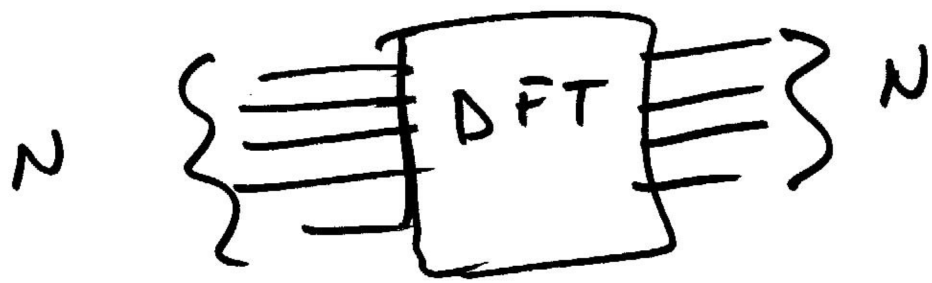
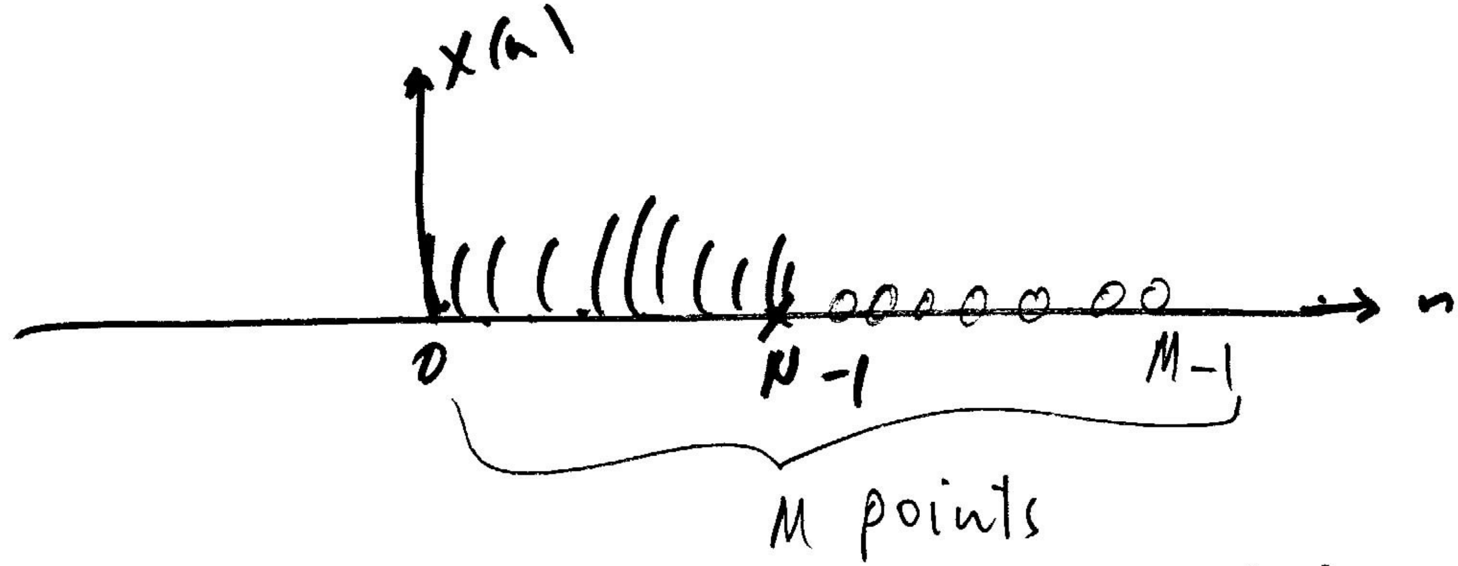
DFT = Discrete Fourier Transform

- finite length N pt seq $x(n)$, N non zero values

$$DTF\{x(n)\} = X(k) = \begin{cases} \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi n k}{N}} & 0 \le k < N \\ 0 & \text{otherwise} \end{cases}$$

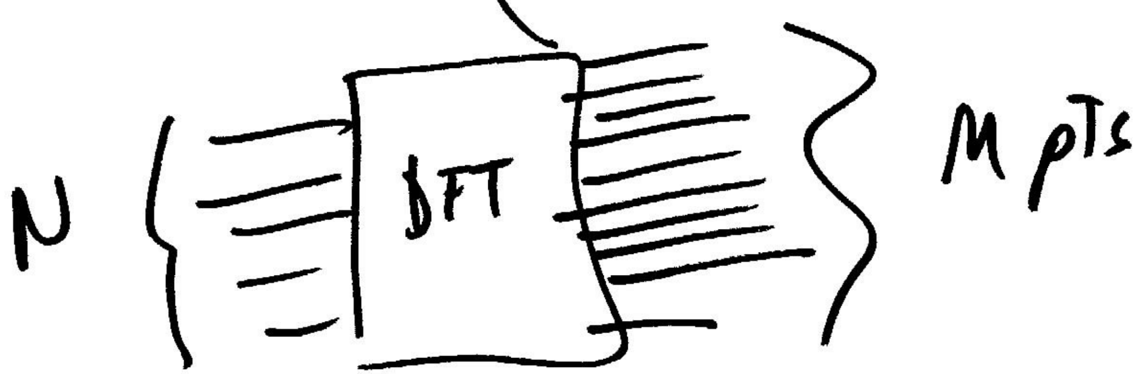


- Possible to pad an N pt seq. with N nonzero values to M points, ie pad it with $M-N$ additional zero's
- Possible to talk about DFT of M pt seq.

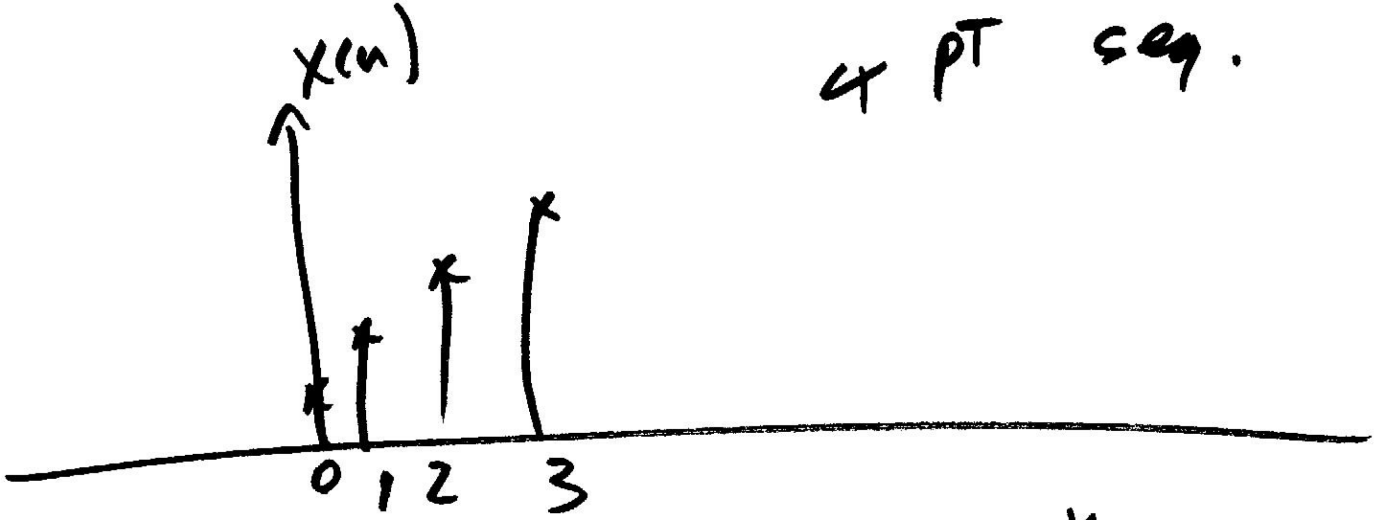


- Can take M pt DFT of $x(n)$:

$$X(k) = \begin{cases} \sum_{n=0}^{M-1} x(n)\, e^{-j2\pi nk/M} & 0 \le k < M \\ 0 & \text{otherwise.} \end{cases}$$



Ex



4 pT seq.

- 4 pT DFT x(n) : $\sum_{n=0}^{3} x(n)\, e^{-j\frac{2\pi n k}{4}}$ 4 #s.

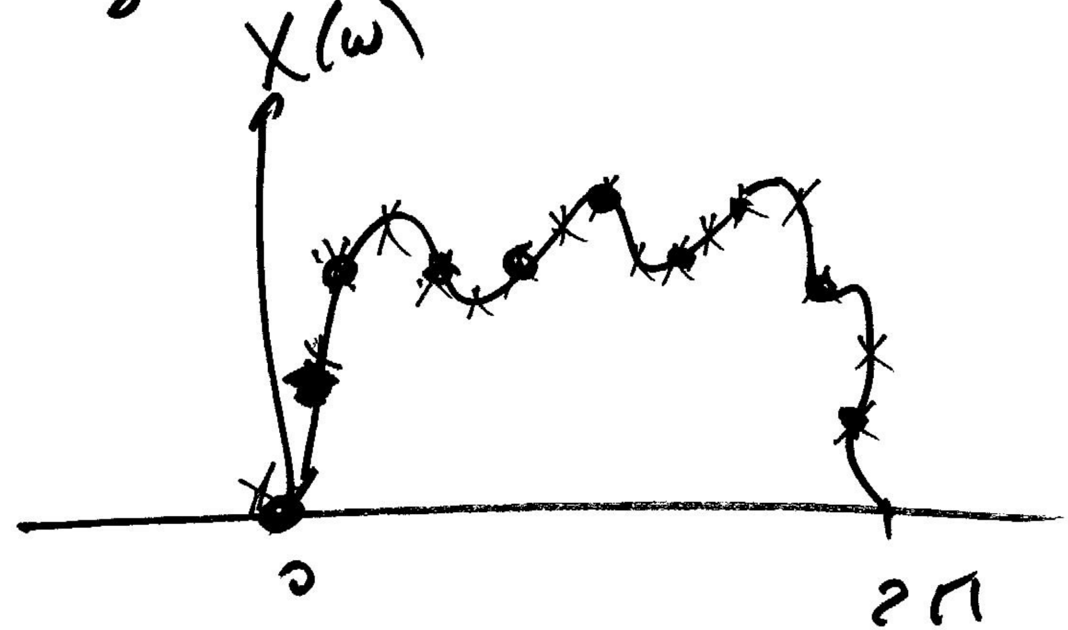
- 5 pt DFT x(n): $\sum_{n=0}^{4} x(n)\, e^{-j\frac{2\pi n k}{5}}$ 5 #s

- 1000 pt DFT of x(n) $\sum_{n=0}^{999} x(n)\, e^{-j\frac{2\pi n k}{1000}}$ 1000 #s

- 8 pt DFT of x(n) $\sum_{n=0}^{7} x(n)\, e^{-j\frac{2\pi n k}{8}}$ 8 #s

$x(n)$ Npt . pad it $M-N$ zeros . $\dot{x}(n)$ Mpt $\longrightarrow$ $X(k)$ Mpt

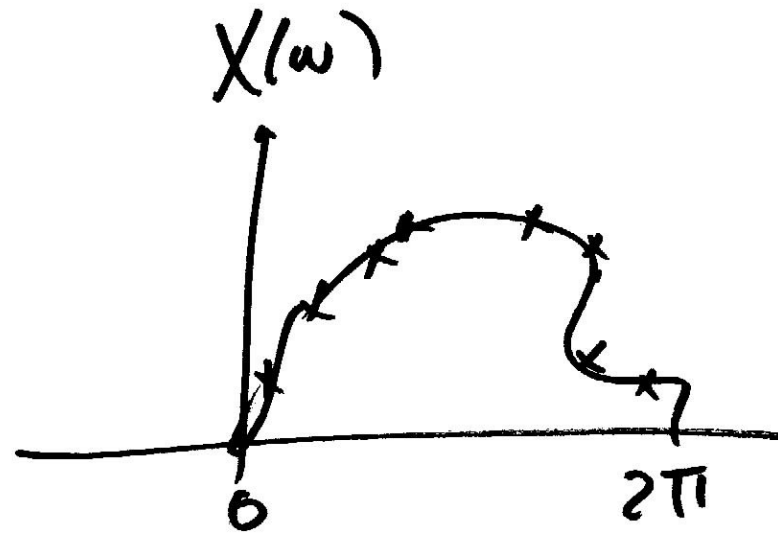
$x(n) \longrightarrow$



$X(n) \longrightarrow \boxed{N\text{pt DFT}} \longrightarrow$ N #s out

$X(n) \longrightarrow \boxed{2N\text{pt DFT}} \longrightarrow$ 2N #s

Npoint seq. Padded it N zeros.



Suppose we have nonuniform samples of $X(\omega)$ at N points. We know $X(\omega)$ is DTFT of an N pt seq.

Can I recover $x(n)$?

<u>Yes</u> solving a linear system of eqns.

$$X(\omega) = \sum x(n) e^{-j\omega n}$$

$$\Big[X(\omega_i)\Big]_{i=0,\dots N-1} = \sum_{n=0}^{N-1} x(n)\, e^{-j\omega_i n}.$$

$$\begin{bmatrix} X(\omega_0) \\ X(\omega_1) \\ \vdots \\ X(\omega_{N-1}) \end{bmatrix} = \begin{bmatrix} e^{-j\omega_0 \cdot 0} & e^{-j\omega_0 \cdot 1} & \cdots & e^{-j\omega_0 (N-1)} \\ \vdots & \ddots & & \vdots \\ e^{-j\omega_{N-1} \cdot 0} & e^{-j\omega_{N-1} \cdot 1} & \cdots & e^{-j\omega_{N-1}(N-1)} \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \end{bmatrix}$$

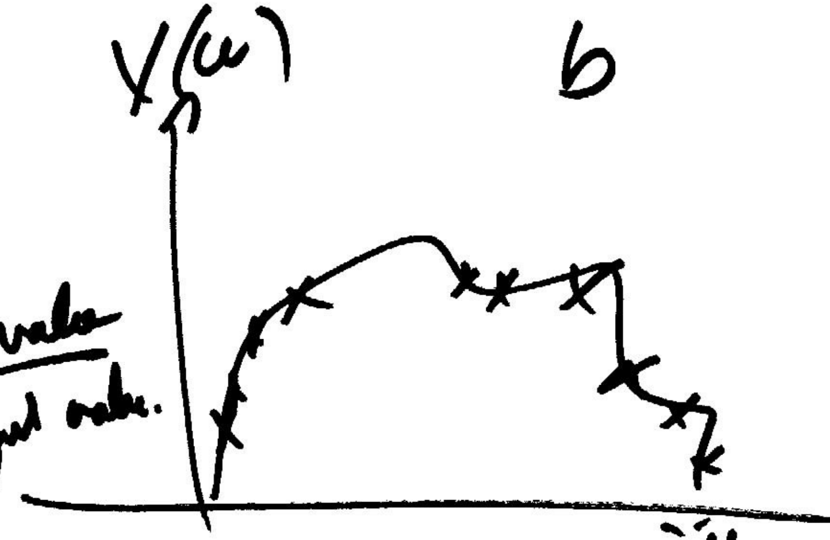
Known. nonuniform sample $N \times N$ Unknown

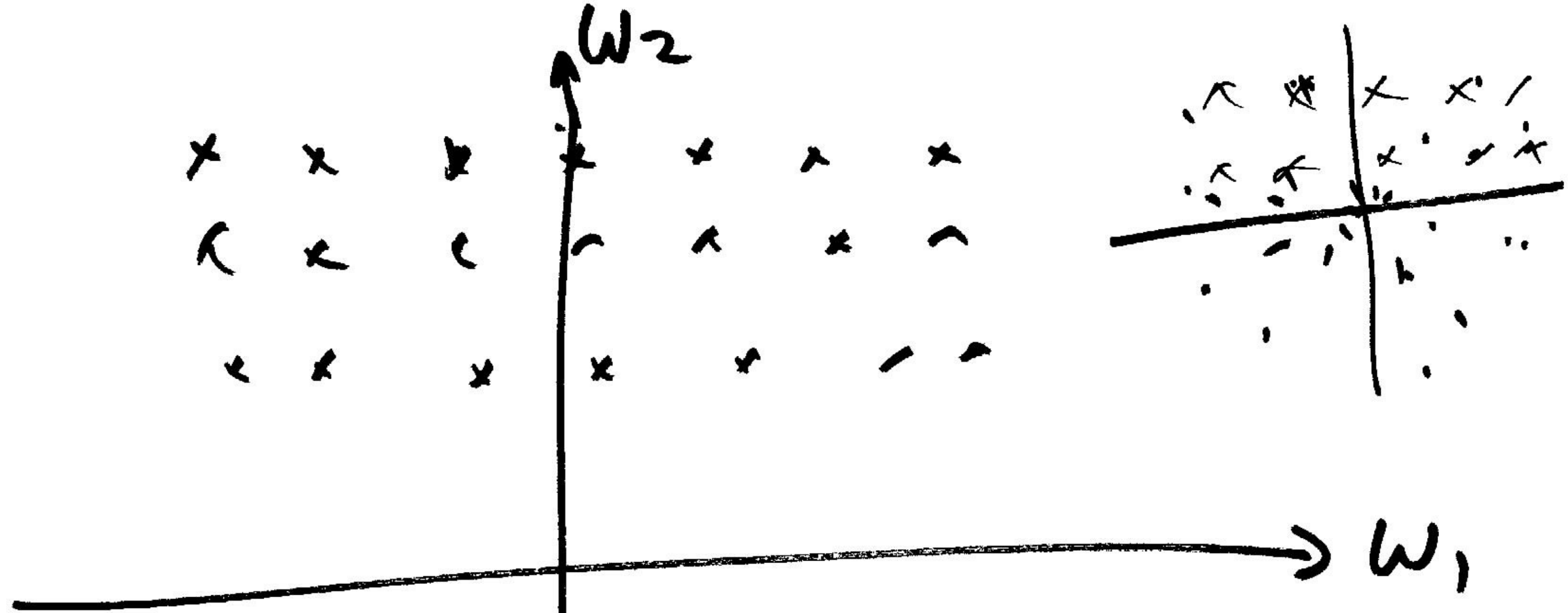
A N pt.

NPt $\vec{x}$ b

$$A\vec{x} = b$$

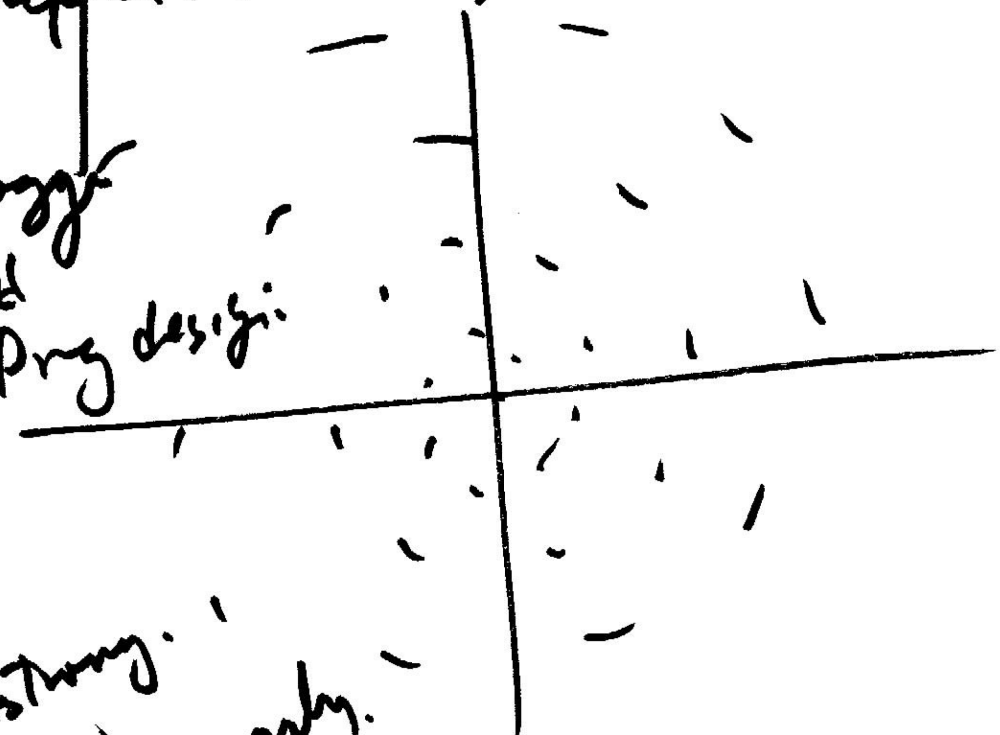
Condition # $= \dfrac{\text{largest singular value}}{\text{smallest singular value}}$





Graph non uniform samples:

Freq.



1. X Ray Crystallography
widely used
biology | Prog design.

2. MRI
3. Radar
3. Radio Astronomy.
4. Tomography.

Thought Exp:

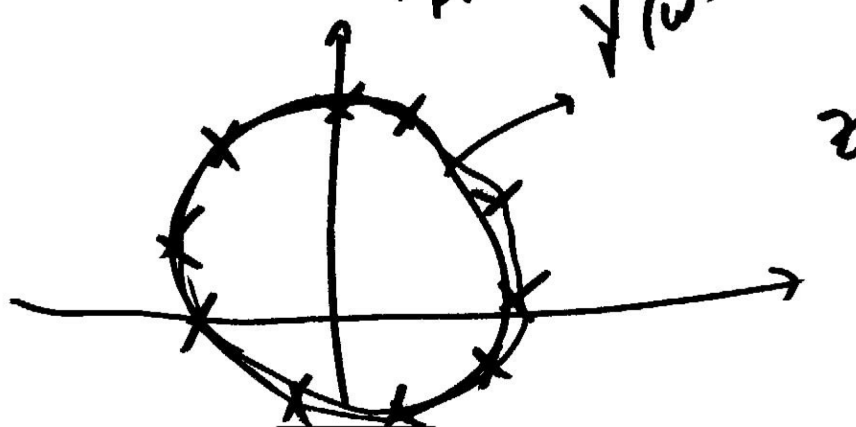
1. $y(n)$: either ∞ extent or finite extent.
2. $Y(\omega) = \text{DTFT}\{y(n)\}$.
3. Sample $Y(\omega)$ at N equally spaced points.

$$\left[Y(\omega)\right]_{\omega = \frac{2\pi k}{N}} = \tilde{X}(k)$$

N pt seq.

4. $$\text{IDTFT}\{\underbrace{\tilde{X}(k)}_{N\text{pt}}\} = x(n)$$

N pt seq.



<u>Answer</u>: periodicize $y(n)$ with period N.

$$\tilde{w}(n) = \sum_{k=-\infty}^{+\infty} y(n + kN)$$

extract one period:

$$x(n) = w(n)\, R_N(n).$$

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proof

$$x(n) = \begin{cases} \frac{1}{N} \sum_{k=0}^{N-1} X(k)\, e^{+j\frac{2\pi nk}{N}} & 0 \le n < N \\ 0 & \text{otherwise} \end{cases}$$

$$x(n) = \begin{cases} \frac{1}{N} \sum_{k=0}^{N-1} \left(\sum_m y(m)\, e^{-j\frac{2\pi km}{N}} \right) e^{+j\frac{2\pi nk}{N}} & 0 \le n < N \\ 0 & \text{otherwise} \end{cases}$$

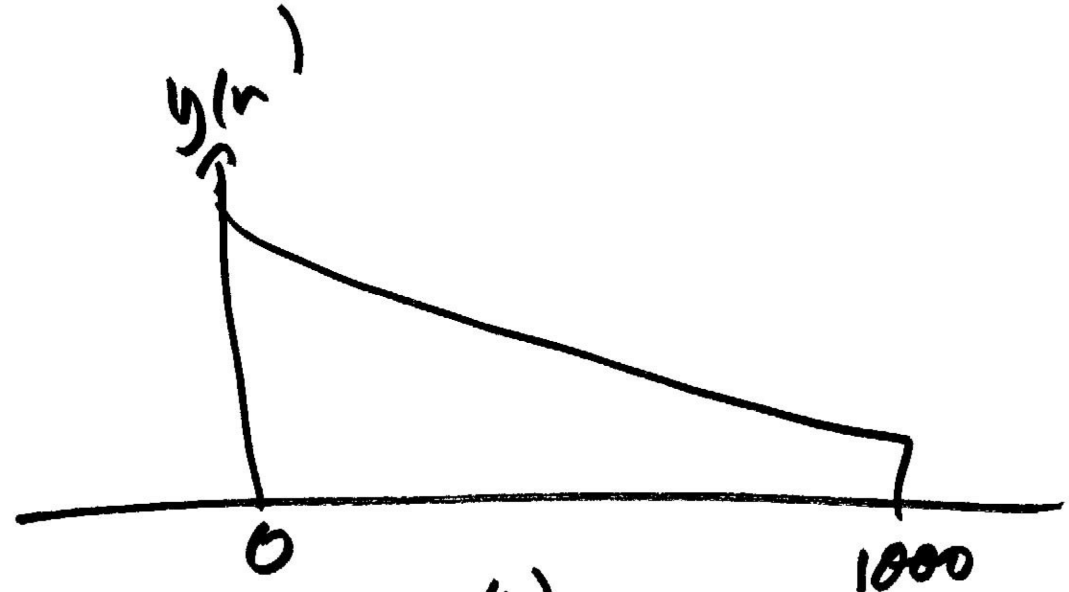
$$x(n) = \begin{cases} \displaystyle\sum_{m=-\infty}^{+\infty} y(m) \underbrace{\frac{1}{N}\sum_{k=0}^{N-1} e^{-j2\pi k\frac{(m-n)}{N}}}_{=\sum_{r=-\infty}^{+\infty}\delta(n-m+rN) \,=\, f(n-m)} & 0 \le n < N \\ 0 & \text{otherwise} \end{cases}$$

$$\text{Call} \quad f(n) = \sum_{r=-\infty}^{+\infty} \delta(n+rN)$$

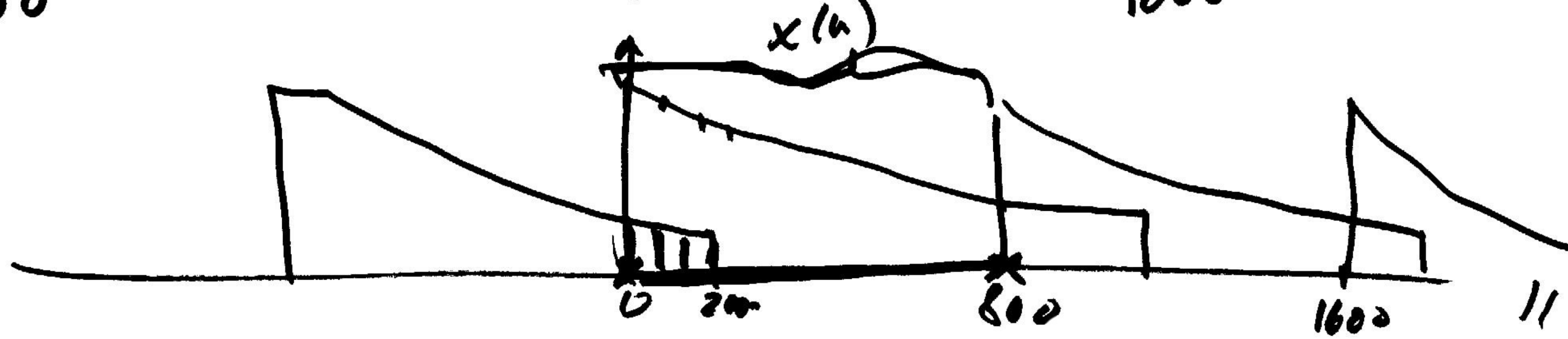
$$x(n = \begin{cases} \displaystyle\sum_{m=-\infty}^{+\infty} y(m)\, f(n-m) & 0 \le n \le N \\ 0 & \text{otherwis..} \end{cases}$$

$$=$$

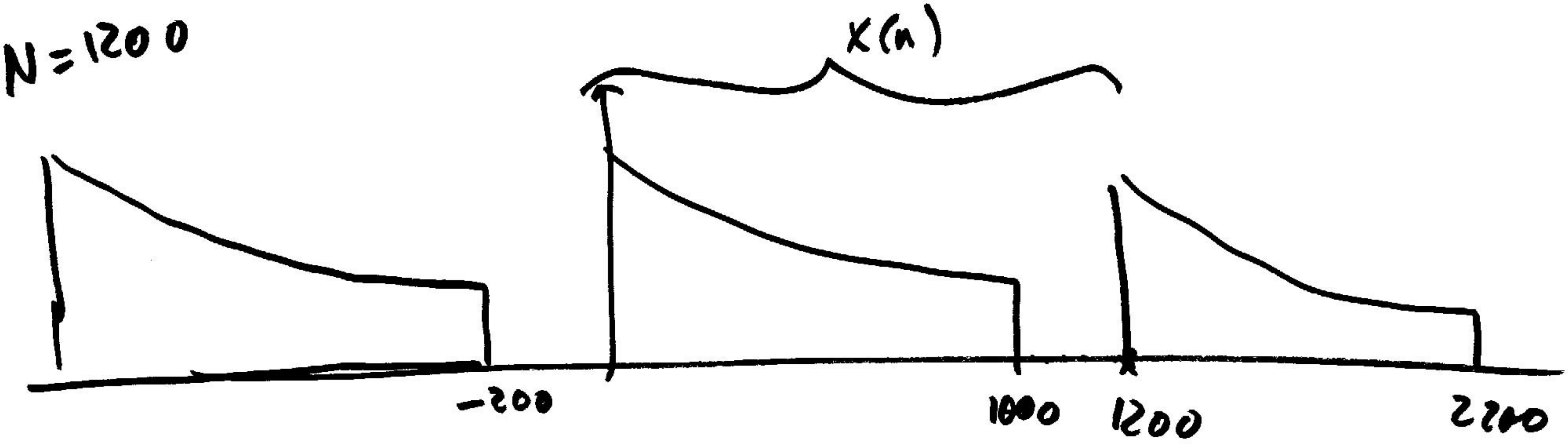
$$x(n) = \begin{cases} \sum_{r=-\infty}^{+\infty} y(n+rN) & 0 \le n \le N \\ 0 & \text{otherwise} \end{cases}$$



N=800



$N = 1200$



To get $y(n)$ back precisely, we need to sample its DTFT, $Y(\omega)$ at a higher rate than # of contiguous nonzero points in $y(n)$.

Properties of DFT.

① Shift property.

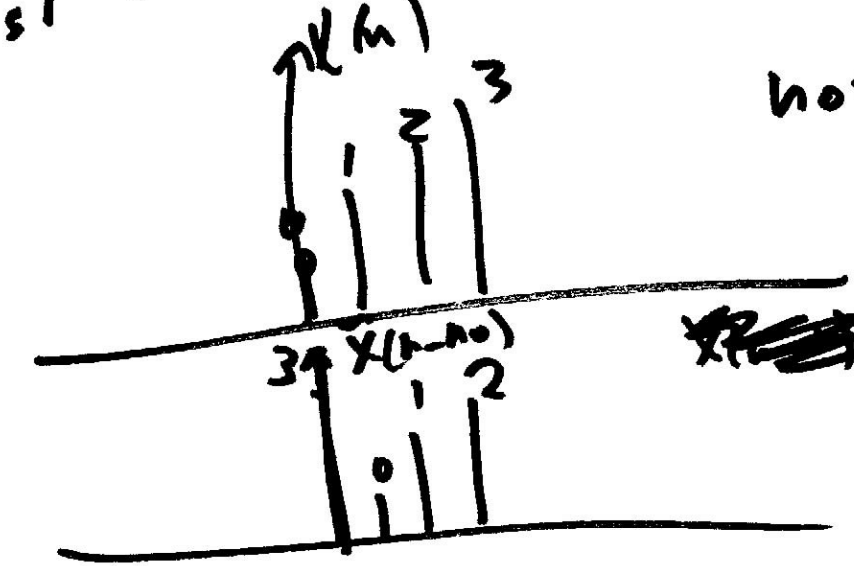
$x(n)$ $\xleftrightarrow{\text{DFT}}$ $X(k)$ Npt $0 \le k < N$

N pt

$\cancel{x(n-n_0)}$ $\xleftrightarrow{\text{IDFT}}$ $X(k)\, e^{-j\frac{2\pi n_0 k}{N}}$ Npt $0 \le k < N$

Should only be nonzero for first N points

$n_0 = 1$



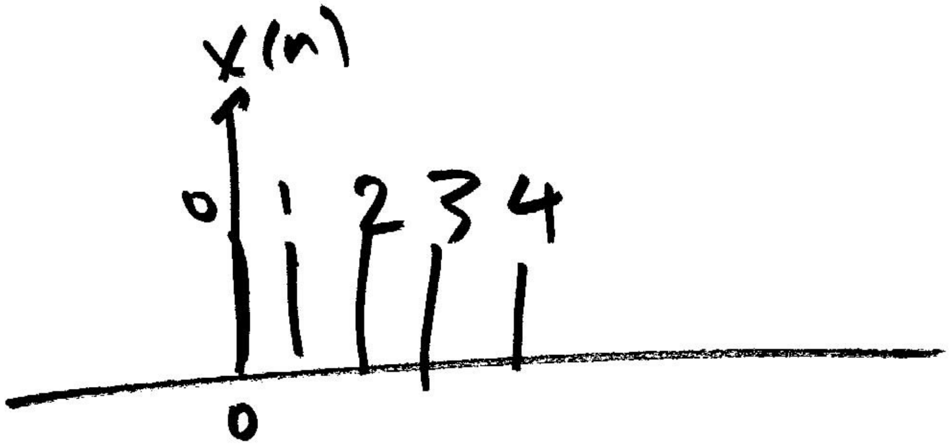
[illegible]

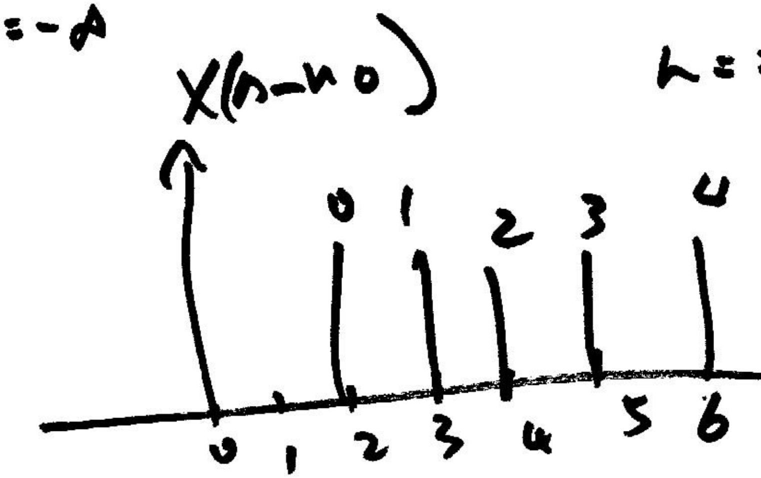
$$\tilde{x}(n-n_0)\,R_N(n) \xleftrightarrow{\text{IDFT}} X(k)\,e^{-j2\pi n_0 k/N}$$

<u>Why?</u>

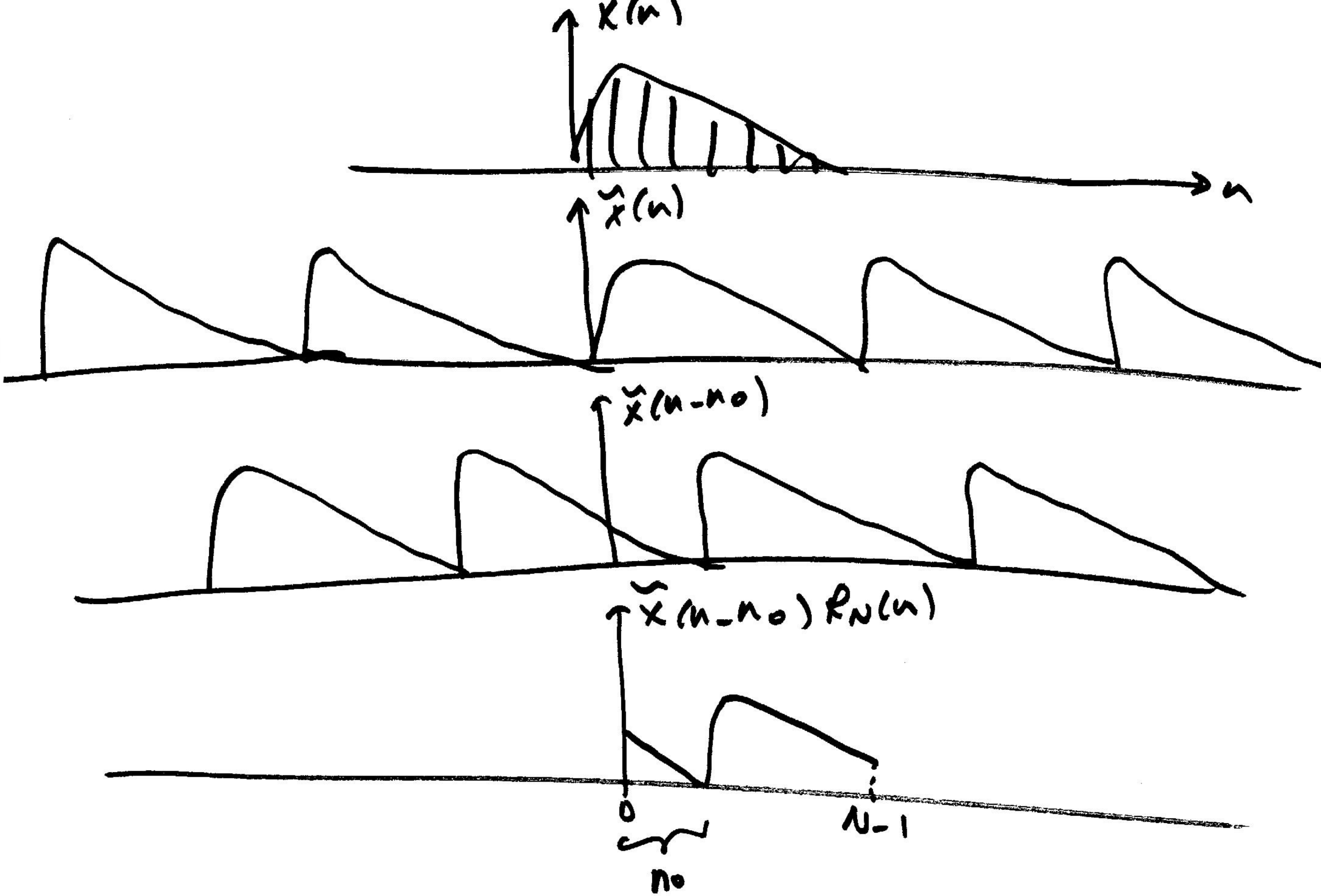
$$x(n) \xleftrightarrow{\text{periodize}} \tilde{x}(n) = \sum_{k=-\infty}^{+\infty} x(n+kN) \xleftrightarrow{\text{DFS}} \underset{\text{DFS}}{\tilde{X}(k)} \longrightarrow \underset{\text{DFT}}{X(k)}$$

$n_0 = 2$





$x(n-n_0)$ is not valid IDFT.



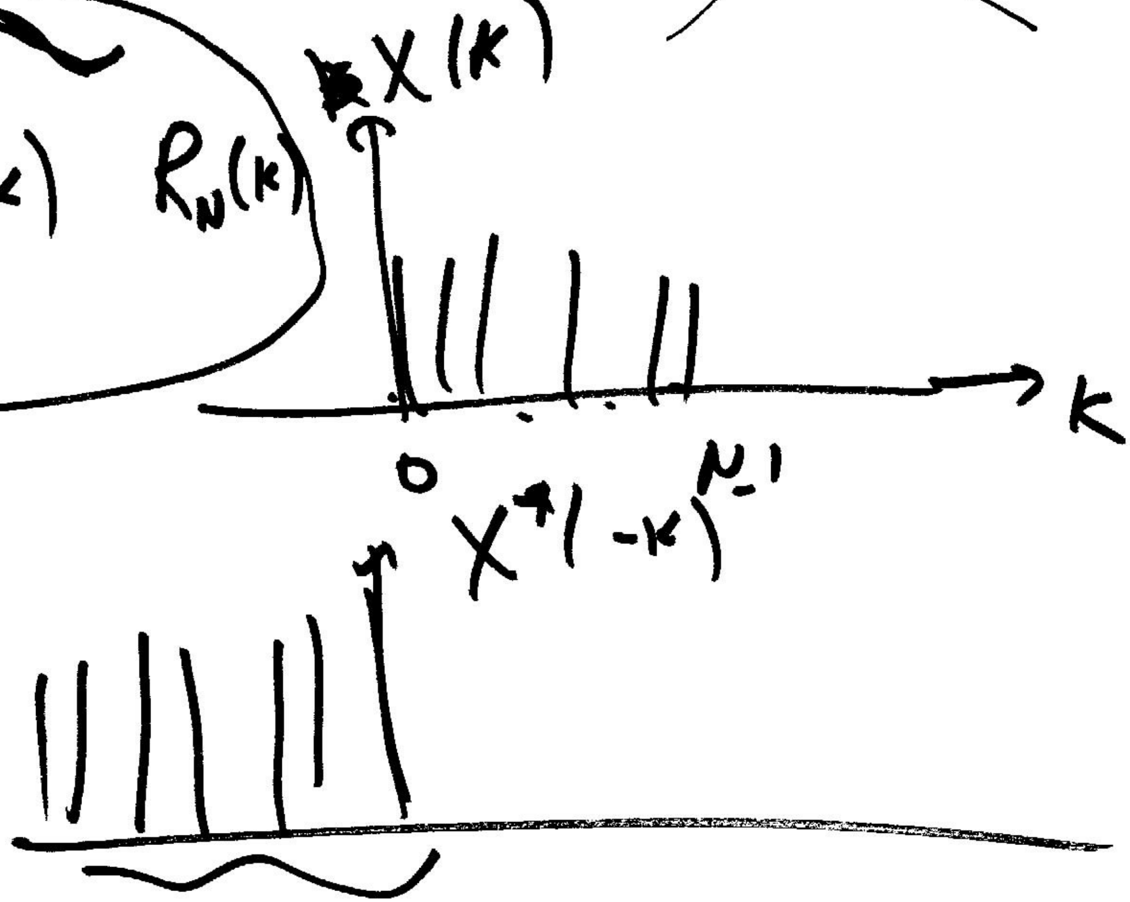
$x(n)$
n
$\tilde{x}(n)$
$\tilde{x}(n-n_0)$
$\tilde{x}(n-n_0)R_N(n)$
0
$N-1$
n_0

Another property of DFT:

$$x(n) \longleftrightarrow \underbrace{X(k)}_{} \rightarrow N\,pt.$$

$$y(n) = x^*(n) \xrightarrow{?} \cancel{X^*(-k)}$$

$$\mathrm{DFT}\{y(n)\} = \overbrace{X^*(-k)} \; R_N(k)$$



Symmetry Prop

$$x(n) \longleftrightarrow X(k)$$

$$\mathrm{Re}\{x(n)\} \longleftrightarrow R_N(k)\,\widetilde{X_e(k)}$$

$$j\,\mathrm{Im}\{x(n)\} \longleftrightarrow R_N(k)\,\widetilde{X_o(k)}$$

$$R_N(n)\,\widetilde{x_e(n)} \longleftrightarrow \mathrm{Re}\{X(k)\}$$

$$R_N(n)\,\widetilde{x_o(n)} \longleftrightarrow j\,\mathrm{Im}\{X(k)\}$$