Time Frequency Wavelets

- 2D Time - Frequency Representation

$S(t,f)$ of signal $x(t)$

→ Spectral characteristics are Time dependent

- Gabor → S.T.F.T.

$$STFT\{x(t)\} = STFT(\tau, f) = \int x(t)\, g^{*}(t-\tau)\, e^{-j2\pi f t}\, dt$$

- Define rms "bandwidth" of g as:

$$g(t) \longrightarrow G(f)$$

$$\Delta f^2 = \frac{\int f^2 |G(f)|^2 \, df}{\int |G(f)|^2 \, df}$$

energy of g

$\Delta f \simeq$ resolution in frequency of STFT

- Similarly spread in Time Δt

$$\Delta t^2 = \frac{\int t^2 \, |g(t)|^2 \, dt}{}$$

$$\int |g(t)|^2 \, dt$$

Δt : two pulse in Time can be differentiated if more than Δt apart

- Heisberg uncertainty principle :

time-bandwith product $\approx \Delta t \, \Delta f \geq \frac{1}{4\pi}$

- Gaussian windows meet the bound with Equaly.

- Observation : Once $g(t)$ is chosen for S.T.F.T, Time and frequency resolution given by Δt, Δf is $\boxed{\text{FIXED}}$ over the entire T.F. plane.

: Same window is used at ALL frequencies

Heisenberg Box for S.T.F.T.

Assume g is real symetric $g(t) = g(-t)$

$\|g\| = 1$

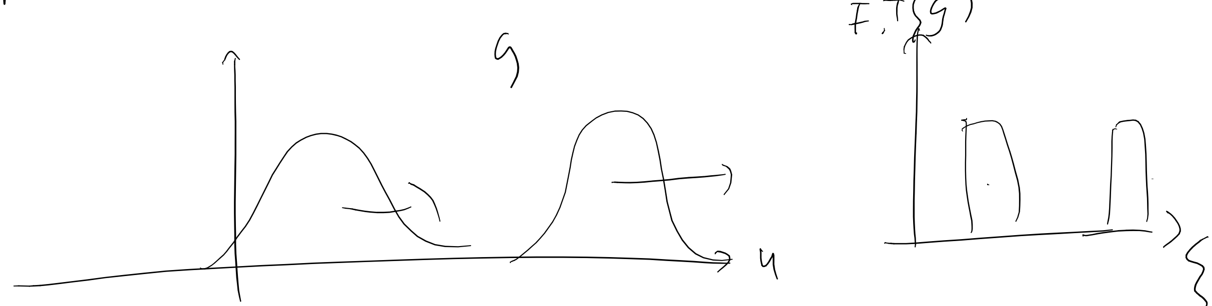
Energy density or Spectrogram

$-j\xi t$ $+\infty$ 2

Energy density

$$P_s f(u, \xi) = \left| \int_{-\infty}^{+\infty} f(t) g(t-u) e^{-j\xi t} dt \right|^2$$

Measures the energy of function f in a Time Freq neighborhood (u, ξ) specified by Heisenberg box of $g_{u,\xi}$



- Time spread around u is indep of u, ξ

$$\sigma_t^2 = \int_{-\infty}^{+\infty} (t-u) |g_{u,\xi}(t)|^2 dt = \int_{-\infty}^{+\infty} t^2 |g(t)|^2 dt$$

$\hat{g}$ = F.T. of g

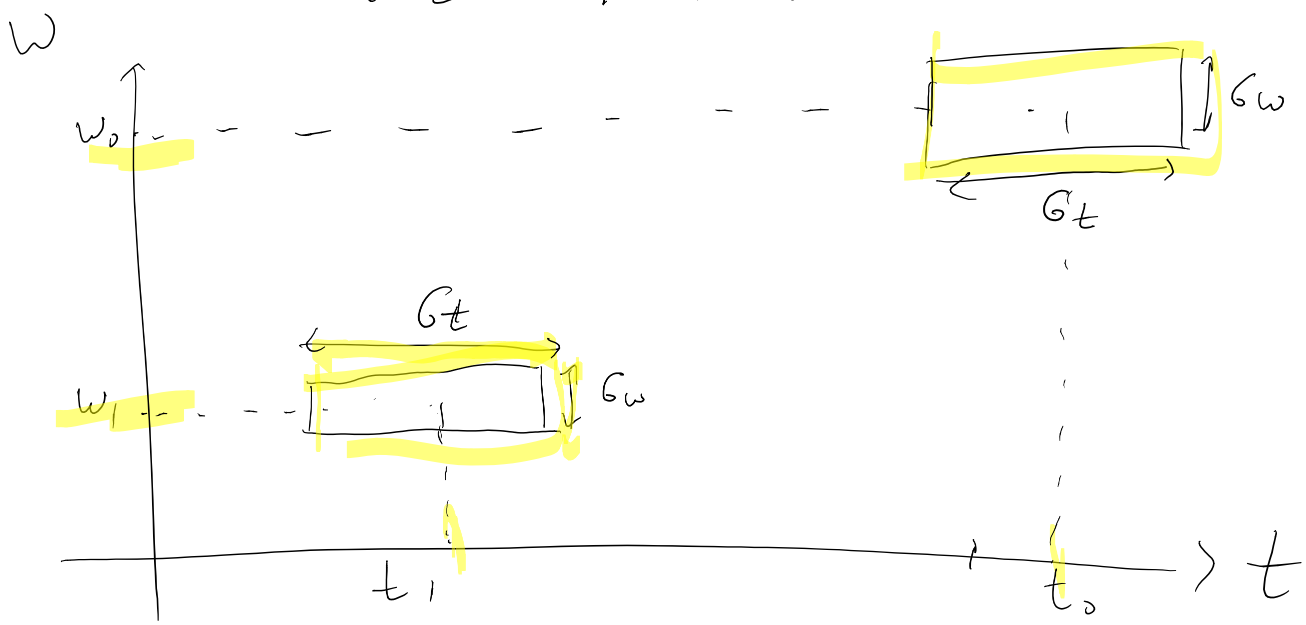
$$\hat{g}_{u,\xi}(\omega) = \hat{g}(\omega - \xi) e^{-ju(\omega - \xi)}$$

Freq spread around ξ is

$$\sigma_\omega^2 = \frac{1}{2\pi} \int (\omega - \xi)^2 |\hat{g}_{u,\xi}(\omega)|^2 d\omega = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \omega^2 |\hat{g}(\omega)|^2 d\omega$$

$g_{u,\xi}$ corresponds To a Heisenberg box

$g_{u,\xi}$ correspond To a Heisberg box of area $\sigma_t \times \sigma_\omega$ centered around (u, ξ)



Continuous Wavelet Transform

- Change the size/support/length of window to analyze signal structure of different size on different "scale". ⟶ Wavelet.
- Wavelet Xform: Decompose signal over dilated and Translated wavelets:
- Wavelet fn: $\psi \in L^2(\mathbb{R})$ ⟶ Continuous differentiable fns

$F.T\{\psi\} = \hat{\psi}$ = bandpass filter $\|\psi\| = 1$

$$\int_{-\infty}^{+\infty} \psi(t)\,dt = 0$$

- Continous Wavelet Transform CWT of $f \in L^2(\mathbb{R})$ at time u, scale s

$$Wf(u,s) = \langle f, \psi_{u,s} \rangle$$

$$= \langle f, \frac{1}{\sqrt{s}} \psi\left(\frac{t-u}{s}\right) \rangle$$

$u \in \mathbb{R}$
$s \in \mathbb{R}^+$

$$= \int_{-\infty}^{+\infty} f(t)\, \psi^*\left(\frac{t-u}{s}\right) dt$$

$$= f * \bar{\psi}_s(u)$$

where $\bar{\psi}_s(u) \triangleq \frac{1}{\sqrt{s}} \psi^*\left(\frac{-t}{s}\right)$

<u>thm</u> Let $\psi \in L^2(\mathbb{R})$ be a real fn. s.t.

$$C_\psi \triangleq \int_0^\infty \frac{|\hat{\psi}(\omega)|}{\omega} d\omega < \infty$$

Then any $f \in L^2(\mathbb{R})$ satisfies

Then any $f \in L^2(\mathbb{R})$

$$f(t) = \frac{1}{C_\psi} \int_0^{\infty} \int_0^{\infty} Wf(u,s) \frac{1}{\sqrt{s}} \psi\left(\frac{t-u}{s}\right) \frac{1}{s^2}\, du\, ds$$

and
$$\int_{-\infty}^{+\infty} |f(t)|^2\, dt = \frac{1}{C_\psi} \int_0^{\infty} \int_{-\infty}^{+\infty} |Wf(u,s)|^2 \frac{1}{s^2}\, du\, ds$$

- $C_\psi < \infty$ admissibility condition.
- If $\hat{\psi}(0) = 0$ is $\hat{\psi}$ is a bandpass function & $\hat{\psi}(\omega)$ is continuously differentiable then admissibility fn is satisfied
- Can show $\hat{\psi}(\omega)$ is continuously differentiable if

$$\int_{-\infty}^{+\infty} (1 + |t|)\, \psi(t)\, dt < \infty$$

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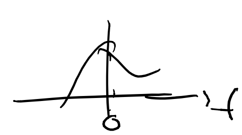
↳ sufficiently time decay

Freq_ time Resolution.

Assume $\psi(t)$ is centered around 0.



Assume $\psi(t)$ is centered around 0. $\to$ ...

$$\psi_{u,s} = \psi\left(\frac{t-u}{s}\right)$$ centered around u

Can show: $$\int_{-\infty}^{+\infty} (t-u)\left|\psi_{u,s}(t)\right| dt = s^2 \sigma_t^2 \leftarrow$$

$$\sigma_t^2 \triangleq \int_{-\infty}^{+\infty} t^2 |\psi(t)|^2 dt$$

Assume $\psi \in L^2(\mathbb{R})$ is analytic i.e.

$$\hat{\psi}(\omega) = 0 \quad \text{if} \quad \omega < 0$$

$\Rightarrow$ "center freq" η of $\psi(\omega)$ is

$$\eta \triangleq \frac{1}{2\pi} \int_0^{\infty} \omega \left|\hat{\psi}(\omega)\right|^2 d\omega$$

$$\hat{\psi}_{u,s}(\omega) = \sqrt{s}\ \hat{\psi}(s\omega)\, e^{-j\omega u} \longrightarrow \text{centerfreq } \frac{\eta}{s}$$

Energy spread of $\hat{\psi}(u,s)$ around $\frac{\eta}{s}$ is

$$\frac{1}{2\pi}\int_0^{\infty} \left(\omega - \frac{\eta}{s}\right)^2 \left|\hat{\psi}_{u,s}(\omega)\right|^2 d\omega = \frac{\sigma_\omega^2}{s^2}$$

$$\sigma_\omega^2 = \frac{1}{2\pi}\int_0^{\infty} (\omega - \eta)\left|\hat{\psi}(\omega)\right|^2 d\omega$$

Combine Energy Spread of wavelet T.F.

$\psi_{u,s}$ concentrates to Heisenberg box centered $\left(u, \frac{\eta}{s}\right)$ of size

$s\,\sigma_t$ along time and $\frac{\sigma_\omega}{s}$ along freq