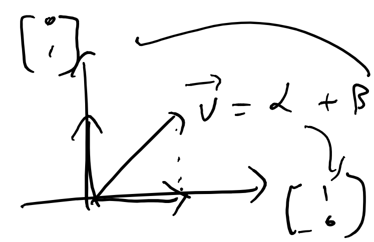
Multi Resolution Expansion

Series Expansion of Signal:

$$f(x) = \sum_{K} \alpha_K \underbrace{\phi_K(x)}_{\text{real valued expansion functions}}$$

α_K = real valued coefficients

$\{\phi_K(x)\}$ is expansion set



If $\exists$ only one set of α_K for any given function $f(x)$, then $\phi_K(x)$ is called the basis function.

- Function Space: $V \triangleq \text{span}\{\phi_K(x)\}$

closed span of expansion set.

If $f(x) \in V \Rightarrow f(x)$ is in closed span of $\{\phi_K(x)\}$.

Can write $f(x) = \sum_k \alpha_k \phi_k(x)$

Dual function $\{ \tilde{\phi}_k(x) \}$ to $\{ \phi_k(x) \}$

$\alpha_k = \langle \tilde{\phi}_k(x), f(x) \rangle$

Def of $\langle \; \rangle$ for f and g:

$\langle f, g \rangle \triangleq \int f^*(x) g(x) dx$

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3 cases:

① Expansion fn forms an orthonormal basis for $\mathcal{V}$:

$$\langle \phi_j(x), \phi_k(x) \rangle = \begin{cases} 0 & \text{if } j \neq k \\ 1 & j = k \end{cases}$$

$\Rightarrow \phi_k(x) = \tilde{\phi}_k(x) \Rightarrow$ basis dual same

$\longrightarrow \alpha_k = \langle \phi_k(x), f(x) \rangle$

② Expansion fn orthogonal But not orthonormal:

$\langle \phi_j(x), \phi_k(x) \rangle = 0 \quad j \neq k$

$$\langle \phi_j(x), \phi_k(x) \rangle = 0 \quad j \neq k$$

$\Rightarrow$ basis fn ϕ and dual $\tilde{\phi}$ are bi-orthogonal

$$\alpha_k = \langle \tilde{\phi}_k(x), f(x) \rangle$$

$$\langle \phi_j(x), \tilde{\phi}_k(x) \rangle = \delta_{jk} = \begin{cases} 0 & j \neq k \\ 1 & j = k \end{cases}$$

(3) more than one set of α_k

$$f(x) = \sum_k \alpha_k \phi_k(x)$$

Exp fn and duals are
"over complete" "Redundant"

Form a "frame"

$$A \, \| f(x) \|^2 \leq \sum_k \left| \langle \phi_k(x), f(x) \rangle \right|^2 \leq B \, \| f(x) \|^2$$

for $A > 0$ $B < \infty$ $\forall f(x) \in V$

If $A = B \Rightarrow$ tight frame:

$$f(x) = \frac{1}{A} \sum_k \langle \phi_k(x), f(x) \rangle \phi_k(x)$$

$f(x) = \frac{1}{A} \sum_{k} \langle \phi_k \ldots$

Scaling Function

- real, square integrable function $\phi(x)$
- Build a set $\phi_{j,k}(x) = 2^{j/2} \phi(2^j x - k)$

$j, k \in \mathbb{Z} \qquad \phi(x) \in L^2(\mathbb{R})$

- If we choose $\phi(x)$ "properly" then $\{\phi_{j,k}(x)\}$ span $L^2(\mathbb{R})$

$$V_j = \overline{\operatorname{Span}_k \{\phi_{j,k}(x)\}}$$

if $f(x) \in V_j \Rightarrow f(x) = \sum_k \alpha_k \phi_{j,k}(x)$

Ex Haar:

$$\phi(x) = \begin{cases} 1 & 0 \le x < 1 \\ 0 & \text{otherwise} \end{cases}$$

See Fig 7.11

Any V_0 expansion ~~s~~ can be decomposed as sum of elements in V_1

If $f(x) \in V_0 \Rightarrow f(x) \in V_1$

all V_0 expansion are contained in V_1

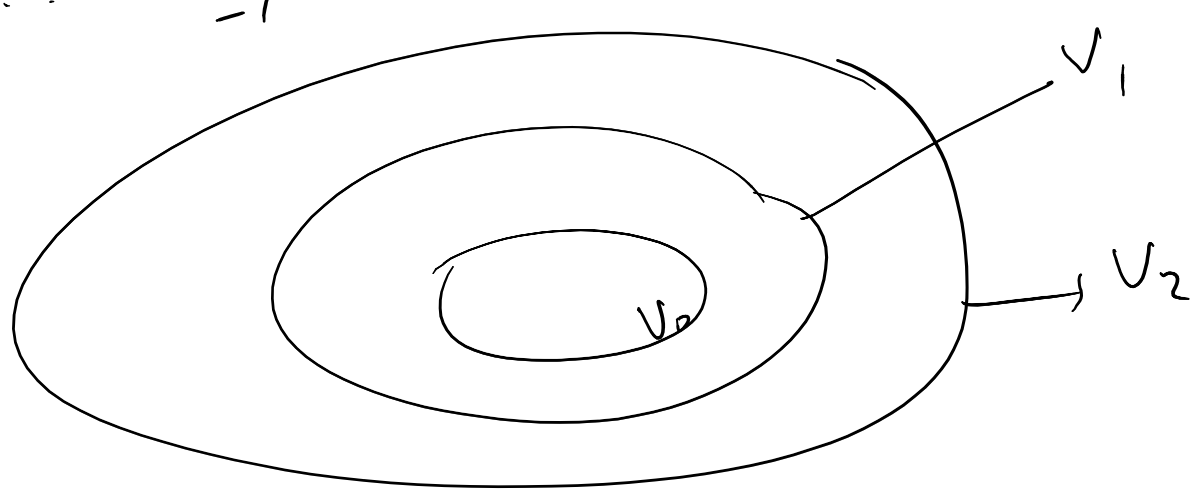
$$V_0 \subset V_1$$

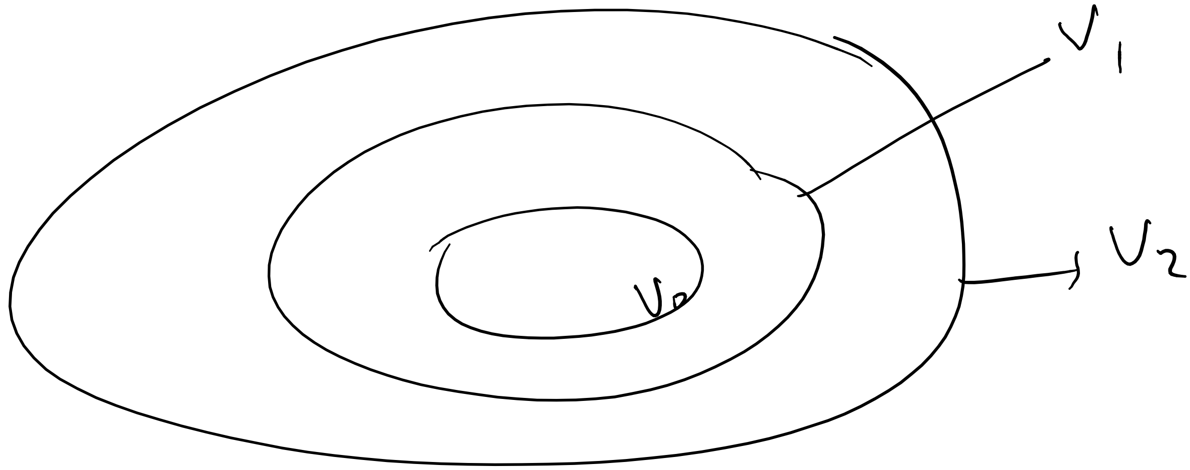
Can show Haar fn obeys 4 requirements of MRA = MultiResolution Analysis

(1) Scaling function $\phi(x)$ is $\perp$ to its integer translates (only for Haar)

(2) subspaces spanned by scaling fn at low scales are nested within those spanned at higher scale.

$$\ldots\ldots V_{-1} \subset V_0 \subset V_1 \subset V_2 \ldots\ldots$$





(3) only fn common to all V_j is

$f(x)=0$ $\qquad V_{-\infty}=\{0\}$

(4) Any fn can be represented with arbitrary precision $\qquad V_{\infty}=\{L^2(\mathbb{R})\}$

Can write ϕ_{jk} as linear combination of $\phi_{j+1,k}$

$$\phi_{jk}(x)=\sum_n \alpha_n \phi_{j+1,n}(x)$$

$$\longrightarrow \quad = \sum_n h_\phi(n)\, 2^{\frac{j+1}{2}}\, \phi(2^{j+1}x-n)$$

Set $j=k=0$ $\qquad \phi_{00}(x)=\phi(x)$

$$\phi(x) = \sum_n h_\phi(n)\sqrt{2}\,\phi(2x-n)$$

- Expand fn of V_j as linear combination of V_{j+1}