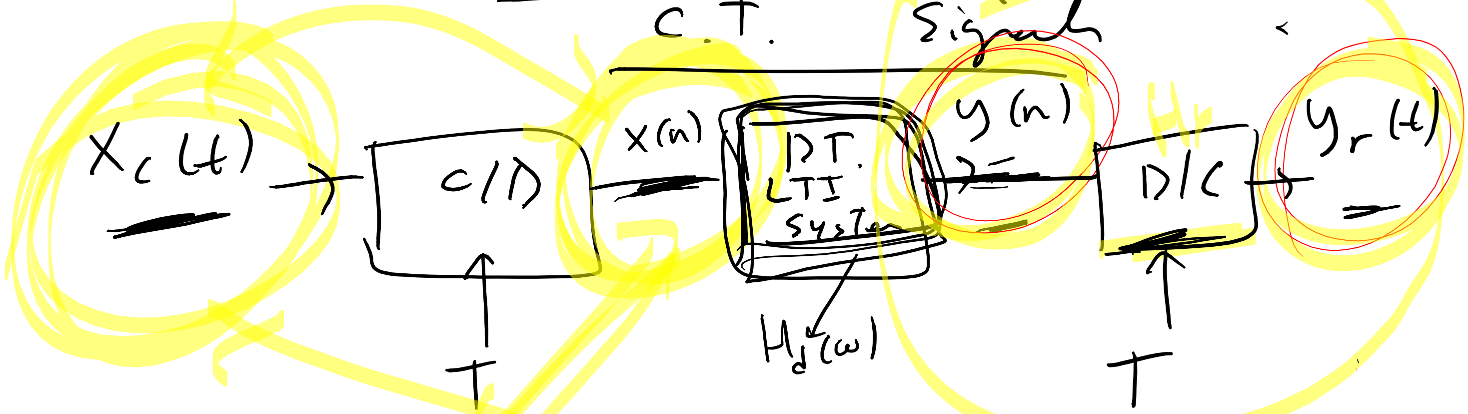
Discretime Processing of C.T. Signals



$x(n) = x_c(nT)$

$$X_d(\omega) = \frac{1}{T}\sum_{k=-\infty}^{+\infty} X_c\left(\frac{\omega}{T} - \frac{2\pi k}{T}\right)$$

DTFT of x(n)

$$y_r(t) = \sum_n y(n) \frac{\sin\left[\pi\frac{(t-nT)}{T}\right]}{\pi\frac{(t-nT)}{T}}$$

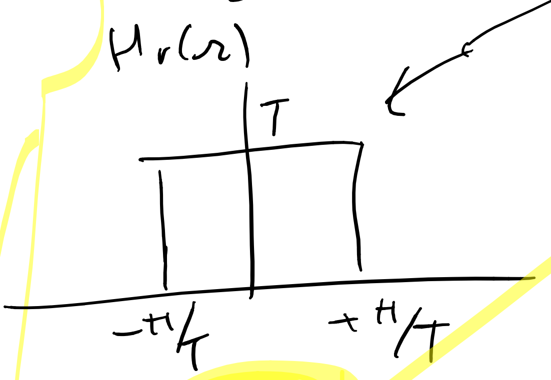
Use ⊕

$$Y_r(\Omega) = H_r(\Omega)\, Y_d(\Omega T)$$

C.TFT $y_r(t)$

ideal L.P.F.

DTFT of y(n)



$$= \begin{cases} T\, Y_d(\Omega T) & |\Omega| < \pi/T \\ 0 & \text{otherwise} \end{cases}$$

$-\pi/T$ $+\pi/T$

$$Y_d(\omega) = H_d(\omega)\, X_d(\omega)$$

filter

Discrete Time

$$Y_r(\Omega) = H_r(\Omega)\, \underbrace{H_d(\Omega T)}_{\text{D.T. Freq.}}\, \underbrace{X_d(\Omega T)}_{\text{DTFT } x(n)}$$

Recont filter ideal LP

$$X_d(\omega) = \frac{1}{T} \sum_k X_c\left(\frac{\omega}{T} - \frac{2\pi k}{T}\right)$$

$$X_d(\Omega T) = \frac{1}{T} \sum_k X_c\left(\Omega - \frac{2\pi k}{T}\right)$$

$$Y_r(\Omega) = H_r(\Omega)\, H_d(\Omega T)\, \frac{1}{T} \sum_k X_c\left(\Omega - \frac{2\pi k}{T}\right)$$

$$Y_r(\Omega) = \begin{cases} X_c(\Omega)\, H_d(\Omega T) & |\Omega| < \pi/T \\ 0 & |\Omega| > \pi/T \end{cases}$$

$$\Big| \; 0 \qquad |\Omega| > \pi/T$$

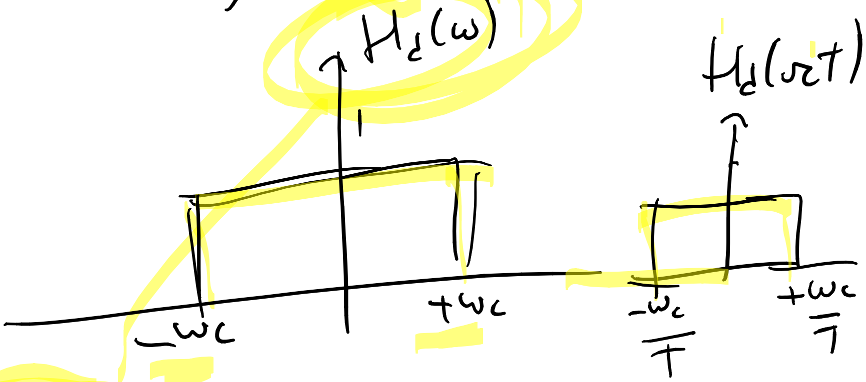
$$Y_r(\Omega) = H_{eff}(\Omega) X_c(\Omega)$$

$$H_{eff}(\Omega) = \begin{cases} H_d(\Omega T) & |\Omega| < \pi/T \\ 0 & |\Omega| > \frac{\pi}{T} \end{cases}$$

Example

$$\underline{H_d(\omega)} = \begin{cases} 1 & |\omega| < \omega_c \\ 0 & \omega_c < |\omega| < \pi \end{cases}$$

DT filter LTI

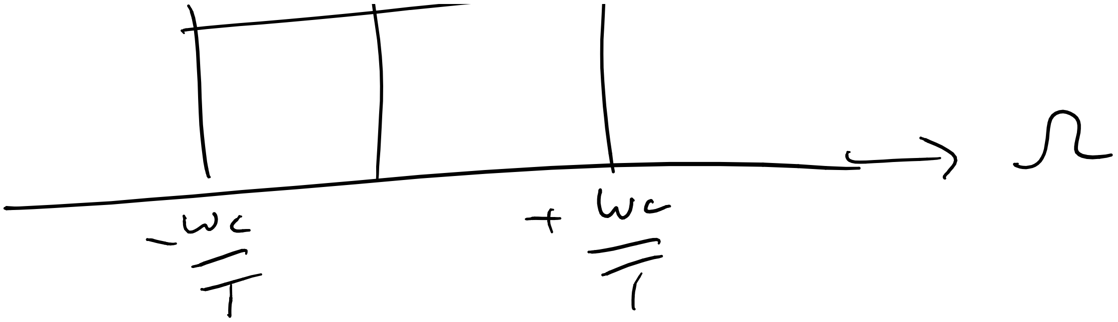


$$H_{eff}(\Omega) = \begin{cases} H_d(\Omega T) & |\Omega| < \pi/T \\ 0 & \text{otherwise} \end{cases}$$

$$|\Omega T| < \omega_c \Rightarrow |\Omega| < \frac{\omega_c}{T}$$

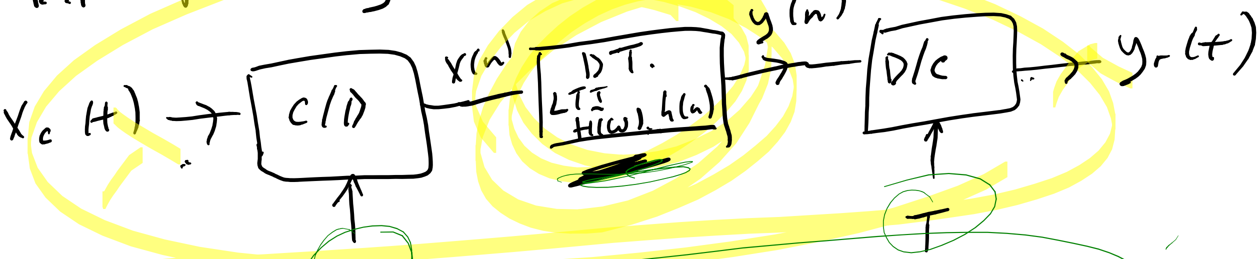
$$|\Omega T| > \omega_c \Rightarrow |\Omega| > \frac{\omega_c}{T}$$

$$\underline{H_{eff}(\Omega)} = \begin{cases} 1 \\ 0 \end{cases}$$



Impulse Invariance:

- Suppose desired C.T. filter $H^c_{desired}(\Omega)$
- D.T. processing



$$H^c_{desired}(\Omega) = \begin{cases} H^*(\Omega) & |\Omega| < \Omega^* \\ 0 & \text{elsewhere} \end{cases}$$

Bandlimited

$$H_{eff}(\Omega) = \begin{cases} H_d(\Omega T) & |\Omega| < \pi/T \\ 0 & \text{elsewhere} \end{cases}$$

$$H_d(\Omega T) = H^c_{desir}(\Omega) \Rightarrow H_d(\omega) = H^c_{desired}\left(\frac{\omega}{T}\right)$$

T must be chosen so that $H^c_{desired} = 0$ for $|\Omega| > \frac{\pi}{T}$

Can show that the impulse response of $h_{desired}(t)$ and $h(n)$ are related as follows

$$h_d(n) = \left[h^c_{desired}(t)\right]_{t = nT}$$

$$H_d(\omega) = \frac{1}{T} \sum^{+\infty} H^c_{desired}\left(\frac{\omega}{T} - \frac{2\pi k}{T}\right)$$

$$H_d(w) = \frac{1}{T} \sum_{K=-\infty} H^c_{desired}\left(\frac{w}{T} - \frac{2\pi k}{T}\right)$$

Assume $H^c_{desire}(\Omega) = 0$ $\quad |\Omega| > \pi/T$

$$H_d(w) = \frac{1}{T} H^c_{desired}\left(\frac{w}{T}\right) \qquad |w| < \pi$$

$\Rightarrow$ If $h_d(n) = T\, h^c_{desired}(nT)$

Then $\Longrightarrow$ $H_d(w) = H^c_{desired}\left(\frac{w}{T}\right) \qquad |w| < \pi$

Ex $\quad H^c_{desired}(\Omega) = \begin{cases} 1 & |\Omega| < \Omega_c \\ 0 & \text{otherwise.} \end{cases}$

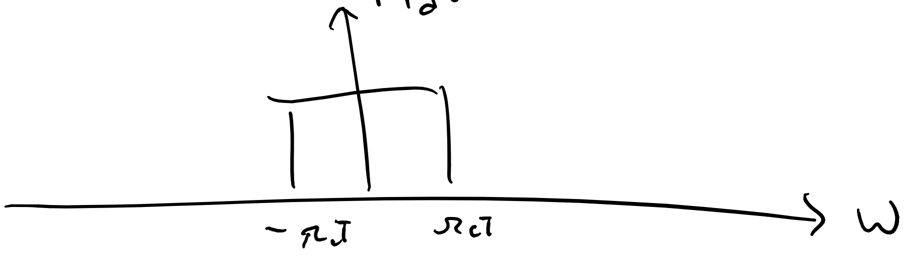
① Sample the impulse response.

$$h(n) = T\left[h^c_{desired}(t)\right]_{t=nT} = T\,\frac{\sin(\Omega_c nT)}{\pi nT}$$

$$H_d(w) = \begin{cases} 1 & |w| < \Omega_c T \\ 0 & \text{elsewhere.} \end{cases}$$

② $H_d(w) = H^c_{desired}\left(\frac{w}{T}\right) \qquad \left|\frac{w}{T}\right| < \Omega_c \Rightarrow |w| < \Omega_c T$

$$= \begin{cases} 1 & \\ 0 & \text{elsewhere.} \end{cases}$$



Example:

$$h^{c}_{desired}(t) = A e^{s_0 t} u(t)$$

$$H_c(s) = \frac{A}{s - s_0} \qquad \mathrm{Re}(s) > \mathrm{Re}(s_0)$$

Apply Impulse Invariance:

$$h(n) = T\left[h(t)\right]_{t=nT} = A T e^{s_0 T n} u(n)$$

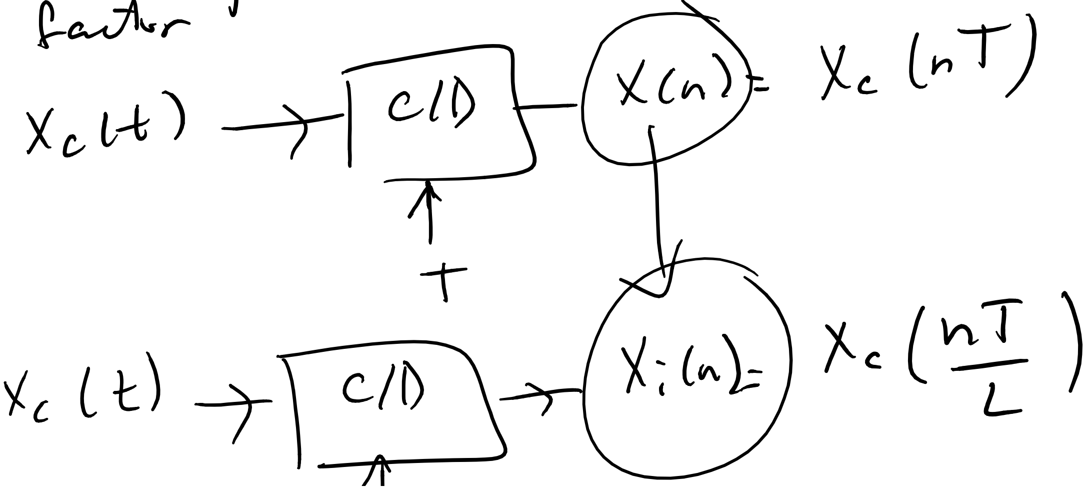
$$\Rightarrow H(z) = \frac{AT}{1 - e^{s_0 T} z^{-1}} \qquad |z| > \left|e^{s_0 T}\right|$$

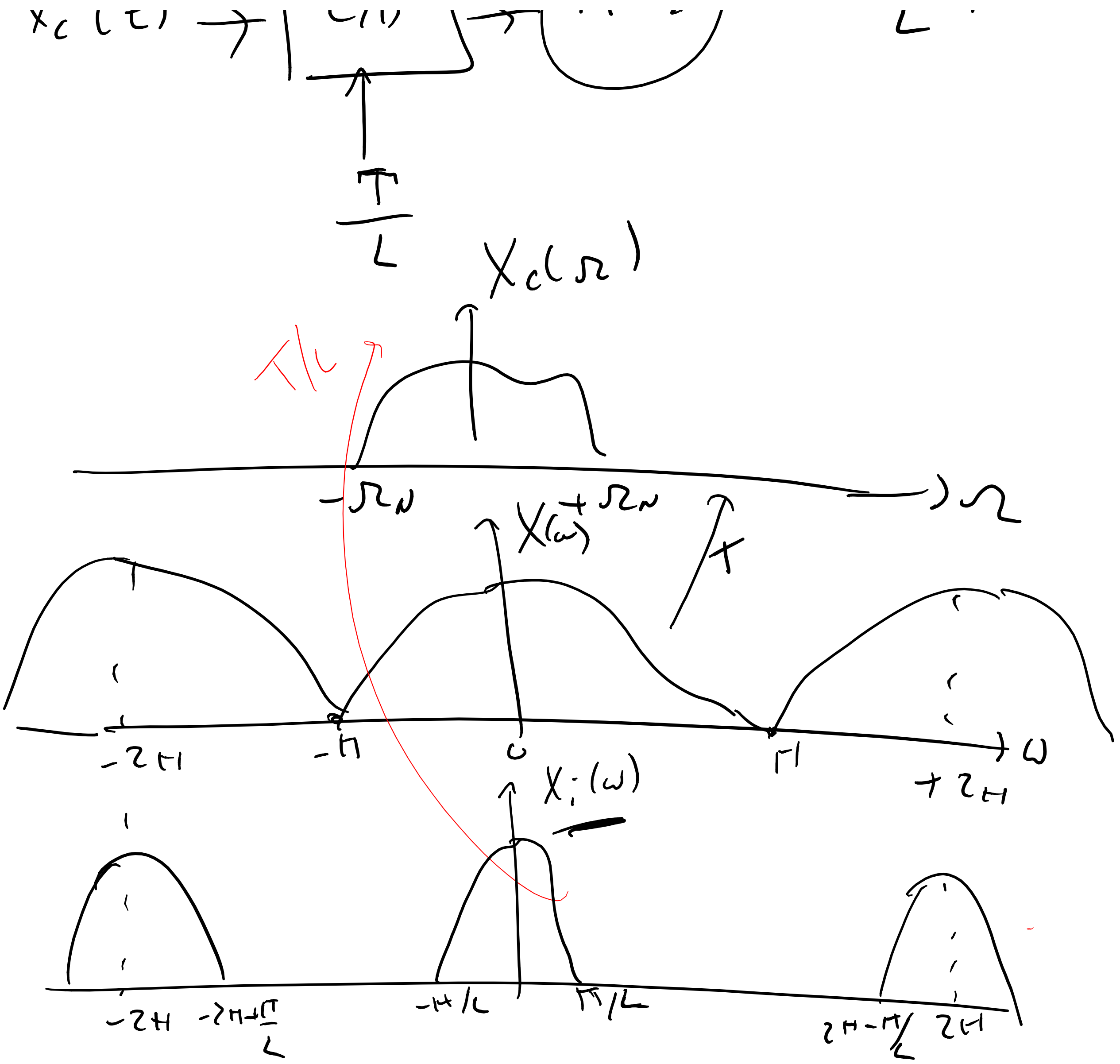
if $\mathrm{Re}[s_0] < 0$

$$\boxed{H_d(\omega) = \frac{AT}{1 - e^{s_0 T} e^{-j\omega}}}$$ — D.T. Filter

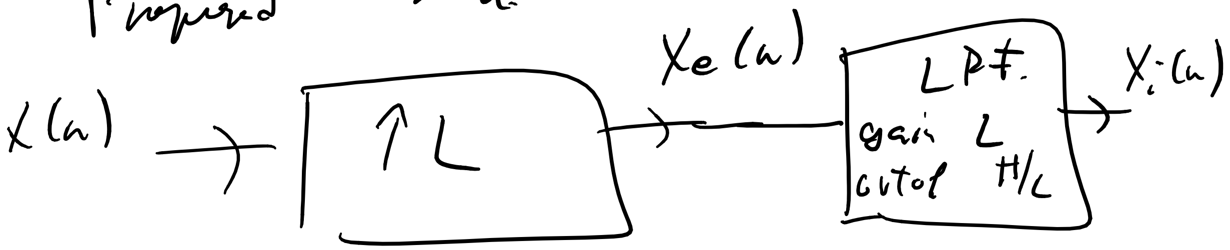
Upsampling

Increasing sample rate by an integer factor

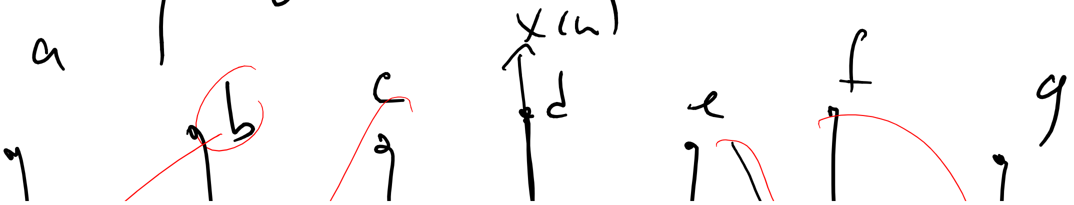


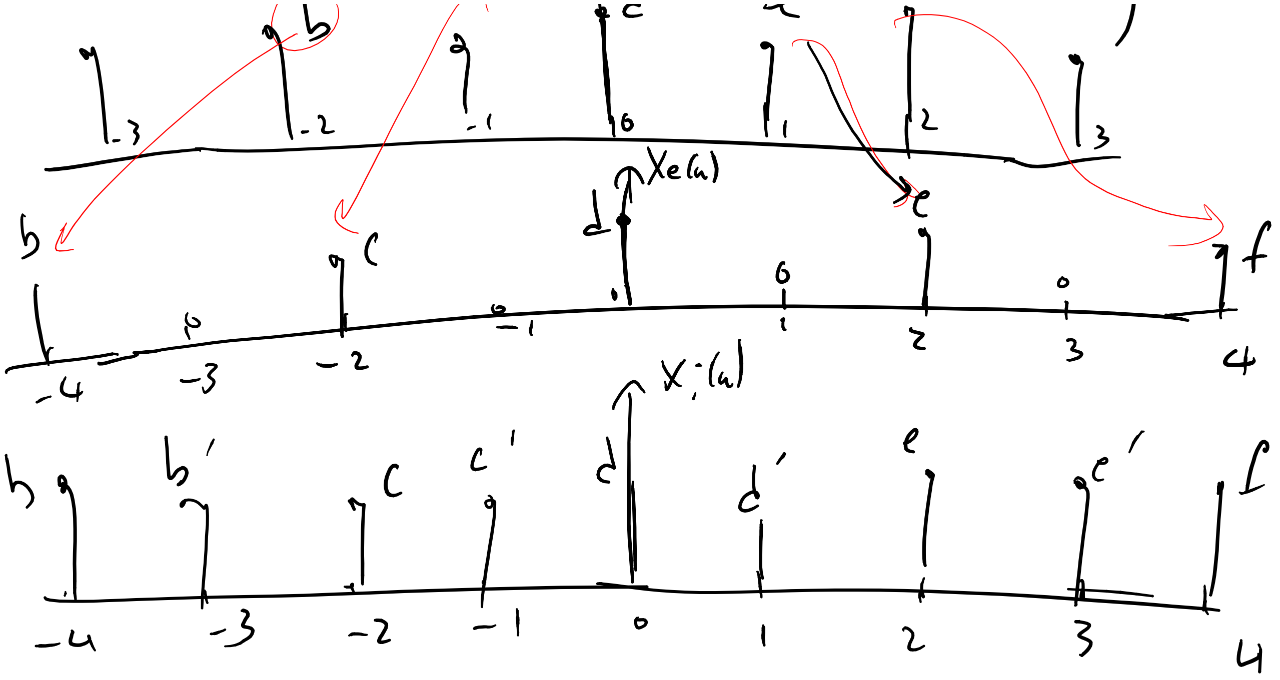


Proposed Soln.



$$x_e(n) \triangleq \begin{cases} x\left(\frac{n}{L}\right) & n=0,\pm L,\pm 2L,\ldots\ldots \\ 0 & \text{, otherwise} \end{cases}$$





$$X_e(n) = \sum_{k} x(k)\,\delta(n-kL)$$

DTFT

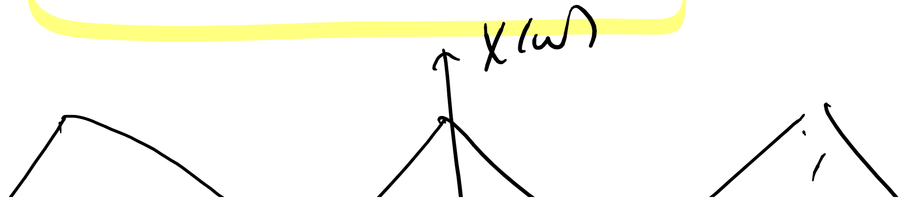
$$X_e(\omega) = \sum_{n} \left(\sum_{k} x(k)\,\delta(n-kL) \right) e^{-j\omega n}$$

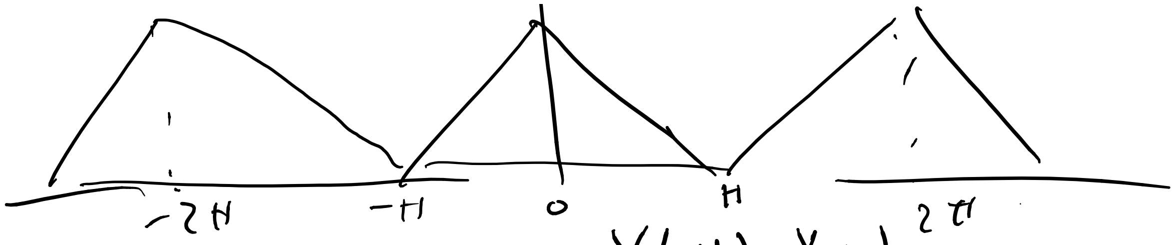
$$= \sum_{k} x(k) \left(\sum_{n} \delta(n-kL)\, e^{-j\omega n} \right)$$

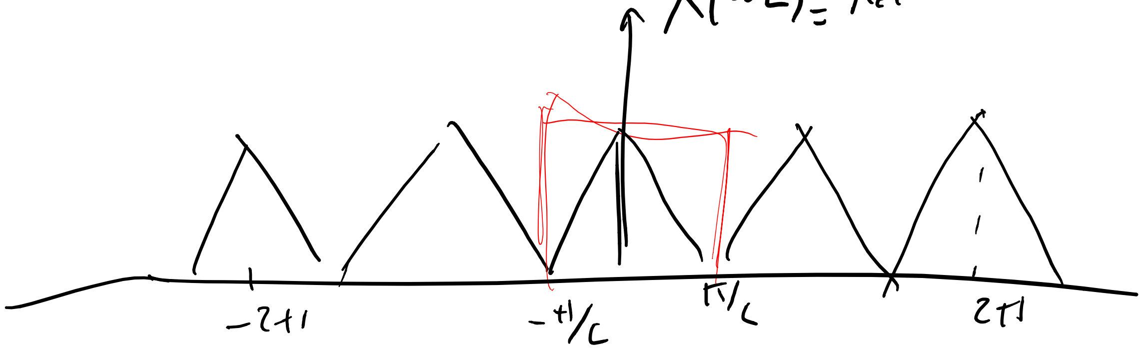
$$\left.\begin{aligned} X_e(\omega) &= \sum_{k} x(k)\, e^{-j\omega kL} \\ X(\omega) &= \sum_{k} x(k)\, e^{-j\omega k} \end{aligned}\right\} \Longrightarrow$$

$$X_e(\omega) = \left[X(\omega) \right]_{\omega \leftarrow \omega L}$$

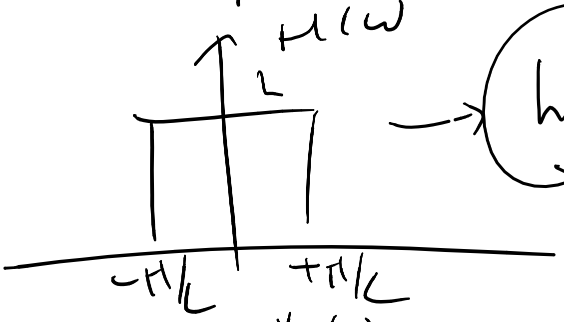
$$\boxed{X_e(\omega) = X(\omega L)}$$



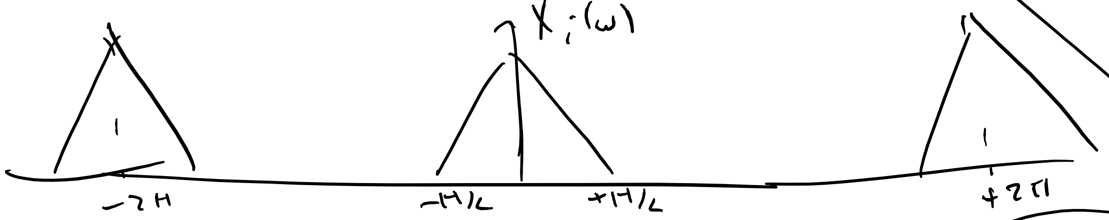




To get $X_i(\omega)$ from $X_e(\omega)$

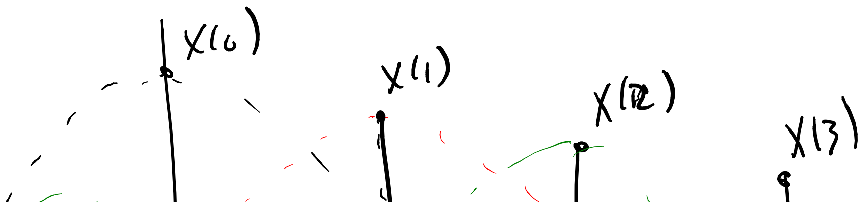


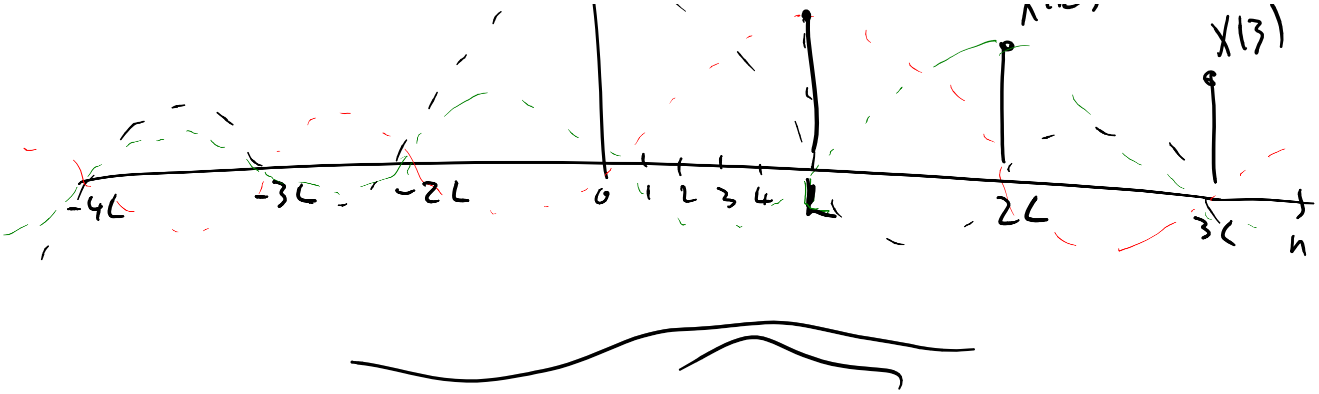
$$h(n) = \frac{\sin\frac{\pi n}{L}}{\pi n / L}$$



$$X_i(n) = X_e * h = \sum_{K} X_e(k)\, h(n-k)$$

$$X_i(n) = \sum_{K} x(K) \frac{\sin\frac{\pi(n-k)}{L}}{\pi\left(\frac{n-K}{L}\right)}$$




X(3)
-4L
-3L
-2L
0
1
2
3
4
L
2L
3L
n