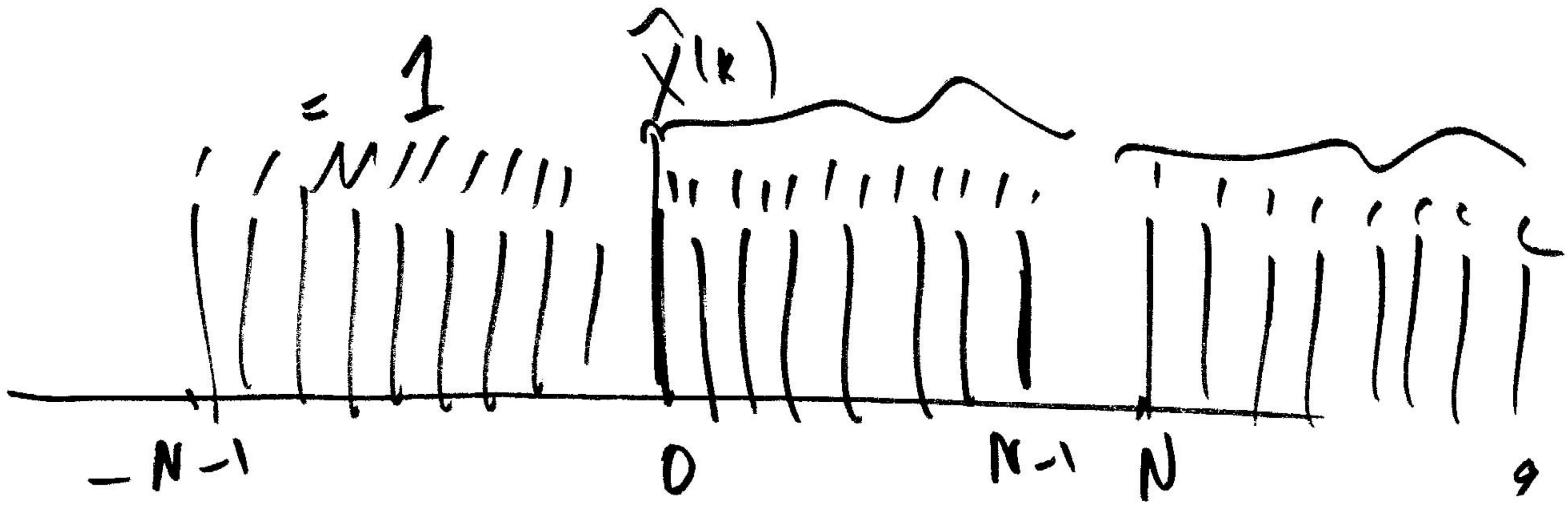
periodic N pt seq $\tilde{x}(n)$ $\longrightarrow$ periodic N point seq in freq. Domain $\tilde{X}(k)$

Ex $\tilde{x}(n) = \sum_{r=-\infty}^{\infty} \delta(n + rN)$



$$\tilde{X}(k) = \sum_{n=0}^{N-1} \left(\sum_{r=-\infty}^{+\infty} \delta(n+rN) \right) e^{-j\frac{2\pi n k}{N}}$$

$= 1$



$$\tilde{x}(n) = \frac{1}{N} \sum_{k=0}^{N-1} e^{j\frac{2\pi n k}{N}}$$

$$\tilde{x}(n) = \sum_{r=-\infty}^{+\infty} \delta(n + rN)$$

equation train.

Shift Property

$$\tilde{x}(n) \longleftrightarrow \tilde{X}(k)$$

$$\tilde{x}(n-n_0) \longleftrightarrow e^{-j\frac{2\pi n_0 k}{N}} \tilde{X}(k)$$

Periodic Convolution:

$$\underset{\text{period } N}{\tilde{x}_1} \underset{\text{period convol}}{\circledast} \underset{\text{period } N.}{\tilde{x}_2} = \underset{\text{period } N.}{\tilde{x}_3}$$

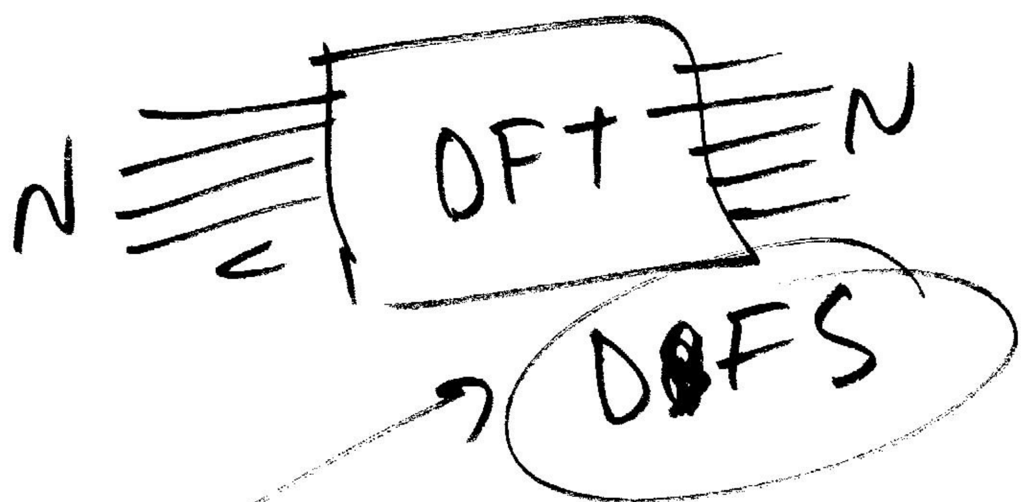
$$\triangleq \sum_{m=0}^{N-1} \tilde{x}_1(m)\, \tilde{x}_2(n-m)$$

$$\tilde{X}_3(k) = \tilde{X}_1(k)\, \tilde{X}_2(k)$$

<u>DFT</u> = Discrete Fourier Trafm.

Npt

$x(n) \longrightarrow X(k)$ — N pt seq



DFT → DFS

DFT → DTFT.

First Approach To DFT via DFS

1. Start with a finite extent seq $x(n)$
   N points long $\quad n=0,\ldots,N-1$

2. "Periodicize" $x(n)$ to get $\tilde{x}(n)$ with period N.

$$x(n) = \tilde{x}(n)\, R_N(n)$$

extracts one period of $\tilde{x}(n)$

$$R_N(n) = \begin{cases} 1 & n=0,\ldots,N-1 \\ 0 & \text{otherwise} \end{cases}$$

$$\tilde{x}(n) = \sum_{r=-\infty}^{+\infty} x(n+rN)$$

$\leftarrow$ periodicization

3. Take DFS of $\tilde{x}(n) \rightarrow \tilde{X}(K)$

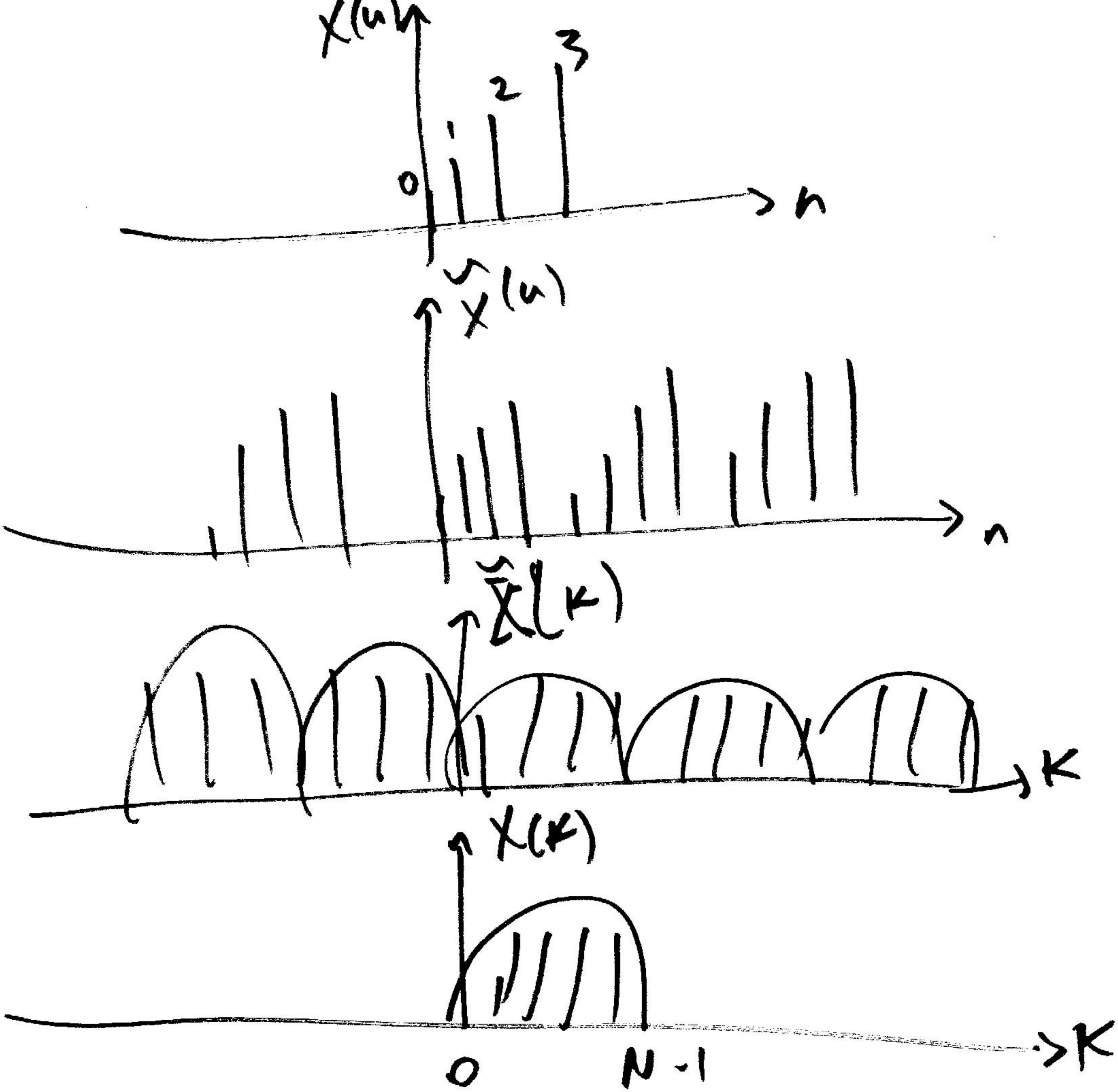
4. Take one period of $\tilde{X}(K)$ To get

$X(K) =$ DFT of $x(n)$

$$X(K) = \tilde{X}(K)\, R_N(K)$$

$$\underset{N\,pt}{x(n)} \longrightarrow \underset{\substack{\text{periodic} \\ N\,pt.}}{\tilde{x}(n)} \xrightarrow{\text{DFS}} \underset{\substack{\text{periodic} \\ N\,pt}}{\tilde{X}(k)} \longrightarrow \underset{N\,pt.}{X(k)}$$

Ex



Defn of DFT

$$X(k) = N\text{pt DFT of } x(n) = \begin{cases} \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi n k}{N}} & \boxed{0 \le k < N} \\ 0 & \text{otherwise} \end{cases}$$

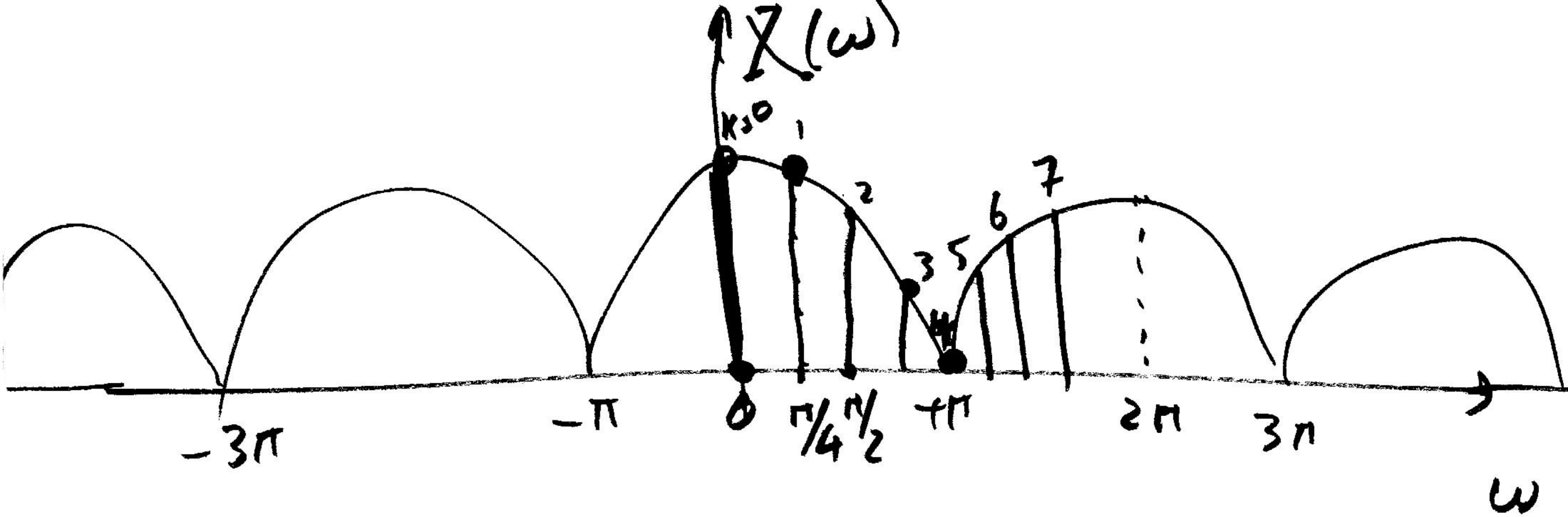
$$x(n) = \begin{cases} \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{j\frac{2\pi n k}{N}} & 0 \le n < N \\ 0 & \text{otherwise} \end{cases}$$

Relate DFT To DTFT:

$$X(\omega) = \sum_{n=0}^{N-1} x(n)\, e^{-j\omega n}$$

$$X(K) = \begin{cases} \left[X(\omega)\right]_{\omega = \frac{2\pi K}{N}} & 0 \le K < N \\ 0 & \text{Otherwise.} \end{cases}$$

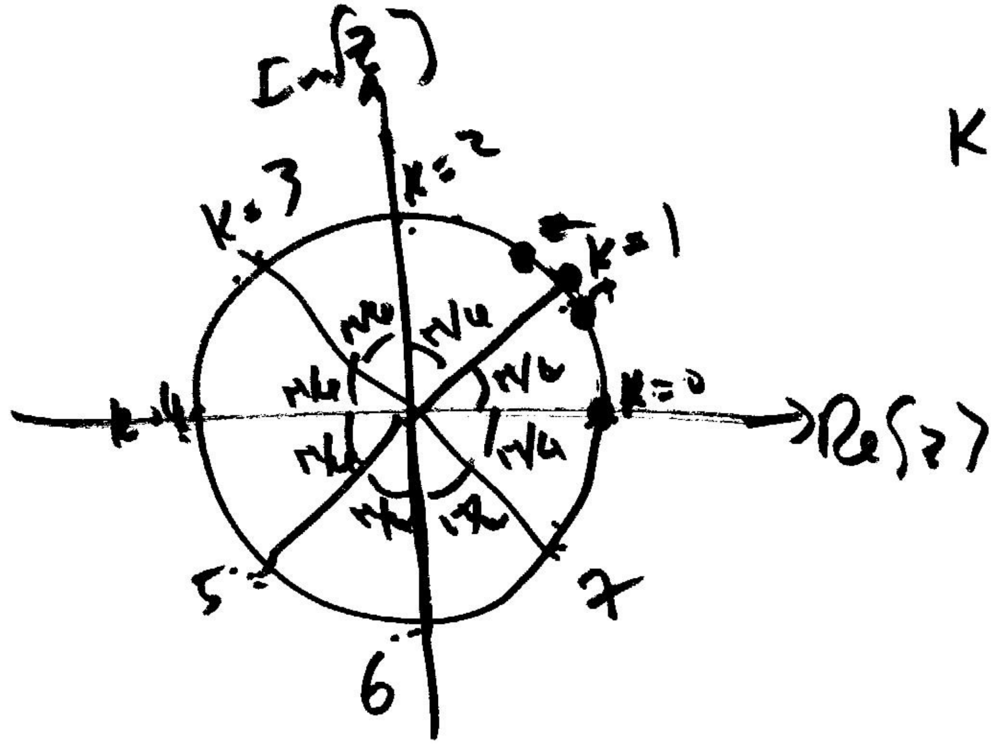
DFT is equally spaced samples of DTFT.



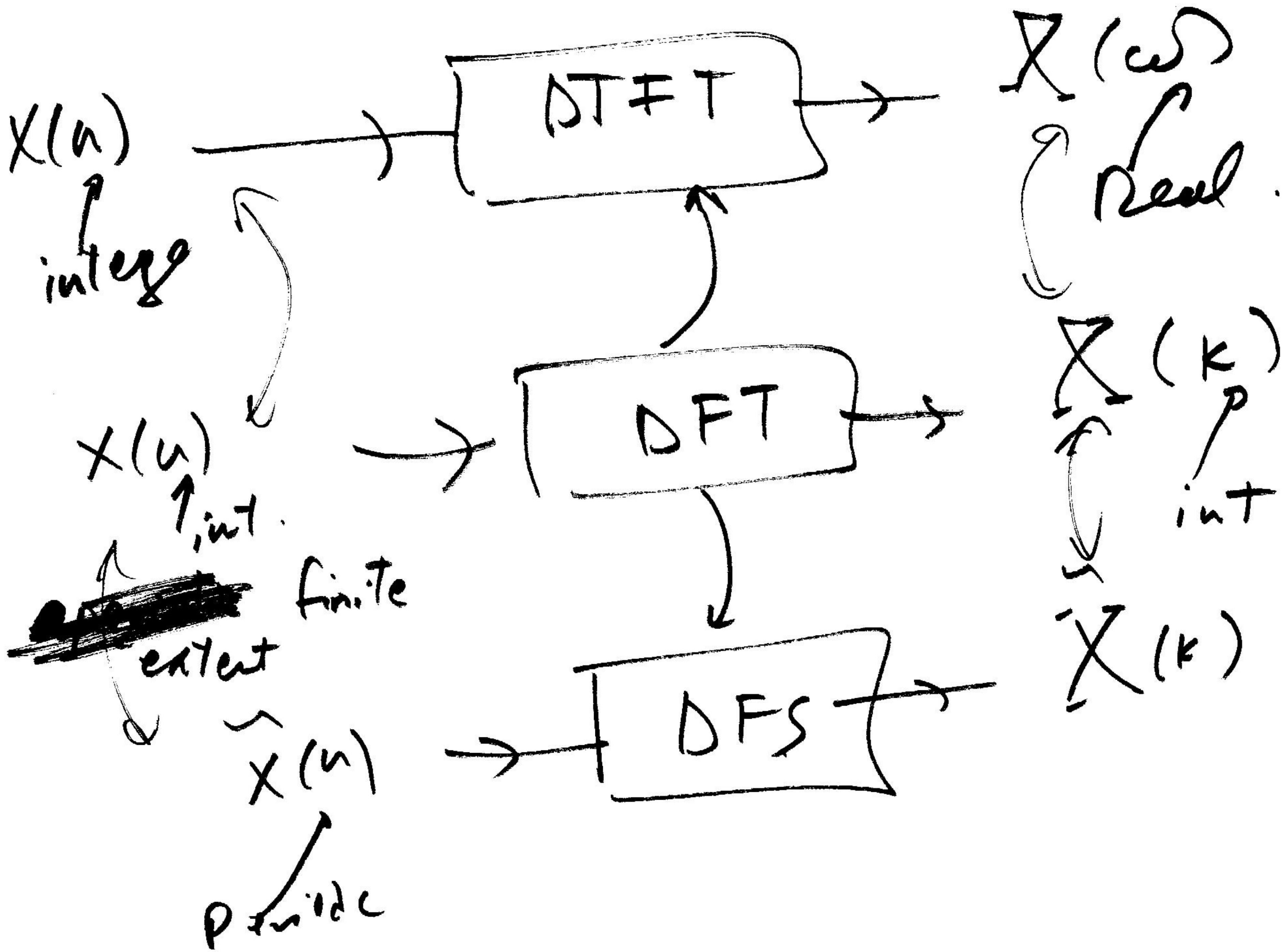
8 pt seq. ⟶ 4pt DFT.

$$\omega = \frac{2\pi k}{8}$$

$k = 0, 1, 2, 3, 4, 5, 6, 7$



z plane


x(n)
integer
DTFT
X(ω)
Real
x(n)
int.
finite
extent
DFT
X(k)
int
x̃(n)
periodic
DFS
X̃(k)