How To use DFT To do Convolution

Convolution: $\longrightarrow$ linear Convolution

$$x_1 * x_2 = x_3$$

$$x_3(n) = \sum_k x_1(k)\, x_2(n-k)$$

$\longrightarrow$ LTI systems.

$$x(n) \rightarrow \boxed{\begin{matrix} LTI \\ h(n) \end{matrix}} \rightarrow y(n) = \sum_k x(k)\, h(n-k)$$

- periodic convolution

$$\underset{N}{\tilde{x}_1} \circledast \underset{N}{\tilde{x}_2} = \underset{N}{\tilde{x}_3(n)}$$

- Circular Convolution :

2 finite length sequences.

x_1 N pt x_2 N pt

$$\underbrace{x_1}_{N\text{pt}} \; \underbrace{\circledast}_{\text{circularly convolution}} \; \underbrace{x_2}_{N\text{pt}} = \underbrace{x_4}_{N\text{pt}}.$$

<u>Def of circular convolution</u> :

$$x_4(n) \triangleq x_1 \circledast x_2 \triangleq \left[\tilde{x}_1 \circledast \tilde{x}_2\right] \cdot R_N(n)$$

Show: Npt DFT $x_4(n) = X_1(k)\, X_2(k)$

$$X_4(k) = X_1(k)\, X_2(k)$$

Npt DFT of x_4 — Npt DFT of x_1 — Npt DFT of x_2

We know from DFS properties:

$$\tilde{x}_1(n) \circledast \tilde{x}_2(n) = \tilde{x}_4(n)$$

$$\tilde{X}_1(k) = \text{DFS}\left\{ \tilde{x}_1(n) \right\} \; . \; . \; . \; . \; .$$

$$\tilde{X}_1(k)\, \tilde{X}_2(k) = \tilde{X}_4(k)$$

$$R_N(k)\left\{\tilde{X}_1(k)\,\tilde{X}_2(k)\right\} = R_N(k)\,\tilde{X}_4(k)$$

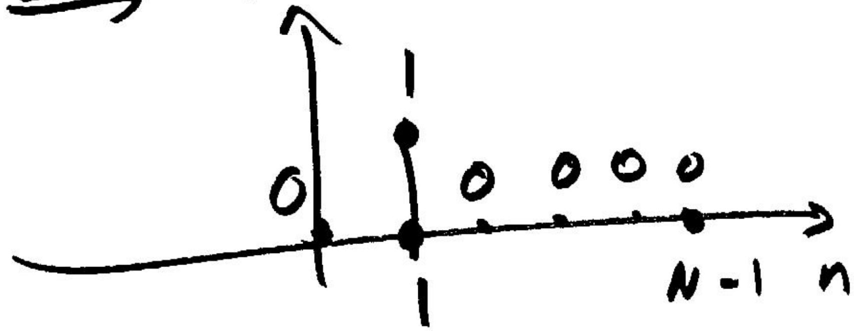
$$X_1(k)\,X_2(k) = X_4(k)$$

$\Rightarrow$ If multiply N pt DFT of 2 N pt sequences, you get DFT of their circular convolution & not their linear convolution

__Ex__ of circular convolution:

$x_1(n) = \delta(n - n_0)$

$n_0 = 1 \Longrightarrow x_1(n) = \delta(n-1)$



$x_2(n)$ an N pt seq.

$$x_1(n) = \begin{cases} 0 & 0 \le n < 1 \\ 1 & n = 1 \\ 0 & 1 < n < N \end{cases}$$

Fig. 8.14 O&S.

$$x_3(n) = x_1(n) ⊛ x_2(n)$$

$$= R_N(n)\left[\tilde{x}_1(n) ⊛ \tilde{x}_2(n)\right]$$

$$= \sum_{m=0}^{N-1} x_2(m)\left[\tilde{x}_1(n-m) R_N(m)\right]$$

Ex2 circular convolution ex.

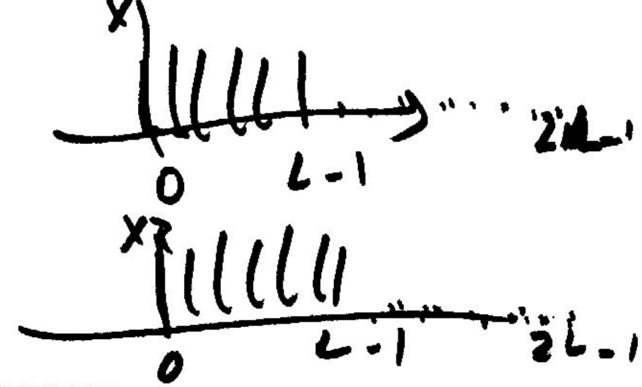
$$x_1(n) = x_2(n) = \begin{cases} 1 & 0 \le n \le L-1 \\ 0 & \text{otherwise} \end{cases}$$

N pt circular convolution x_1 & x_2

2 Cases of N:

(1) $N = L$

(2) $N = 2L$



Case (1) $N = L$

use DFT do our circ. conv..

Take $N_{pT} = L$ pt. DFT of x_1

$$X_L(k) = \sum_{n=0}^{L-1} x_1(n)\, e^{-j\frac{2\pi nk}{L}}$$

x_1 & x_2 $\to$ $X_L(k) = \begin{cases} L & k = 0 \\ 0 & \text{otherwise.} \end{cases}$

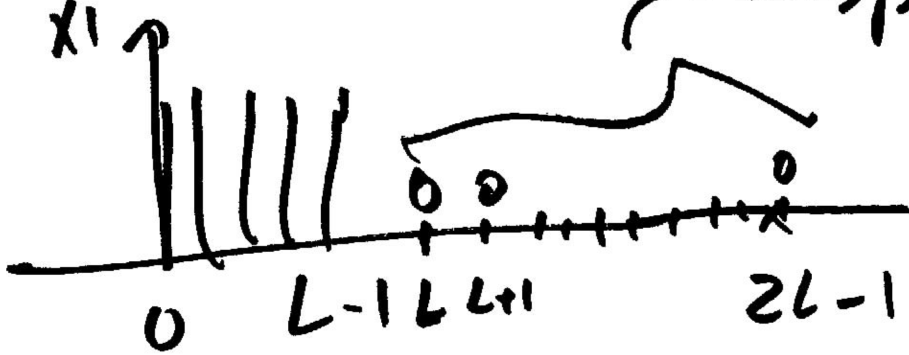
$X_3(k) =$ L pt. circ. Conv. of x_1 & x_2

$$= X_L(k)\, X_L(k) = \begin{cases} L^2 & k = 0 \\ 0 & \text{otherwise.} \end{cases}$$

$$L\text{pt IDFT}\{X_3(k)\} = \begin{cases} L & 0 \le n \le L-1 \\ 0 & \text{otherwise} \end{cases}$$

Case 2 $\quad N = 2L.$

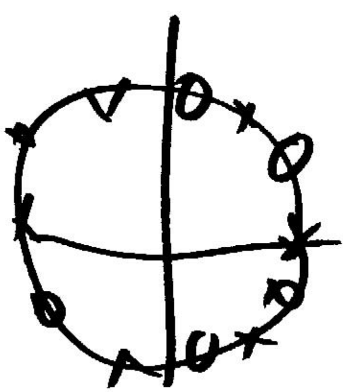
- $2L$ pT. circ. con of x_1 & x_2.



→ pad L extra zeros & I get a $2L$ point seq.

Compute $2L$ pt DFT of x_1

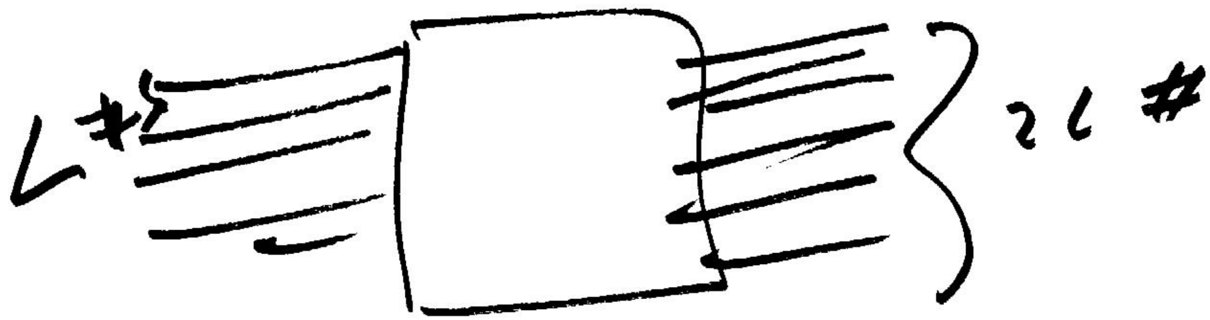
zpla



$$= \sum_{n=0}^{2L-1} x_1(n)\, e^{-j\frac{2\pi nk}{2L}}$$

$$= \sum_{n=0}^{L-1} x_1(n)\, e^{-j\frac{2\pi nk}{2L}}$$

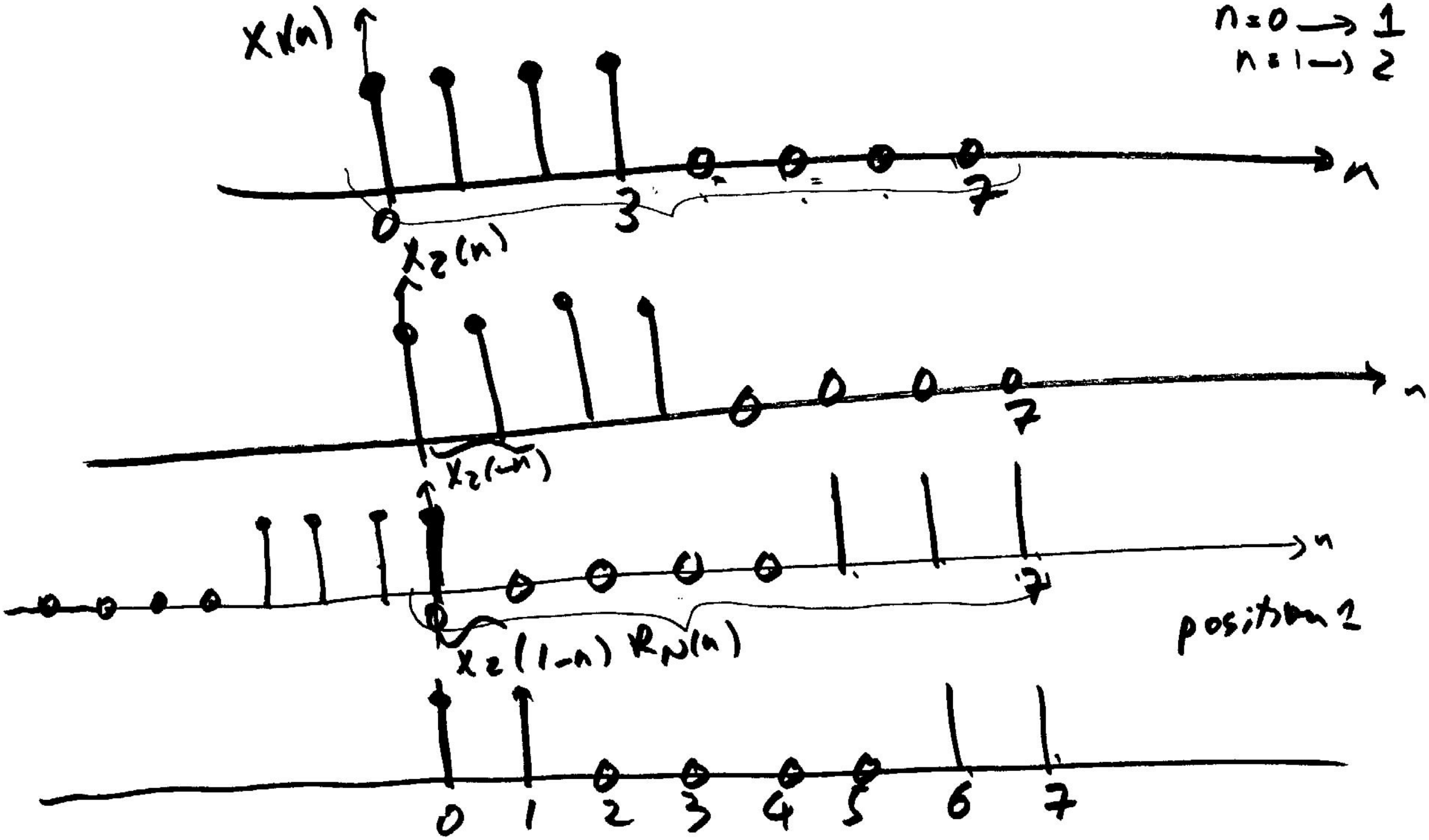
$$= X_{2L}(k)$$



2L pT DFT of x_1

2L pT circ. conv =

$$\mathrm{IDFT}_{2L}\left\{ X_{2L}(k)\, X_{2L}(k) \right\} = x_1 \underset{2L\,pT.}{\circledast} x_2$$



$x(n)$
$n=0 \to 1$
$n=1 \to 2$
n
0
3
7
$x_2(n)$
n
7
$x_2(-n)$
n
7
position 2
$x_2(1-n)\,R_N(n)$
0
1
2
3
4
5
6
7

Goal: prove: padding with enough zeros can convert circ. to linear conv.

$x_1(n) \longrightarrow L$ pt. $\qquad N > L$

$x_2(n) \longrightarrow P$ pt. $\qquad N > P$

goal: $x_3(n) = x_1 * x_2$ = linear convolution.

To find linear convolution.

$$x_3(n) = \sum x_1(m)\, x_2(n-m)$$

$$\text{DTFT}\{x_3(n)\} = X_3(\omega) = \sum_n x_3(n)\, e^{-j\omega n}$$

$$X_3(\omega) = X_1(\omega)\, X_2(\omega)$$

DTFT of x_2

DTFT of x_1

Sample $X_3(\omega)$ at N equally spaced pt.

$= Y(k)$

$$Y(k) = \left[X_3(\omega) \right]_{\omega = \frac{2\pi k}{N}}$$

$$\text{IDFT}\{Y(k)\} = \begin{cases} \sum_{r=-\infty}^{+\infty} X_3(n + rN) & 0 \leq n < N \\ 0 & \text{otherwise} \end{cases}$$

N pt.

Thought exp.

$$Y[k] = \underbrace{\left[X_1(\omega)\right]_{\omega=\frac{2\pi k}{N}}}_{N\text{pt DFT of } x_1} \underbrace{\left[X_2(\omega)\right]_{\omega=\frac{2\pi k}{N}}}_{N\text{pt DFT of } x_2}$$

$$\boxed{\underset{N\text{pt}}{\mathrm{IDFT}}\left[Y(k)\right] = \underbrace{x_1 \circledast x_2}_{N\text{pt circular convolution}}}$$

$$\underbrace{x_1 \circledast x_2}_{N\text{pt circular}} = \begin{cases} \sum_{r=-\infty}^{+\infty} x_3(n+rN) & 0 \le n < N \\ 0 & \text{otherwise} \end{cases}$$

<u>Words</u>: N pT circ conv of x_1 & x_2 is same as than linear conv, x_3, periodicized with period N, Take one period.

$$x_3 \longrightarrow L+P-1$$

$$N \geq L+P-1$$

To ensure no aliasing.

$\Rightarrow$ x_3 i.e. linear convolution can be computed just using circ. conv.

Know DFT can do
circ. convolution

DFT can do linear convolution

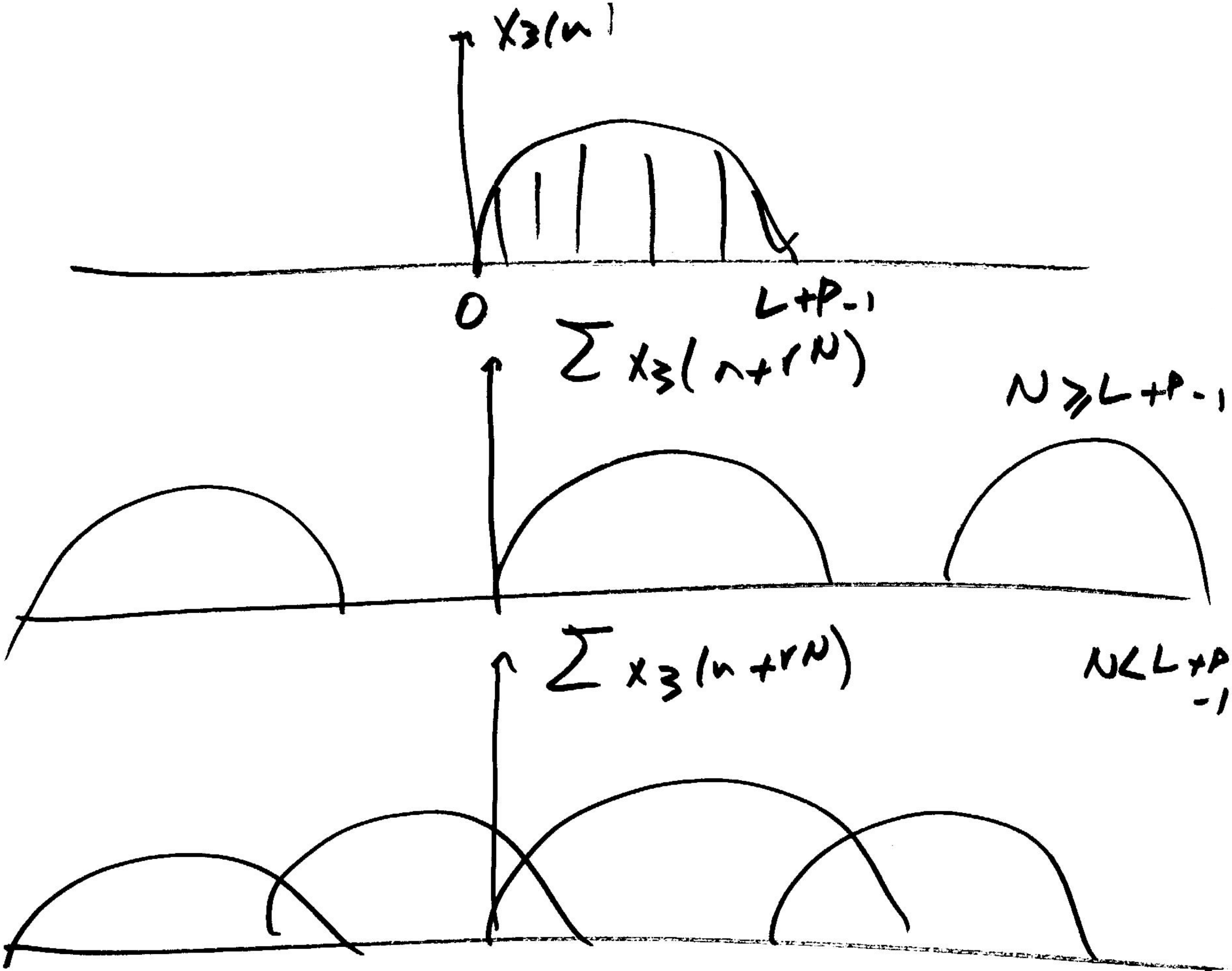
$$N \geq L + P - 1$$

$$x_1 \circledast x_2 = x_3(n)$$

N pt

linear conv.

N pt circ conv of x_1 & x_2 same as linear convolution as long as N larger than the extent of linear convolution



$x_3(n)$
0
$L+P-1$
$\sum x_3(n+rN)$
$N \geq L+P-1$
$\sum x_3(n+rN)$
$N < L+P-1$

x_1 L pts.

x_2 P points.

$L=10$
$P=5$
$N=20$

① Pad x_1 $N-L$ zeros To get N pt.

② Pad x_2 with $N-P$ zeros To get N pt seq.

③ Take N pt DFT of the new seq we get in step 1 & 2.

③ Take IDFT Npt of the seq in step 3 $\longrightarrow$ linear conv $N \geq P+L-1$