DTFT

$x(n) \xrightarrow{DTFT} X(\omega) = \sum_{n=-\infty}^{+\infty} x(n) e^{-j\omega n}$

n integ

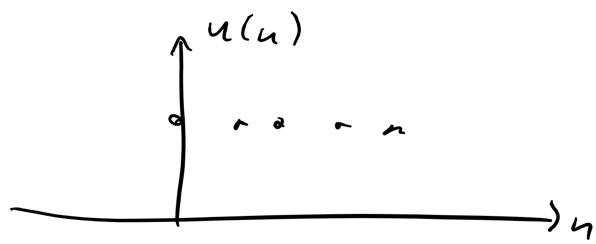
Continuous Real valued

If $x(n)$ is absolutely summable

$\sum_n |x(n)| < \infty$ Then F.T. exists

and converge To a continuous fn.

Ex $x(n) = a^n u(n)$

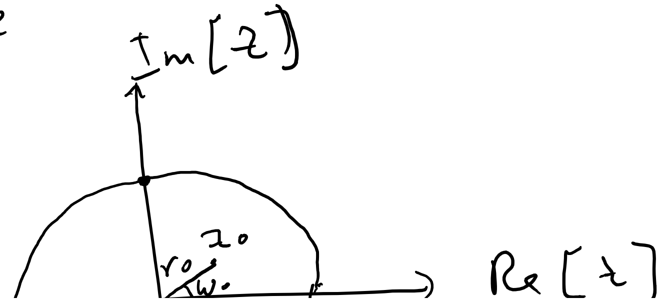


$x(n) \longleftrightarrow \mathcal{Z}\{x(n)\} \triangleq X(z) = \sum_{n=-\infty}^{+\infty} x(n) z^{-n}$

$X(\omega) = \sum_n x(n) e^{-j\omega n}$

$X(\omega) = \left[X(z)\right]_{z=e^{j\omega}}$

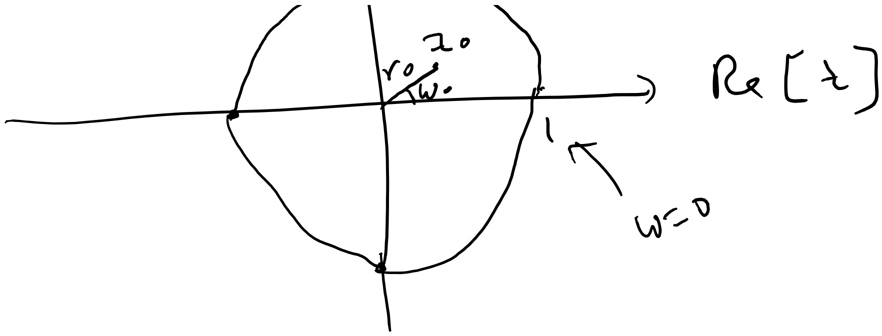
$z_0 = r_0 e^{j\omega_0}$



$z_0 = r_0 e$

$|z_0| = r_0$

$\measuredangle z_0 = w_0$



What values of z does $X(z)$ converge?

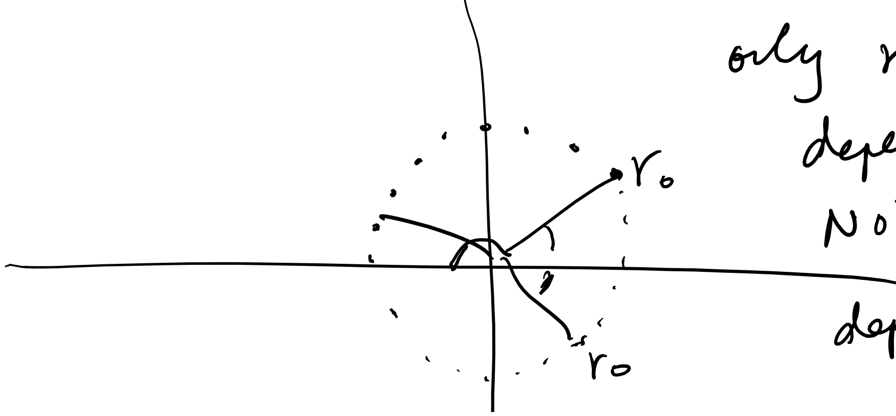
$$X(z) = \sum_{n=-\infty}^{+\infty} x(n) \left(r e^{jw} \right)^{-n}$$

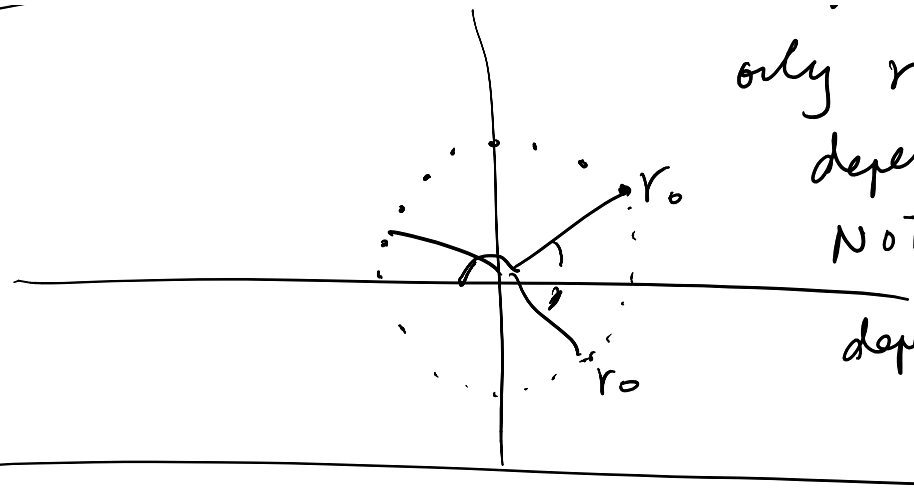
$$X(z) = \sum_{n=-\infty}^{+\infty} \underbrace{\left(x(n) r^{-n} \right)}_{\text{DTFT of } x(n) r^{-n}} e^{-jwn}$$

$X(z)$ converges if $x(n) r^{-n}$ is absolutely summable i.e

$$\sum_n \left| x(n) r^{-n} \right| < \infty$$

ROC Z. Transform is only r dependent NOT w dependent





only r dependent
NOT ω dependent

Possibilities for Region of Convergence ROC

- Inside of some circle
- outside " " "
- Between 2 circles.
- point
- at ∞

$X(z)$ by itself does not uniquely specify a sequence

$X(z) +$ ROC $\longleftrightarrow x(n)$

$X(z) +$ ROC 1 $\longleftrightarrow x_1(n)$

$X(z) +$ ROC 2 $\longleftrightarrow x_2(n)$

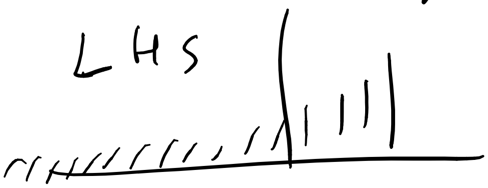
Example $X(z) = \dfrac{1}{1 - az^{-1}}$

ROC 1 → LHS
Left handed

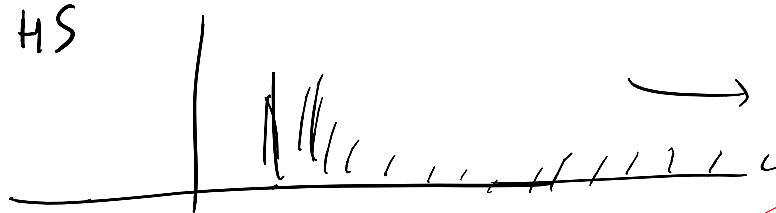
ROC 1 $\rightarrow$ LHS

Left handed sequen

LHS



ROC 2 $\rightarrow$ RHS

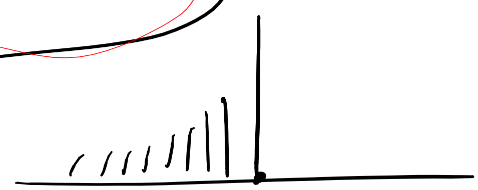


$x(n) = a^n u(n)$ RHS

$$X(z) = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} (a z^{-1})^n = \frac{1}{1 - a z^{-1}}$$

if $|a z^{-1}| < 1 \Rightarrow |z| > |a| \Rightarrow$ ROC2

Ex $x(n) = -a^n u(-n-1)$ Left hand Seq

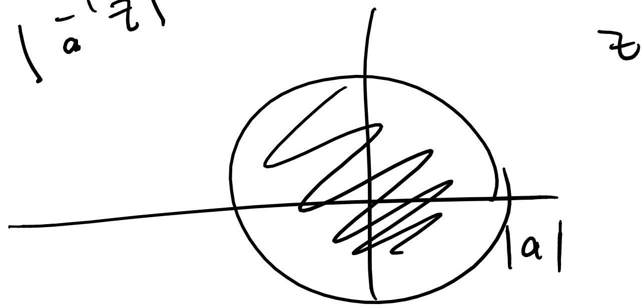


$$X(z) = \sum_{n=-\infty}^{-1} a^n z^{-n} = 1 - \sum_{m=0}^{\infty} (a^{-1} z)^m$$

change of vrial $m = -n$

Converge $|a^{-1} z| < 1 \Rightarrow |z| < |a|$

Con... $|a^{-1}z| < 1$



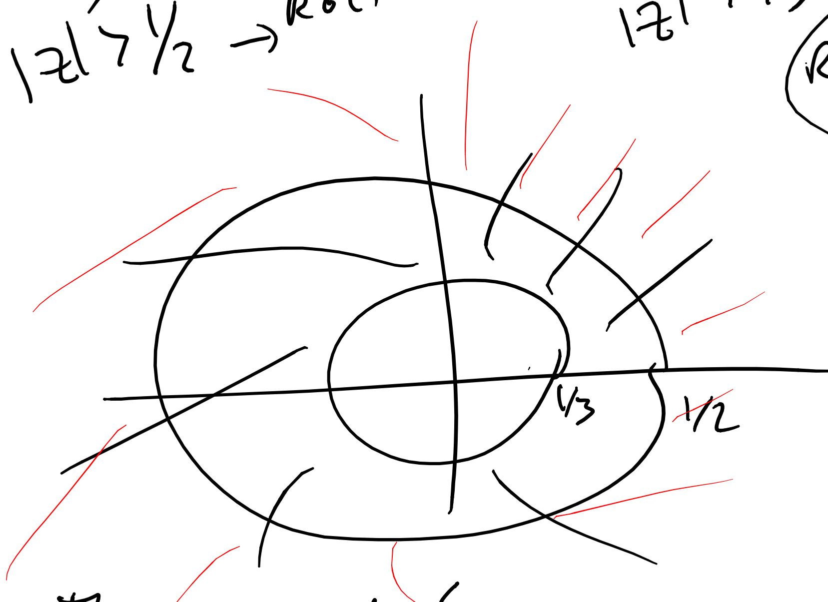
Ex

Ex $x(n) = \underbrace{\left(\frac{1}{2}\right)^n u(n)}_{} + \underbrace{\left(-\frac{1}{3}\right)^n u(n)}_{}$

converge $|z| > 1/2 \rightarrow$ ROC1

$|z| > 1/3 \rightarrow$ ROC2

ROC $|z| > 1/2$



Intersect ROC1 and ROC2

Ex Both Handed Seq.

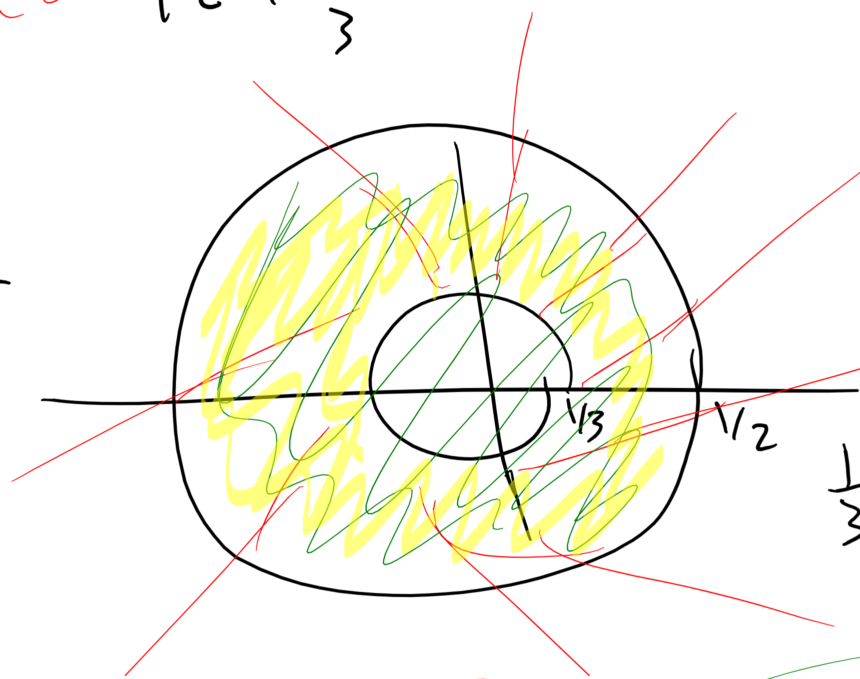
$x(n) = \left(-\frac{1}{3}\right)^n u(n) - \left(\frac{1}{2}\right)^n u(-n-1)$

ROC1 $|z| > \frac{1}{3}$

$|z| < \frac{1}{2}$ ROC2

ROC1 $|z| > \frac{1}{3}$ $\qquad$ $|z| < \frac{1}{2}$ ROC2

Ring



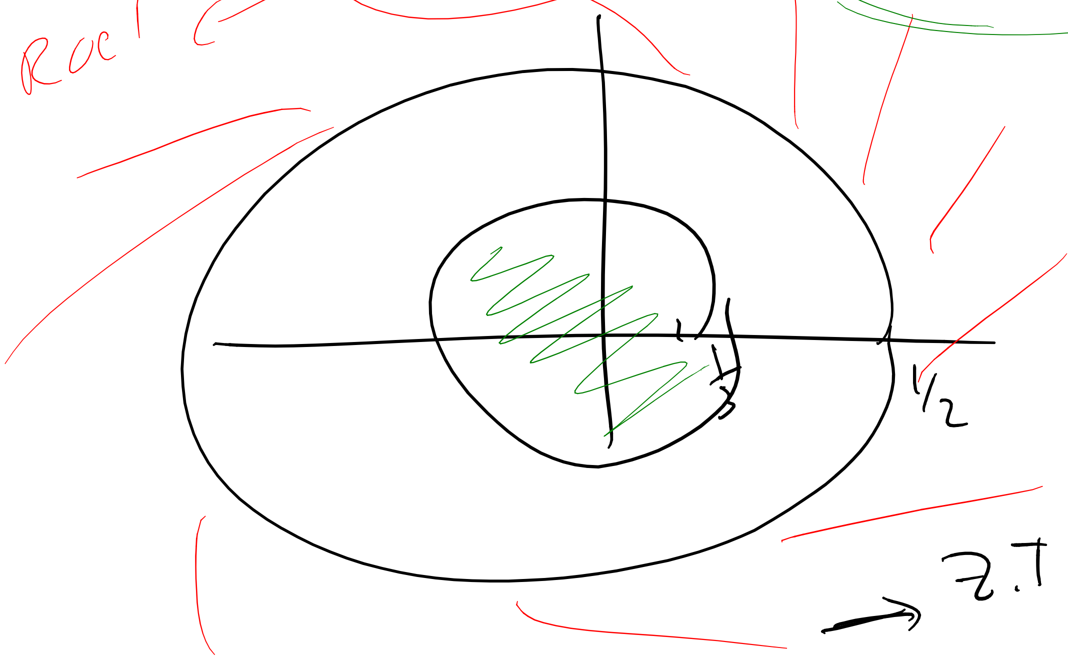
ROC is a ring

ROC1 $\cap$ ROC2

$\frac{1}{3} < |z| < \frac{1}{2}$

$$x(n) = \left(-\frac{1}{2}\right)^n u(n) - \left(\frac{1}{3}\right)^n u(-n-1)$$

ROC1 $\qquad$ ROC2



$\Rightarrow$ there is no ROC for

$\rightarrow$ Z.T. Does not exist

Ex $\quad x(n)$ Finite length seq.

$$X(z) = \sum^{N} x(n) z^{-n} = x(0) z^{0} + x(1) z^{-1} + \dots$$

$$X(z) = \sum_{n=M}^{N} x(n) z^{-n} = x(0) z^{0} + x(1) z^{-1} + x(-1) z^{1} + x(-2) z^{2} + \cdots$$

Converge everywhere
except at zero and ∞

Ex $x(n) = \begin{cases} a^n & 0 \leq n \leq N-1 \\ 0 & \text{otherwise} \end{cases}$

$$X(z) = \sum_{n=0}^{N-1} x(n) z^{-n} = \frac{1 - (a z^{-1})^N}{1 - a z^{-1}}$$

$$= 1 + a z^{-1} + a^2 z^{-2} + \cdots + a^{N-1} z^{-N+1}$$

$$= 1 + \frac{a}{z} + \frac{a^2}{z^2} + \cdots$$

ROC : everywhere except for $z = 0$

Ex $x(n) = \begin{cases} a^n & -N \leq n \leq 0 \\ 0 & \text{elsewhere} \end{cases}$

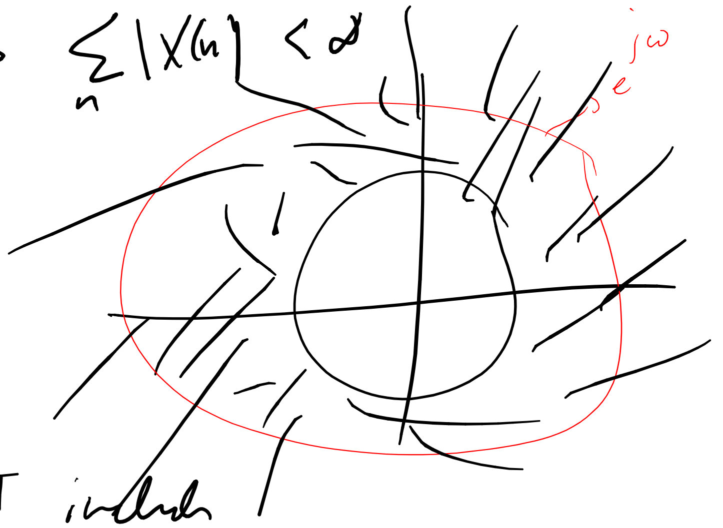
$$X(z) = \sum_{n=0}^{-N} a^n z^{-n} = 1 + a^{-1} z + a^{-2} z^{2} + \cdots$$

ROC : everywhere except $z = \infty$

Properties of ROC:

1. ROC is a fn of r and not ω
2. F.T. exists $\sum_n |x(n)| < \infty$

   if ROC include unit circle.



3. ROC cannot include a pole

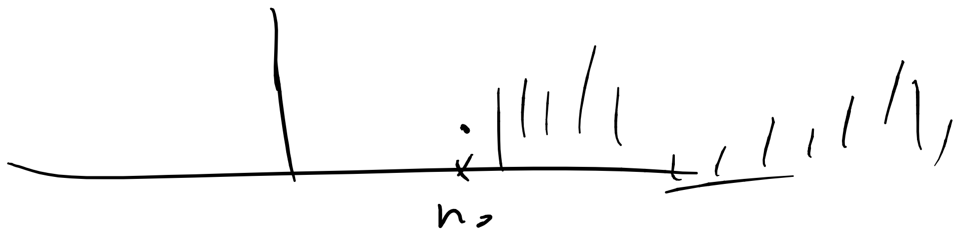
   $$\frac{1}{1-\frac{1}{2}z}$$

4. Finite length seq.

   ROC everywhere except for $z=0$ or $z \to \infty$

Show: If seq RHS $\longrightarrow$ ROC outside of some circle.

- $x(n)$ RHS $\rightarrow$ non zero for some n_0 and large indices than n_0



- ROC outside of some circle $\Longleftrightarrow$

If $X(z)$ converge for r_0 it also converge for $\forall\, r > r_0$

Assume $X(z)$ converge for some r_0.
Show it also converge for $r > r_0$

$$\Rightarrow \sum_{n=-\infty}^{\infty} \left| x(n)\, r_0^{-n} \right| < \infty \qquad \checkmark \text{ True}$$

Invoke RHS

$$= \sum_{n=n_0}^{\infty} \left| x(n)\, r_0^{-n} \right| < \infty \qquad \checkmark \text{ True.}$$

Show if $r > r_0$ show: $\displaystyle\sum_{n=n_0}^{\infty} \left| x(n) r^{-n} \right| < \infty$

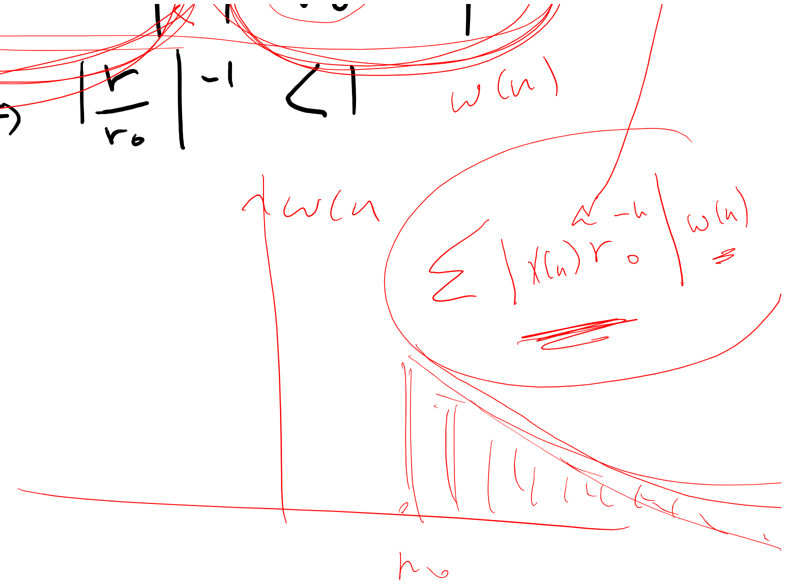
want to show

$$\left| x(n) r^{-n} \right| = \left| x(n)\, r_0^{-n} \left(\frac{r}{r_0}\right)^{-n} \right|$$

$$= \left| x(n)\, r_0^{-n} \right| \left| \left(\frac{r}{r_0}\right)^{-n} \right|$$

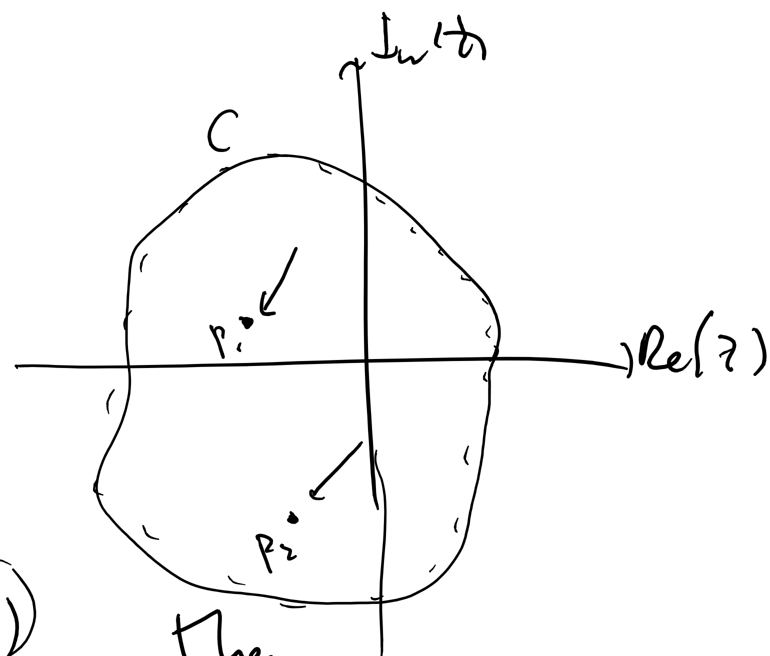
know $n > n_0$ $\quad |r_0| < 1 \quad |r|^{-1} < 1 \quad x(n)$

know
$r > r_0 \Rightarrow \left|\frac{r_0}{r}\right| < 1 \Rightarrow \left|\frac{r}{r_0}\right|^{-1} < 1$



Cauchy's Residue Theorem

$$\frac{1}{2\pi j}\oint_C F(z)\,dz = \sum_{\text{poles}} \text{Residue of } F(z) \text{ at each pole inside contour } C$$



If $F(z) = \frac{\phi(z)}{(z-z_0)^s}$ then

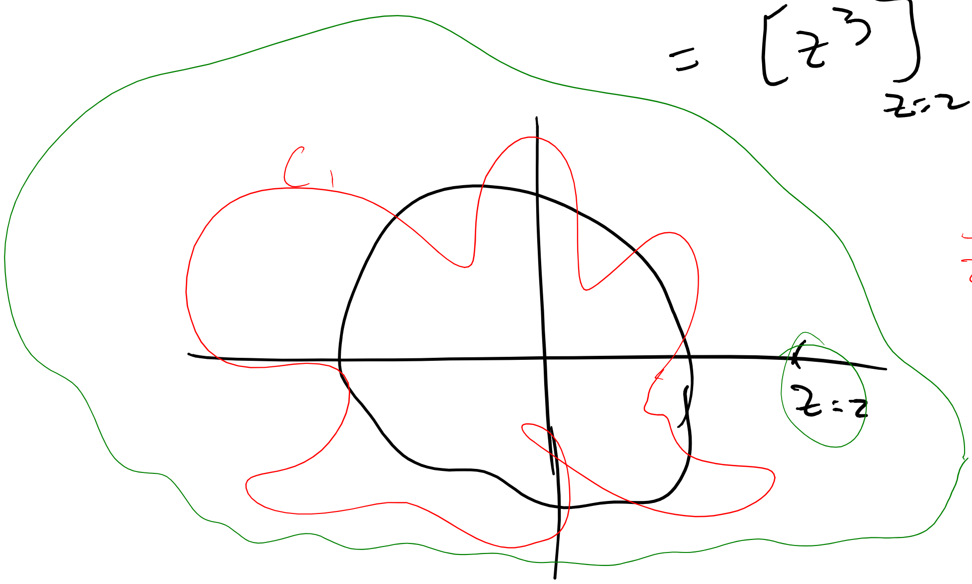
$$\text{Residue}\left[F(z)\right]_{z=z_0} \triangleq \frac{1}{(s-1)!}\left[\frac{d^{s-1}\phi(z)}{dz^{s-1}}\right]_{z=z_0}$$

Ex $F(z) = \dfrac{1}{(z-2)^2}$ $\qquad \phi(z) = 1$, $s = 2$, $z_0 = 2$

$$\text{Residue}\left[F(z)\right]_{z=2} = \frac{1}{(2-1)!}\,\frac{d}{dz}\left[1\right] = 0$$

Ex $F(z) = \dfrac{z^3}{z-2}$ $\qquad \phi(z) = z^3$, $z_0 = 2$, $s = 1$

$$\text{Residue}\left[F(z)\right]_{z=2} = \frac{1}{1}\left[\phi(z)\right]_{z=z_0} = \left[z^3\right]_{z=2} = 8$$



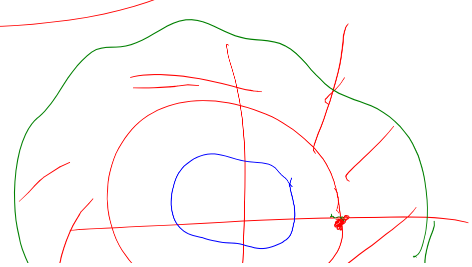
$$\frac{1}{2\pi j}\oint_{C_1} F(z) = 0$$

$$\frac{1}{2\pi j}\oint_{C_2} F(z) = 8$$

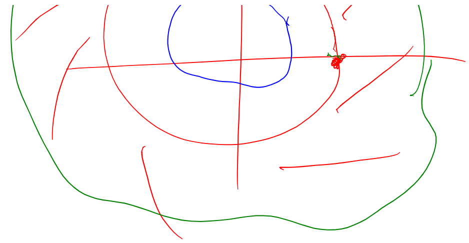
Inverse Z Transform:

$$X(n) = \frac{1}{2\pi j}\oint_C X(z)\, z^{n-1}\, dz$$

C: closed contour in ROC of



contour in ROC of $X(z)\, z^{n-1}$



Observation

$$\frac{1}{2\pi j} \oint_C z^n \, dz = \begin{cases} 1 & n=-1 \\ 0 & \text{otherwise} \end{cases}$$

C: any closed contour encircling origin

(1) $n \geq 0 \rightarrow z^n$ has no poles inside C

$$\Rightarrow \frac{1}{2\pi j} \oint z^n \, dz = 0$$

(2) $n = -2$ $\quad \frac{1}{2\pi j} \oint z^{-2} \, dz$

pole $z=0$, order $2 = s$ $\quad \phi(z) = 1$

$$\text{Res}[z=0] = \frac{1}{(s-1)!} \left[\frac{d}{dz} \phi(z) \right] = 0$$

Same argument $n = -3, -4, \ldots$

(3) $n=-1$ simple pole, order 1 @ $z=0$

$$\frac{1}{2\pi j} \oint z^{-1} \, dz =$$

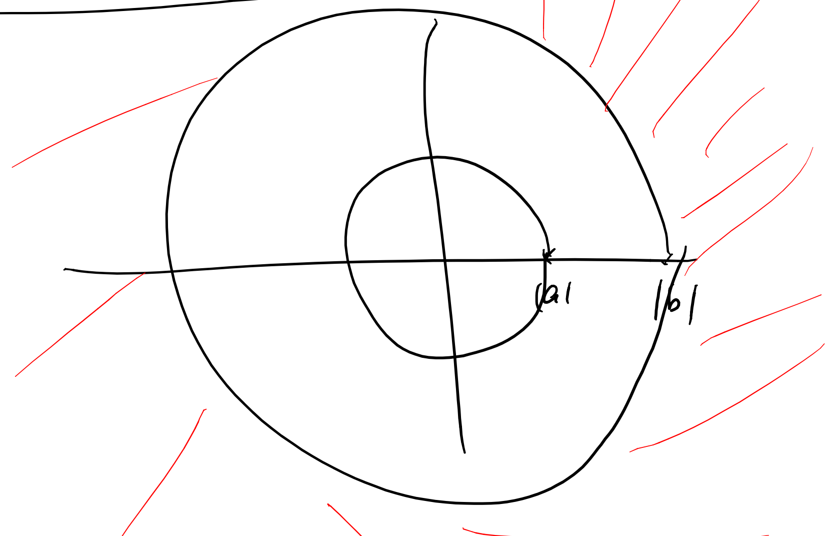
$$\text{Res}[z=0] = \frac{1}{(1-1)!} \left[\phi(z) \right]_{z=0} = 1$$

$$\text{Res}[z=0] = \frac{1}{(1-1)!} \Big|_{z=0}$$

$$\Longrightarrow \frac{1}{2\pi j} \oint_C z^n \, dz = \delta(n+1)$$

Ex $X(z) = \dfrac{1}{(1-az^{-1})(1-bz^{-1})}$ $\quad |b| > |a|$

ROC : $|z| > |b|$



$$x(n) = \frac{1}{2\pi j} \oint \underbrace{\frac{z^{n+1}}{(z-a)(z-b)}}_{F(z)} dz$$

Consider:

① $n \geq -1 \Longrightarrow$ 2 ploes $z=a, z=b$

② $n < -1 \Longrightarrow$ Repeated poles at the origin and $z=a, z=b$

① $n \geq -1$

$$x(n) = \text{Res}[F(z)]_{z=a} + \text{Res}[F(z)]_{z=b}$$

$+$

$z=a$

$$= \qquad + $$

$$= \left[\frac{z^{n+1}}{(z-b)}\right]_{z=a} + \left[\frac{z^{n+1}}{(z-a)}\right]_{z=b}$$

② $n < -1$ $\qquad n = -2, -3 \ldots$

$n=-2 \longrightarrow z=0, \quad z=a, \quad z=b$

$n=-3 \longrightarrow$ double pole at $z=0 \rightarrow s=2$

$z=a, \quad z=b$

$n=-4 \longrightarrow$ triple pole $z=0 \quad s=3$

$z=a, \quad z=b.$

$n=-2$

$$x(n) = \left[\frac{1}{(z-a)(z-b)}\right]_{z=0} + \left[\frac{1}{z(z-b)}\right]_{z=a} + \left[\frac{1}{(z)(z-a)}\right]_{z=b}$$

$$= \ldots\ldots = 0$$

$n=-3, -4, -5 \ldots \implies x(n)=0$

$$x(n) = u(n)\left(\frac{a^{n+1}}{a-b} + \frac{b^{n+1}}{b-a}\right)$$

R.H.S.

$$X(z) = \frac{1}{(1-az^{-1})(1-bz^{-1})} = \frac{A}{\;} + \frac{B}{\;} = \frac{P(z)}{\;}$$

$$= \frac{A}{(1-az^{-1})} + \frac{B}{(1-bz^{-1})} = \frac{P(z)}{Q(z)}$$

Partial Fraction Expansion

Initial Value Theorem

If $x(n)=0$ for $n<0$ Then

$$x(0) = \lim_{z\to\infty} X(z)$$

$$\lim_{z\to\infty} X(z) = \lim_{z\to\infty}\left[\sum_{n=0}^{+\infty} x(n) z^{-n}\right]$$

$$= \lim_{z\to\infty}\left[x(0) + \cancel{x(1)z^{-1}} + \cancel{x(2)z^{-2}} + \cdots\right]$$

$$= x[0]$$

Methods for Inverse Z.T.

1. Cauchy Residue Th
2. Partial Fraction Exp
3. Direct way: inspect.

$$X(z) = 5 + 20z^{6}$$

$$X(z) = \sum x(n) z^{-n} = x(0) + x(1)z^{-1} + \cdots$$

$$X(z) = \sum_n x(n) z^{-n} = \overline{x(0) + x(1) z^{-1} +} \\ \underline{x(2) z^{-2} + \cdots\cdots}$$

$$x(n) = \begin{cases} 5 & n = 0 \\ 20 & n = 6 \end{cases}$$

(4) Table look up

(5) Use properties.

(6) Series Expansion.

$$X(z) = \log(1 + a z^{-1}) \qquad ROC \ |a z^{-1}| < 1$$

$$\log(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

$|x| < 1$

$$\Rightarrow \log(1 + a z^{-1}) = a z^{-1} - \frac{(a z^{-1})^2}{2} + \frac{(a z^{-1})^3}{3} - \frac{(a z^{-1})^4}{4} + \cdots$$

$$X(z) = \sum_n x(n) z^{-n} = \cdots + x(-2) z^2 + x(-1) z + x(0) + x(1) z^{-1} + x(2) z^{-2} + x(3) z^{-3} + \cdots$$

$$x(2) = -\frac{a^2}{2} \qquad x(1) = a$$

$$x(3) = \frac{a^3}{3}$$

$$x(n) = \begin{cases} 0 & n \le 0 \\ a & n = 1 \\ -\frac{a^2}{2} & n = 2 \\ \frac{a^3}{3} & n = 3 \end{cases}$$

$$x(n) = \underline{u(n-1)} \frac{a^n}{n} \underline{(-1)^{n+1}}$$

Convergence Issue

$$\text{D.T.F.T.}\{x(n)\}$$

$$\sum_{n=-\infty}^{+\infty} x(n)\, e^{-j\omega n}$$

Q Does it converge? if yes To what?
$\longrightarrow Y(\omega)$.

Q If converged To $Y(\omega)$, plug into inverse

$$\int Y(\omega)\, e^{j\omega n}\, d\omega \overset{???}{=} x(n)$$

Def Uniform Convergence.

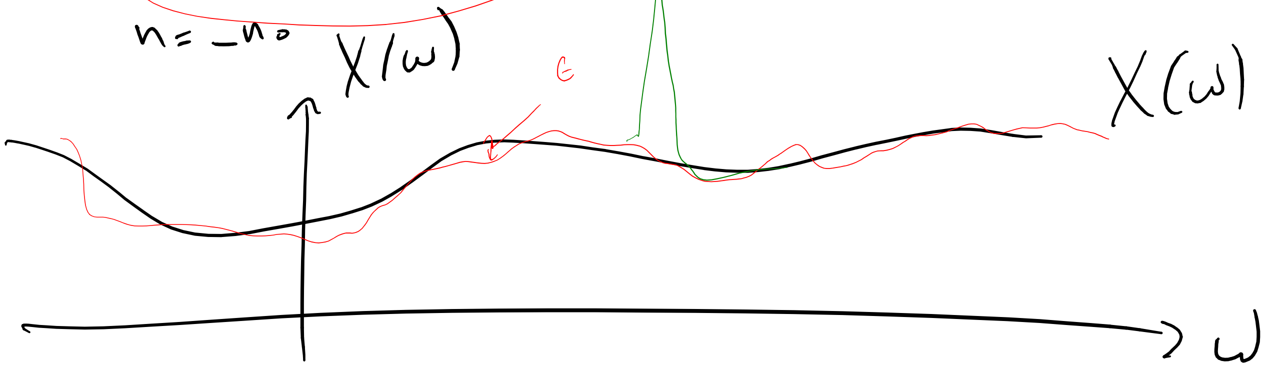
$\sum_n x(n) e^{-j\omega n}$ converges uniformly if

$\sum_n x(n)e$ ...

$\exists$ a continuous function of ω called $X(\omega)$ s.t.

$\forall\ \epsilon > 0,\ \omega\ \ \exists\ n_0$ s.t.

$$\left| \sum_{n=-n_0}^{+n_0} x(n) e^{-j\omega n} - X(\omega) \right| < \epsilon$$

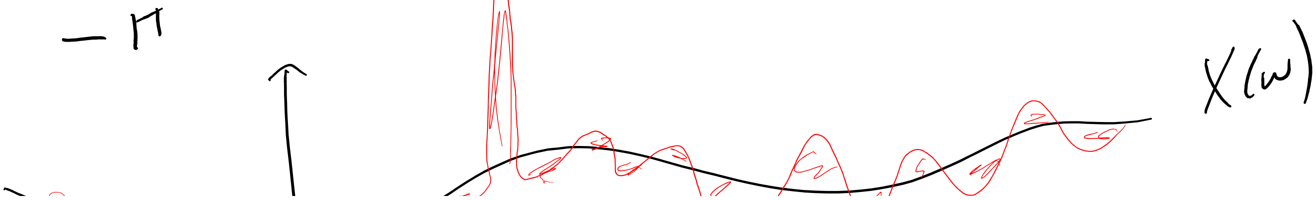


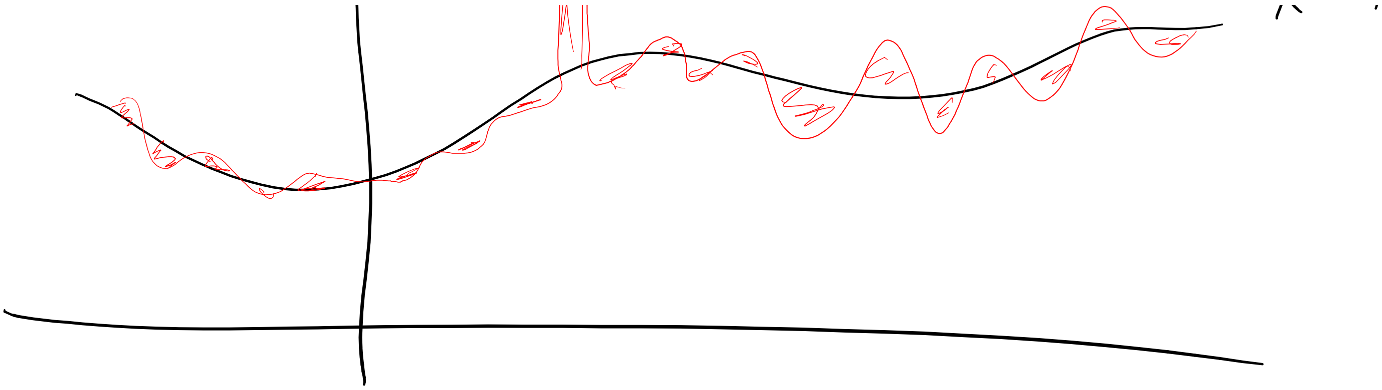
<u>MSE</u> Convergence (MSE)

$\sum_n x(n) e^{-j\omega n}$ converges in MSE sense to a known given function $X(\omega)$ if

$\forall \epsilon\ ,\ \exists n_0$ s.t.

$$\int_{-\pi}^{+\pi} \left| \sum_{n=-n_0}^{+n_0} x(n) e^{-j\omega n} - X(\omega) \right|^2 d\omega < \epsilon$$





① If Seq $x(n)$ is absolutely sumable

$\sum_n |x(n)| < \infty \Longrightarrow$ Then $\sum_n x(n) e^{-j\omega n}$

Converges uniformly $\Longrightarrow$ D.T.F.T exist and

D.TFT is a continuous function

AND whatever it converges to $\sum_n x(n) e^{-j\omega n}$

if I plug into $\int X(\omega) e^{j\omega n} d\omega$ get $x(n)$ back

Ex $\left(\frac{1}{2}\right)^n u(n) = x(n) \longrightarrow \dfrac{1}{1-\frac{1}{2} e^{-j\omega}}$

$2^n u(n)$ $x(n) \longrightarrow$ F.T doesn't exist

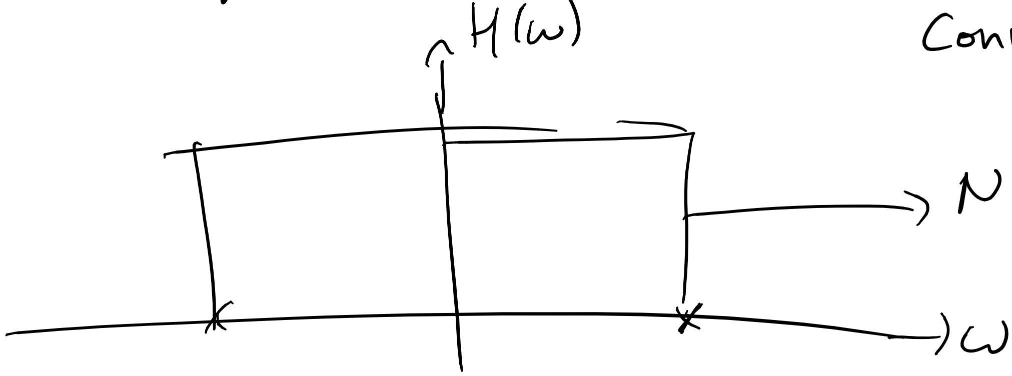
---

② If $x(n)$ is Square sumable

$\sum x^2(n) < \infty$ $\quad \sum x(n) e^{-j\omega n}$

$$\sum_n x^2(n) < \infty \longrightarrow \sum_n x(n) e^{-j\omega n}$$

Converges in mse sense



Not a continuous fn

$$\int_{-\pi}^{+\pi} H(\omega) e^{j\omega n} \, d(\omega) = \frac{\sin \omega_c n}{\pi n} = h(n)$$

$\sum_n \frac{1}{n}$ diverges so does $\sum \left| \frac{\sin \omega_c n}{\pi n} \right|$ diverge

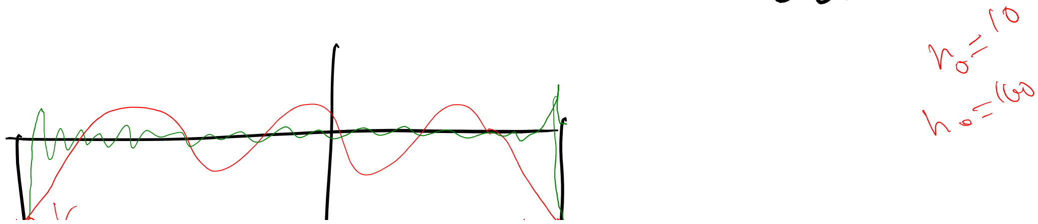
$h(n)$ is not absolutely summable

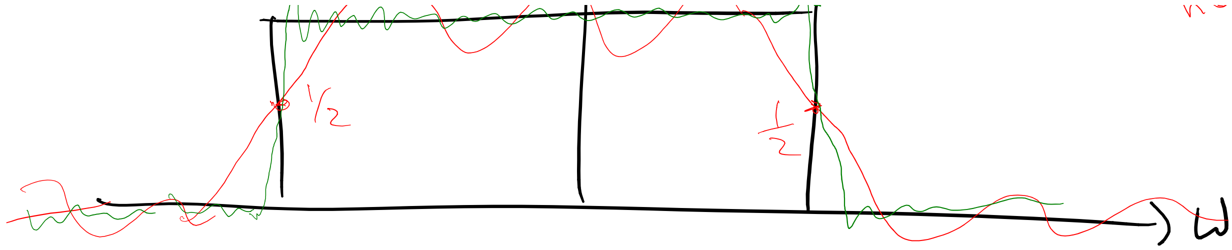
However $h(n)$ square sum

$$\sum_n \frac{\sin^2 \omega_c n}{\pi^2 n^2}$$ converges because $\sum \frac{1}{n^2}$ converges

$$\sum_{-n_0}^{+n_0} h(n) e^{-j\omega n}$$ converges in a mse sense.

At $\omega = \omega_c$ $\sum h(n) e^{-j\omega n}$ converges $\rightarrow \frac{1}{2}$ then.





Discrete Fourier Series

$x(t)$ — real, continuous $\xrightarrow{\text{C.T.F.T.}}$ $X(\Omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\Omega t}\, dt$ — Ω real

$x(n)$ — int., sequence, ∞ or finite $\xrightarrow{\text{D.T.F.T}}$ $X(\omega) = \sum_{n} x(n)\, e^{-j\omega n}$ — ω real

$\xrightarrow{\text{Z.T.}}$ $X(z) = \sum_{n} x(n)\, z^{-n}$ — z complex

$x(n)$ — ∞ long / finite, int

$\tilde{x}(n)$ — periodic, int $\xrightarrow{\text{D.F Series}}$ $\tilde{X}(k) = \sum_{n \text{ over one period}} \tilde{x}(n)\, e^{-j\frac{2\pi nk}{N}}$ — k int

$x(n)$ — N finite duration, int $\xrightarrow{\text{D.F. Transform}}$ $X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi nk}{N}}$

DFS = Discrete Fourier Series.

Discrete Time signal with period N

DFS = ...

$\tilde{x}(n)$ periodic discrete time signal with period N

$$\tilde{x}(n) = \tilde{x}(n + kN)$$

$k \rightarrow$ any integer

$$e_k(n) = e^{j\frac{2\pi nk}{N}} \quad k = 0, \dots N-1$$

$$\tilde{x}(n) = \frac{1}{N} \sum_{l=0}^{N-1} \tilde{X}(k) \, e^{j\frac{2\pi nk}{N}}$$

Decompose $\tilde{x}(n)$ along exponentials $e_k(n)$

Show: $e_k(n)$ is periodic:

$$e_k(n) = e_{k+rN}(n)$$

$r \rightarrow$ arbitrary int.

$$e_0(n) = e_N(n) = e_{2N}(n) = \dots$$

$$e_1(n) = e_{N+1}(n) = e_{2N+1}(n) = \dots$$

Proposal: $\tilde{X}(k) \overset{??}{=} \sum_{n=0}^{N-1} \tilde{x}(n) \, e^{-j\frac{2\pi nk}{N}}$ $\leftarrow$ RHS

Proof.

$$\text{RHS} = \sum_{n=0}^{N-1} \left(\frac{1}{N} \sum_{l=0}^{N-1} \tilde{X}(l) \, e^{j\frac{2\pi nl}{N}} \right) e^{-j\frac{2\pi nk}{N}}$$

$$= \sum_{l=0}^{N-1} \tilde{X}(l) \underbrace{\left[\frac{1}{N} \sum_{n=0}^{N-1} e^{j\frac{2\pi (l-k)}{N}} \right]}_{A = \delta(l-k-rN)}$$

$r \leftarrow$ any r

$\tilde{X}(k+rN)$ $\quad \tilde{X}(k) = \tilde{X}(k+N)$

$= \tilde{X}(k+rN) = \tilde{X}(k) = \tilde{X}(k+N)$

$= \tilde{X}(k+2N) = \tilde{X}(k+3N)$

$= LHS$

Show A: Case 1: $l-k = rN$ is an integer multiple of N

$$A = \frac{1}{N}\sum_{n=0}^{N-1} e^{j\frac{2\pi rN}{N}} = 1$$

Case 2: $l-k \neq rN$

$$A = \frac{1}{N}\sum_{n=0}^{N-1} e^{j\frac{2\pi(l-k)n}{N}}$$

Recall: $$\sum_{n=0}^{N-1} \alpha^n = \frac{1-\alpha^N}{1-\alpha}$$ ← math ....

$$A = \frac{1}{N}\,\frac{1-e^{j\frac{2\pi(l-k)N}{N}}}{1-e^{j\frac{2\pi(l-k)}{N}}} = 0$$

$$A = \delta(l-k-rN)$$

$$\tilde{x}(n) = \sum_{k=0}^{N-1} \tilde{X}(k)\, e^{j\frac{2\pi nk}{N}}$$ ← Synthesis

$$\tilde{X}(k) = \frac{1}{N}\sum_{n=0}^{N-1} \tilde{x}(n)\, e^{-j\frac{2\pi nk}{N}}$$ ← analysis

DFS Pair

DFS Pair

analogy

periodic seq $\tilde{x}(n)$ $\longleftrightarrow$ $\tilde{X}(k)$ periodic N point in freq. domain.

Shift Property.

$$\tilde{x}(n) \longleftrightarrow \tilde{X}(k)$$

$$\tilde{x}(n-n_0) \longleftrightarrow e^{-j\frac{2\pi n_0 k}{N}} \tilde{X}(k)$$

Period Convolution:

$$\tilde{x}_1 \circledast \tilde{x}_2 = \tilde{x}_3 \longrightarrow \text{periodic } N.$$

period. N — periodic N.

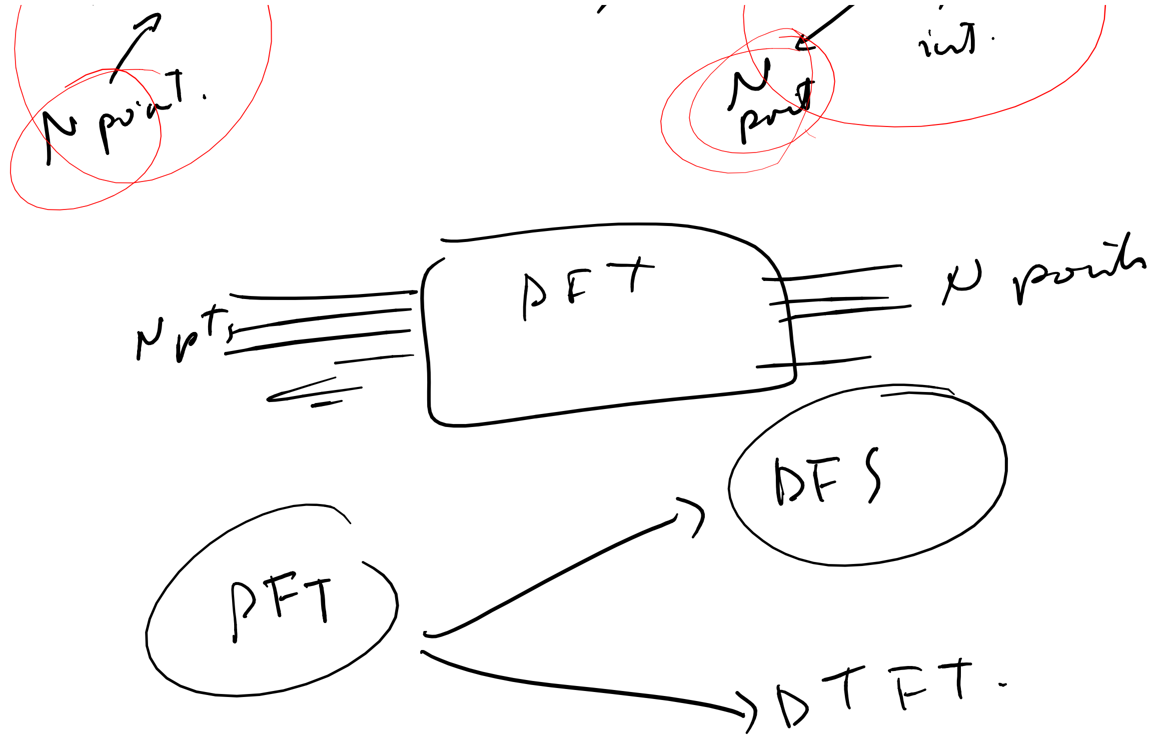
$$\triangleq \sum_{m=0}^{N-1} \tilde{x}_1(m)\, \tilde{x}_2(n-m)$$

$$\tilde{X}_3(k) = \tilde{X}_1(k)\, \tilde{X}_2(k)$$

DFT = Discrete Fourier Transform.

$x(n) \longrightarrow X(k)$

$T.$ N int.



First Appr: DFT $\leftrightarrow$ DFS

1. Start with $x(n)$

2. "periodicize" $x(n)$ To get $\tilde{x}(n)$

$$\tilde{x}(n) = \sum_{r=-\infty}^{+\infty} x(n+rN)$$

$$x(n) = \tilde{x}(n) \underbrace{R_N(n)}_{\text{Extract one period}}$$

$$R_N(n) = \begin{cases} 1 & n = 0, \ldots, N-1 \\ 0 & \text{otherwise} \end{cases}$$

3. Compute DFS of $\tilde{x}(n) \longrightarrow \tilde{X}(k)$

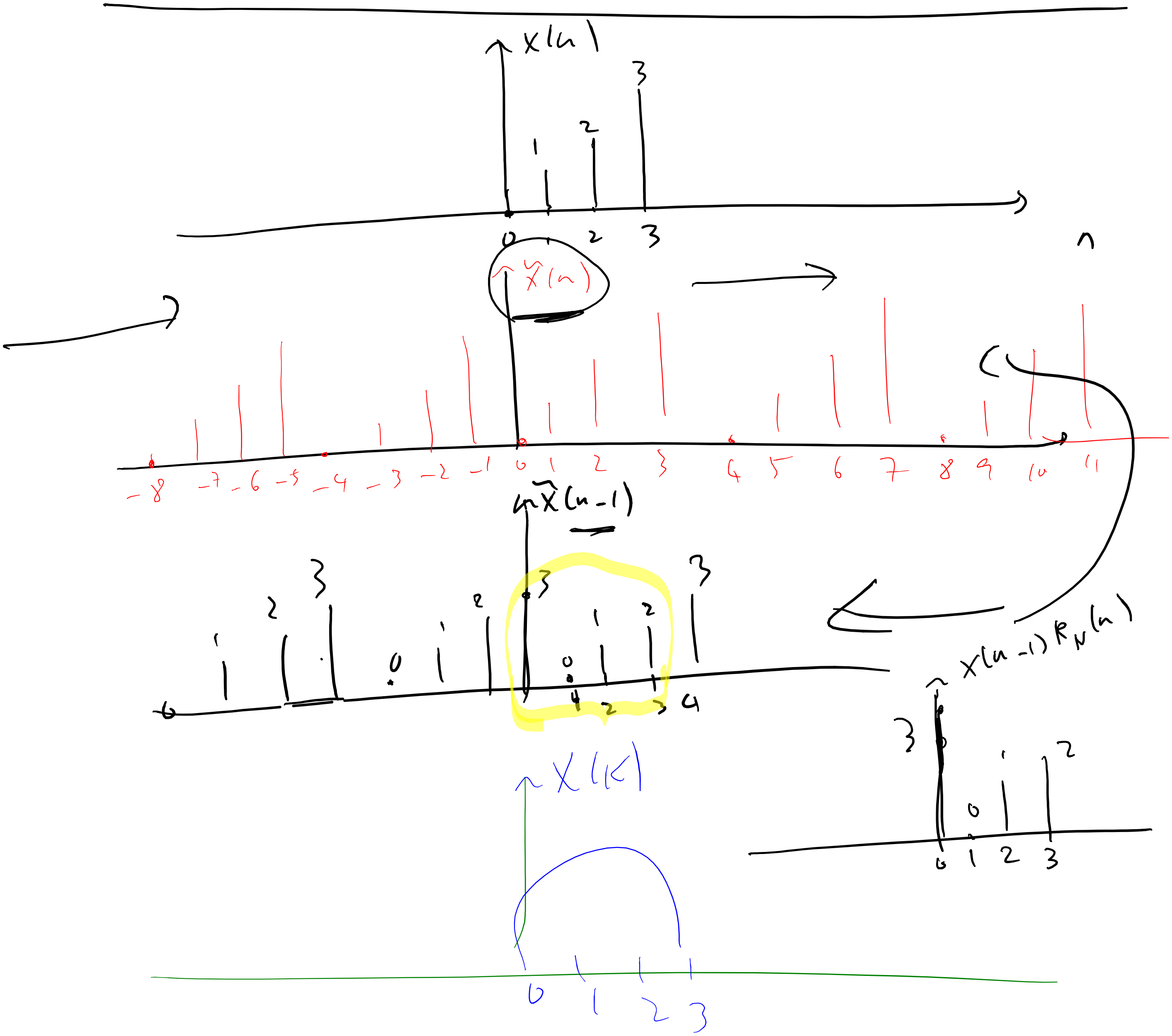
4. Take one period of $\tilde{X}(k)$ To get $X(k)$ DFT of $x(n)$

$$X(k) = \text{DFT of } x(n)$$

$$X(k) = \tilde{X}(k)\, R_N(k)$$

$$x(n) \longrightarrow \tilde{x}(n) \xrightarrow{\text{DFS}} \tilde{X}(k) \longrightarrow X(k)$$

N pt — periodic — periodic — N pt



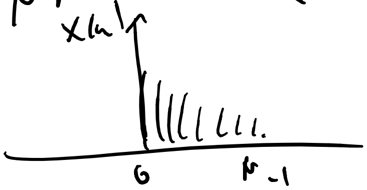
N pt ⟶ N pt seq

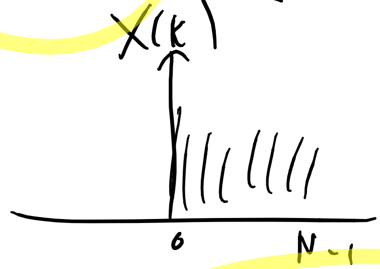
Properties of DFT

① Shift property

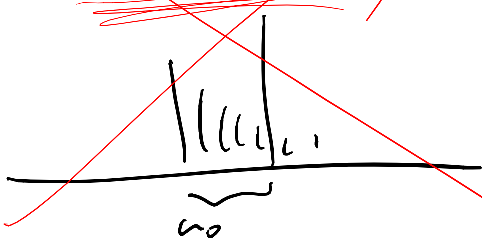
$$x(n) \xrightarrow{\text{DFT}} X(k) \qquad 0 \le k < N$$

N pt $\quad 0 \le n < N$

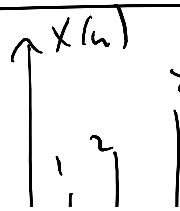




$$\cancel{x(n-n_0)\ ??} \xleftarrow{\text{IDFT}} X(k)\, e^{-j\frac{2\pi n_0 k}{N}}$$



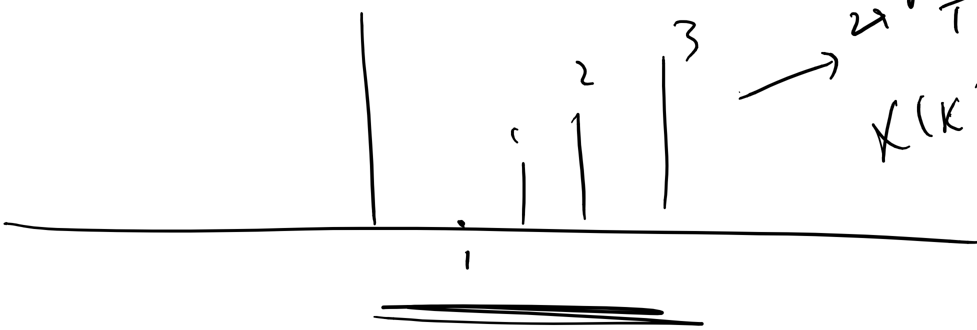
$$x(n-n_0)\, R_N(n) \xrightarrow{\text{DFT}} X(k)\, e^{-j\frac{2\pi n_0 k}{N}}$$



→ 4 pt seq.
4 pt DFT.

4 pt DFT

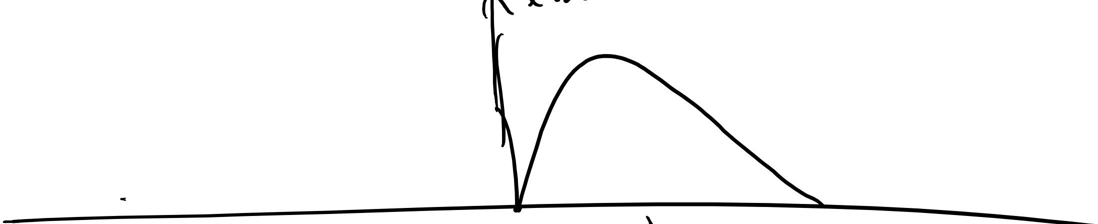
$x(n-1)$



4 pt DFT of This is DFT

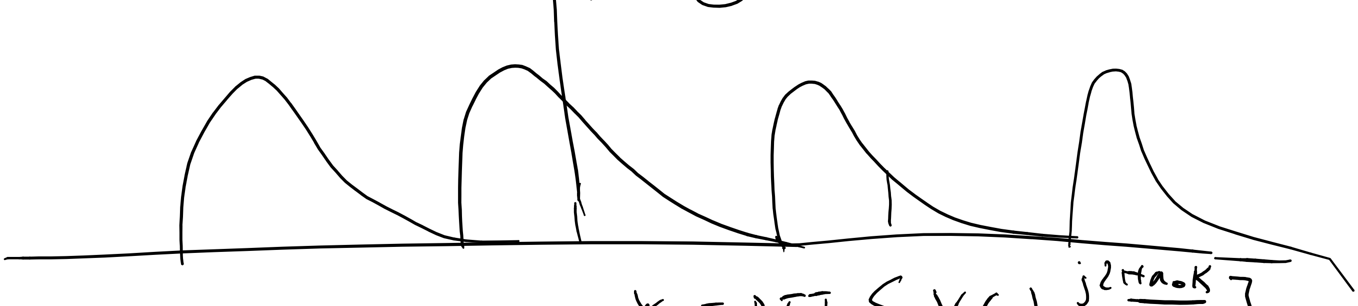
$X(k)\, e^{-j\ldots}$

$x(n)$

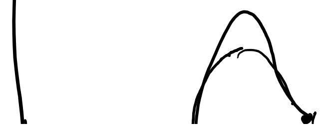


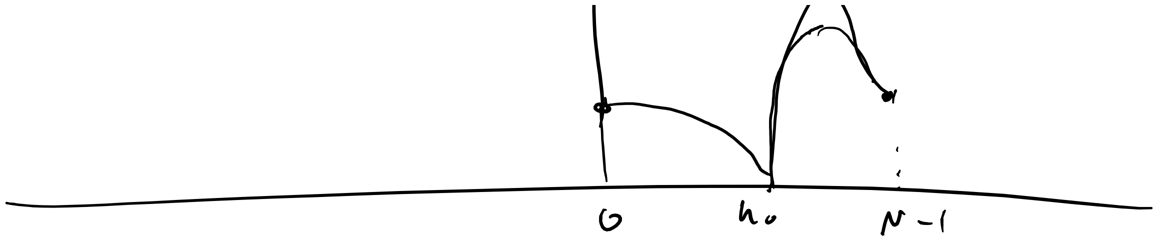
$\tilde{x}(n)$

$\tilde{x}(n-n_0)$



x IDFT $\left\{ X(k)\, e^{j2\pi n_0 k / N} \right\}$





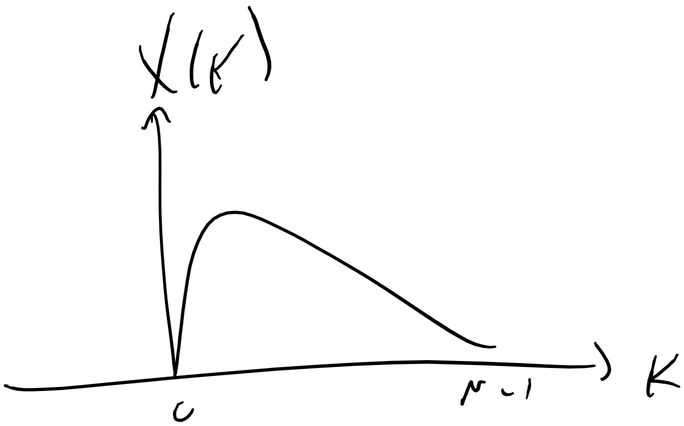
Properties of DFT.

$x(n) \xleftrightarrow{DFT} X(k) \quad 0 \le k < N$

$0 \le n < N$

$y(n) = x^*(n) \longleftrightarrow \cancel{X^*(-k)}$ wrong.

$X^*(-k)\, R_N(k)$



$X^*(-k)\, R_N(k)$

Symmetry Properties

$x(n) \xleftrightarrow{DFT} X(k)$

$$x(n) \xleftrightarrow{DFT} X(k)$$

$$Re\{x(n)\} \xleftrightarrow{DFT} R_N(k)\,\tilde{X}_e(k)$$

$$j\,Im\{x(n)\} \longleftrightarrow R_N(k)\,\tilde{X}_o(k)$$

$$R_N(n)\,\tilde{x}_e(n) \xleftrightarrow{DFT} Re\{X(k)\}$$

$$R_N(n)\,\tilde{x}_o(n) \longleftrightarrow j\,Im\{X(k)\}$$

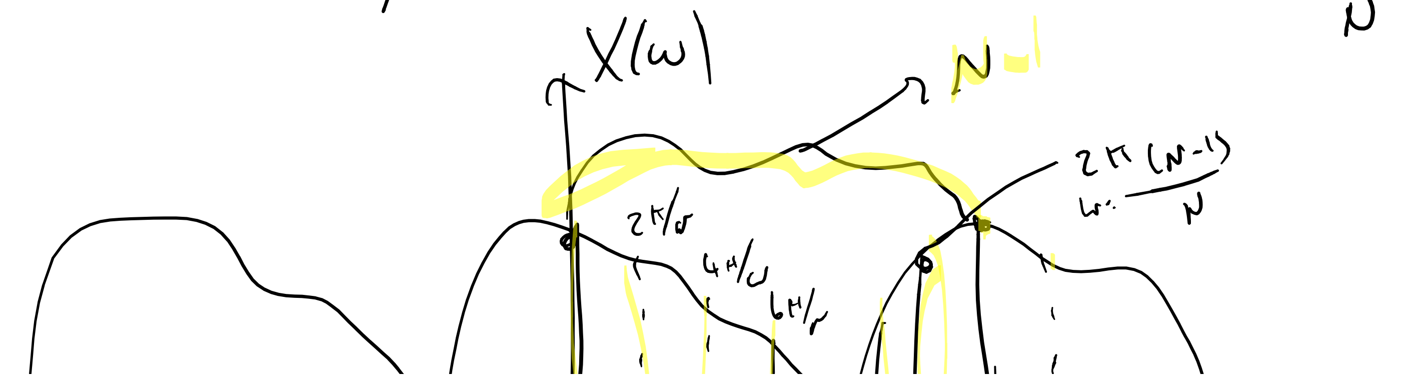
DFT To DTFT.

$$x(n) \longleftrightarrow X(k)$$

$$x(n) \longrightarrow X(\omega) = \sum_n x(n)\, e^{-j\omega n}$$

$$\boxed{X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi nk}{N}}} \leftarrow \text{DFT pair.}$$

$$X(k) = DFT = \left[X(\omega)\right]_{\omega = \frac{2\pi k}{N}}$$



$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\, e^{+j\frac{2\pi nk}{N}}$$ Synthesis

$$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi nk}{N}}$$ analysis

Defn of DFT

$$X(k) = N\text{ pt DFT of } x = \begin{cases} \sum \ldots\ldots & 0 \le k < N \\ 0 & \text{otherwise} \end{cases}$$

$$x(n) = \begin{cases} \frac{1}{N} \sum \ldots\ldots & 0 \le n < N \\ 0 & \text{otherwise} \end{cases}$$

DFT $\longleftrightarrow$ DTFT

$$X(\omega) = \sum x(n)\, e^{-j\omega n}$$

$$X(\omega) = \sum_n x(n)\, e^{-j\omega n}$$

$$X(k) = \sum_n x(n)\, e^{-j\frac{2\pi n k}{N}}$$

$0 \le k < N$

$$X(k) = \begin{cases} \left[X(\omega)\right]_{\omega = \frac{2\pi k}{N}} & 0 \le k < N \\ 0 & \text{otherwise} \end{cases}$$

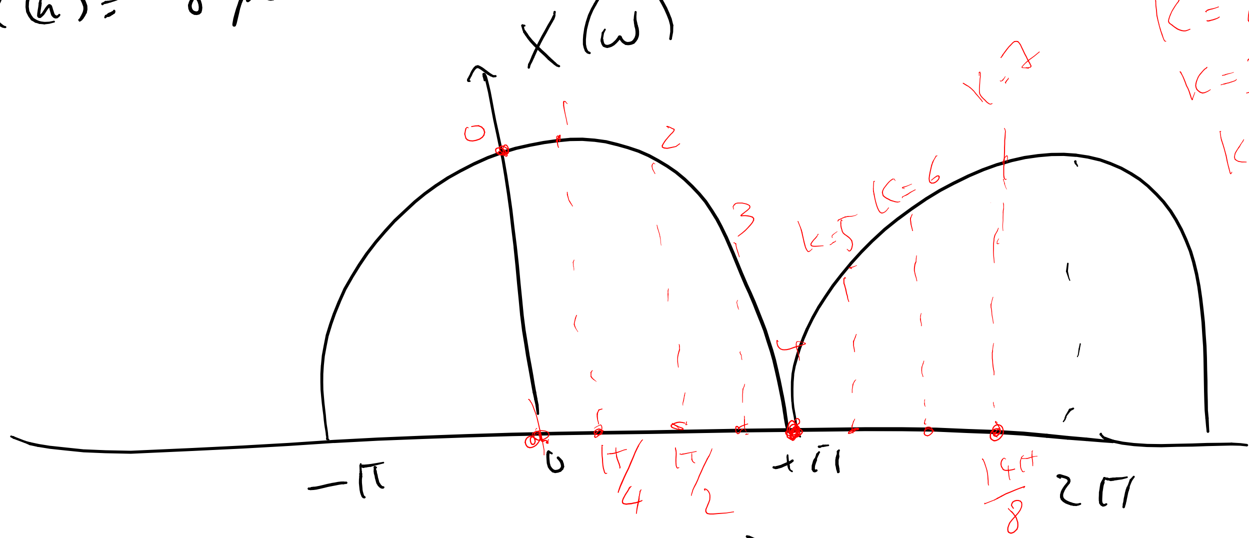
$k = 0, \ldots, 7$

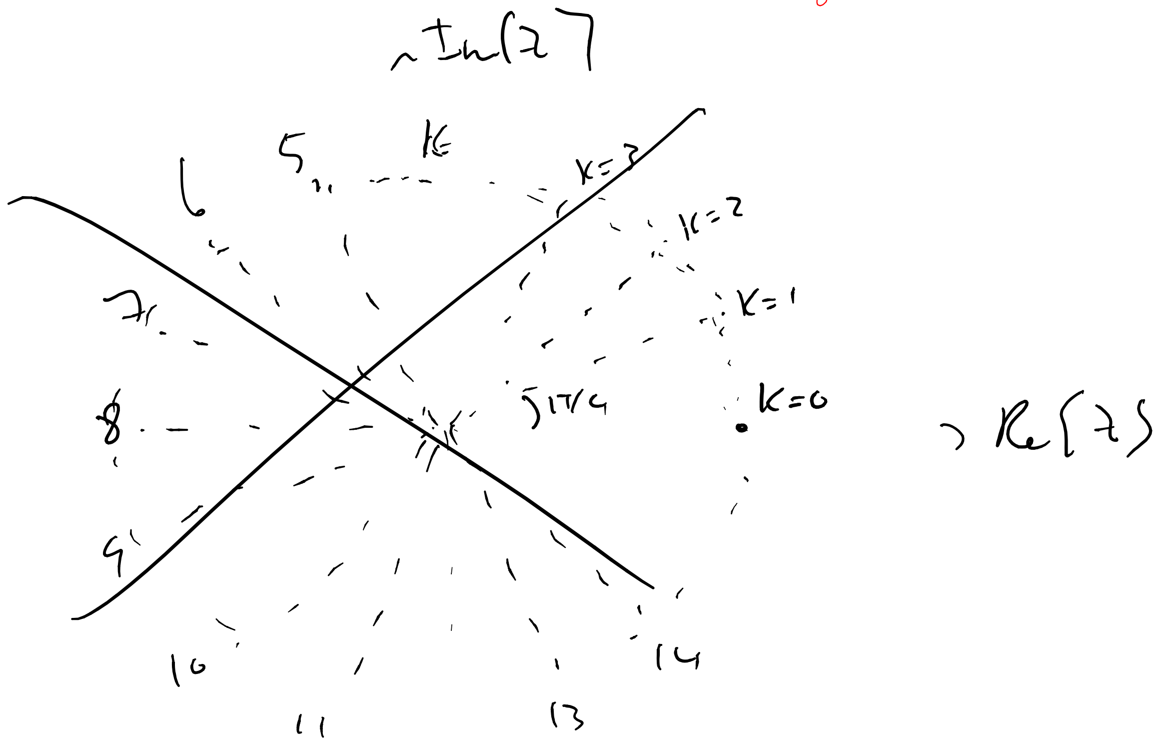
$k = 2$

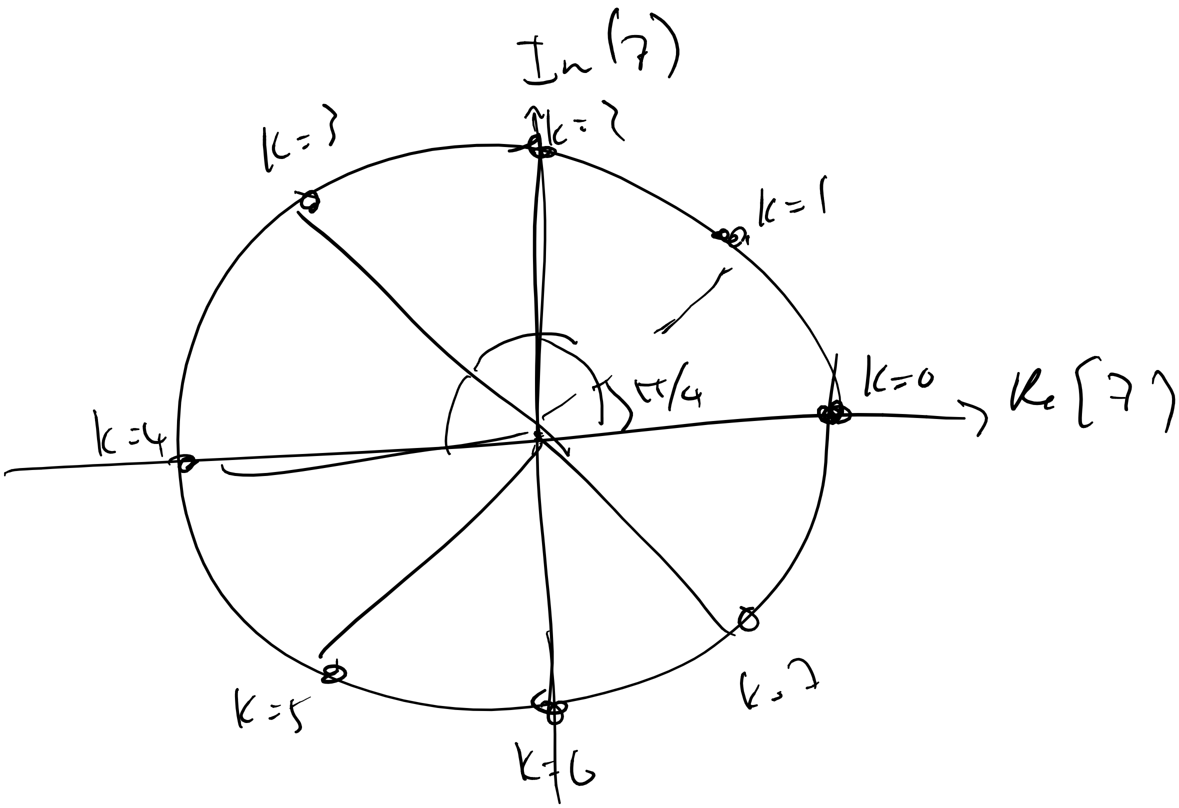
$k = 3$

$k = 4$

$x(n) = 8$ pnt.





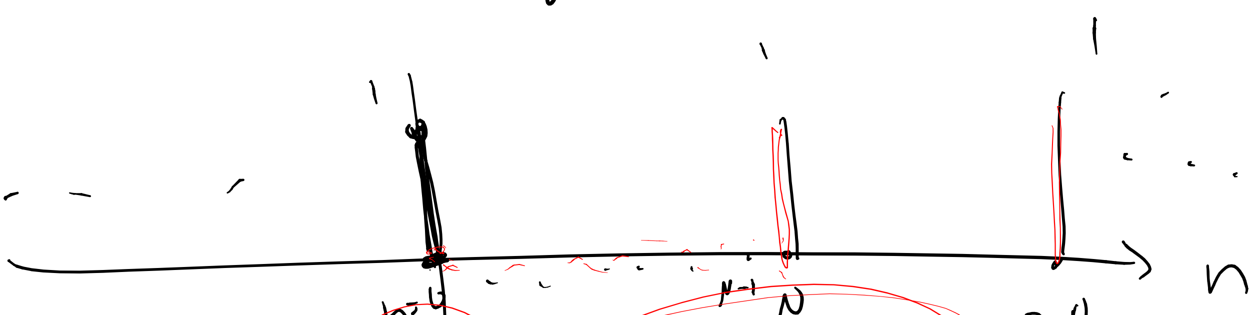


DFT

DFS

DTFT

$$\tilde{x}(n) = \sum_{r=-\infty}^{\infty} \delta(n + rN)$$



$\sim \sum^{N-1} \left(\sum^{+\infty} \delta(\quad) \right) \quad -j\frac{2\pi nk}{N}$

$$\tilde{X}(k) = \sum_{n=0}^{N-1} \left(\sum_{r=-\infty}^{+\infty} \delta(n + rN) \right) e^{-j\frac{2\pi nk}{N}}$$

$k=0$ $\quad \tilde{X}(k) =$

$k=0$

$k=1$

$k=N-1$

$$\boxed{\tilde{X}(k) =}$$

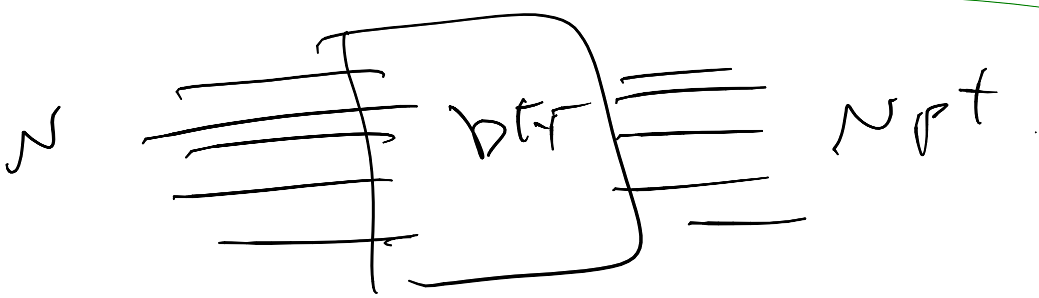
$\tilde{X}(k)$

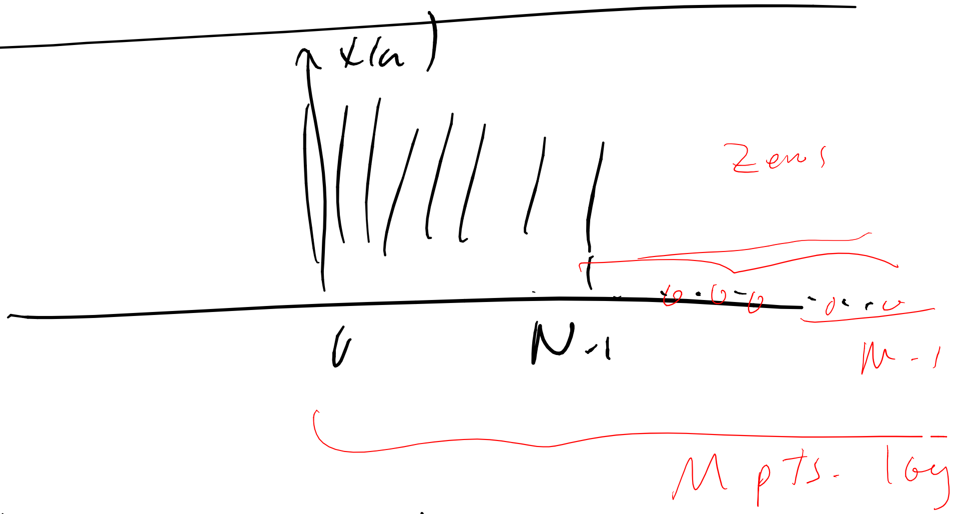
$$\boxed{\tilde{x}(n) = \frac{1}{N} \sum_{k=0}^{N-1} e^{j\frac{2\pi nk}{N}} = \sum_{r=-\infty}^{+\infty} \delta(n + rN)}$$

identity

Train of δ

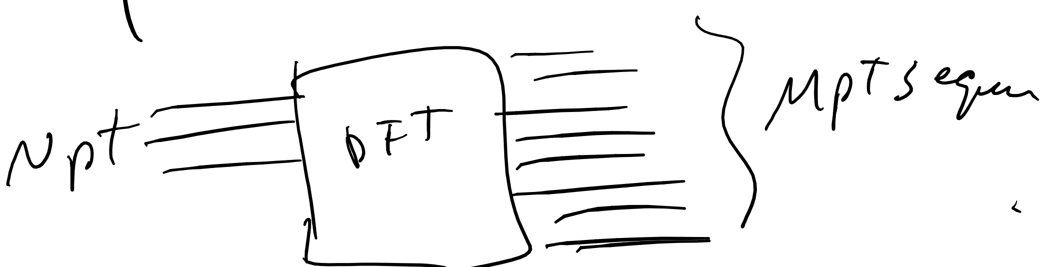
fun.



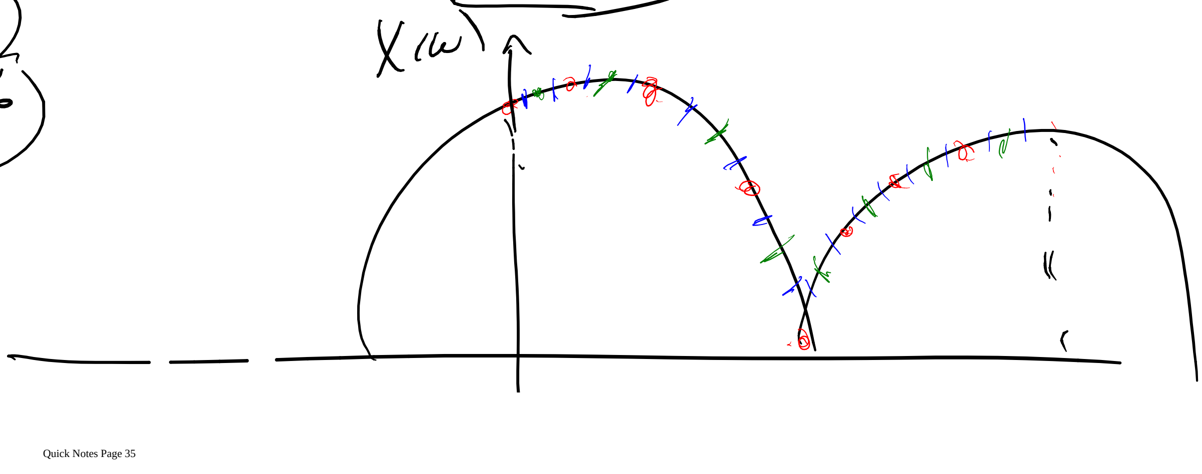


M pt DFT w/ x(n)

$$X(k)=\begin{cases}\displaystyle\sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi nk}{M}} & 0\le k<M \\ 0 & \text{otherwise}\end{cases}$$



N=8
M=16



$$X(\omega) = \sum_{n} x(n)\, e^{-j\omega n}$$

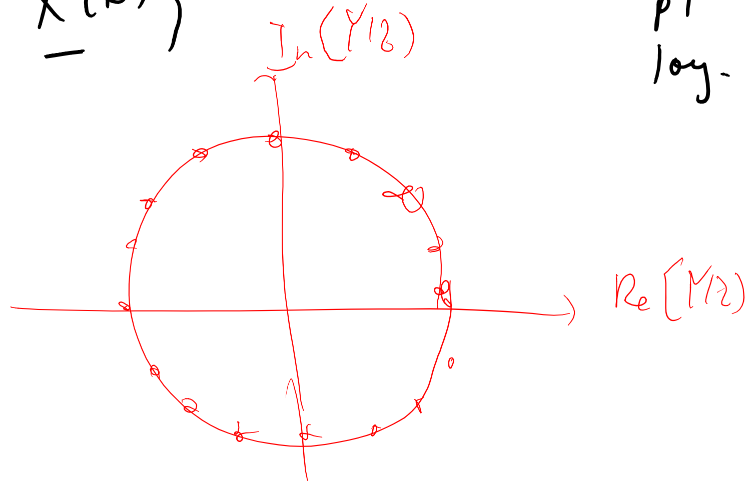
$$= \sum_{n=0}^{N-1} x(n)\, e^{-j\omega n} = \sum_{n=0}^{M-1} x(n)\, e^{-j\omega n}$$

Thought Exp

1. $y(n)$ either ∞ or finite $\leftarrow$ length unknown
2. $Y(\omega) = \text{DTFT}\{y(n)\}$
3. Sample $Y(\omega)$ at N equally spaced points

$$[Y(\omega)]_{\omega = \frac{2\pi k}{N}} = X(k) \leftarrow N \text{ pt long}$$

4. $\underset{N\,pt}{\text{IDFT}}\{X(k)\} = x(n) \leftarrow N$ pt long.



Answer: $x(n) = \tilde{W}(n)\, R_N(n)$

Answer:

$$x(n) = \tilde{w}(n) R_N(n)$$

$$\tilde{w}(n) = \sum_{k=-\infty}^{+\infty} y(n+kN)$$

$$x(n) = \begin{cases} \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{j\frac{2\pi nk}{N}} & 0 \le n < N \\ 0 & \text{otherwise} \end{cases}$$

$$x(n) = \begin{cases} \frac{1}{N} \sum_{k=0}^{N-1} \left(\sum_{m=-\infty}^{+\infty} y(m) e^{-j\frac{2\pi k}{N} m} \right) e^{j\frac{2\pi k}{N}} & 0 \le n < N \\ 0 & \text{otherwise} \end{cases}$$

$$x(n) = \begin{cases} \sum_{m=-\infty}^{+\infty} y(m) \frac{1}{N} \sum_{k=0}^{N-1} e^{-j\frac{2\pi k(m-n)}{N}} & 0 \le n < N \\ 0 & \text{otherwise} \end{cases}$$

apply identity

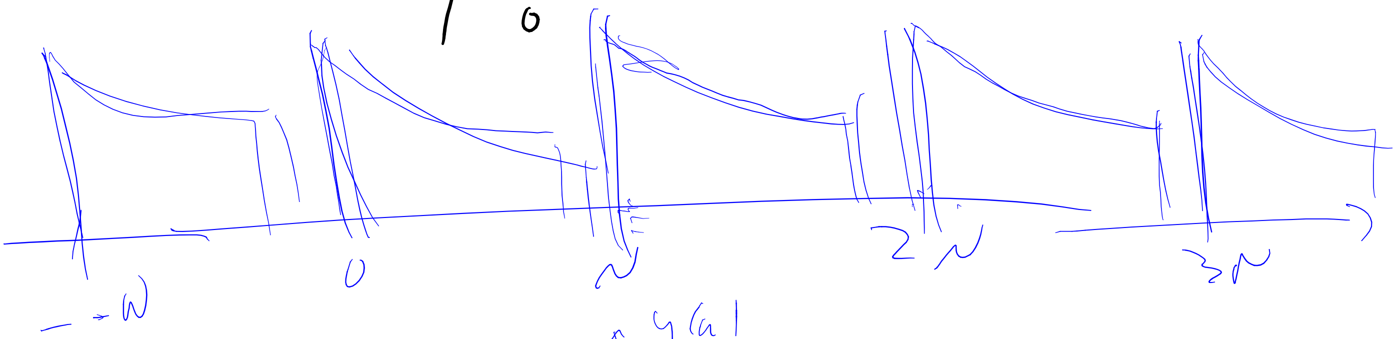
$$= \sum_{r=-\infty}^{+\infty} \delta(n-m+rN) = f(n-m)$$

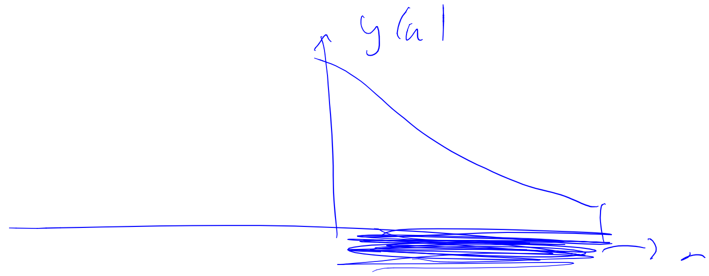
name: $f(n) = \sum_{r=-\infty}^{+\infty} \delta(n+rN)$

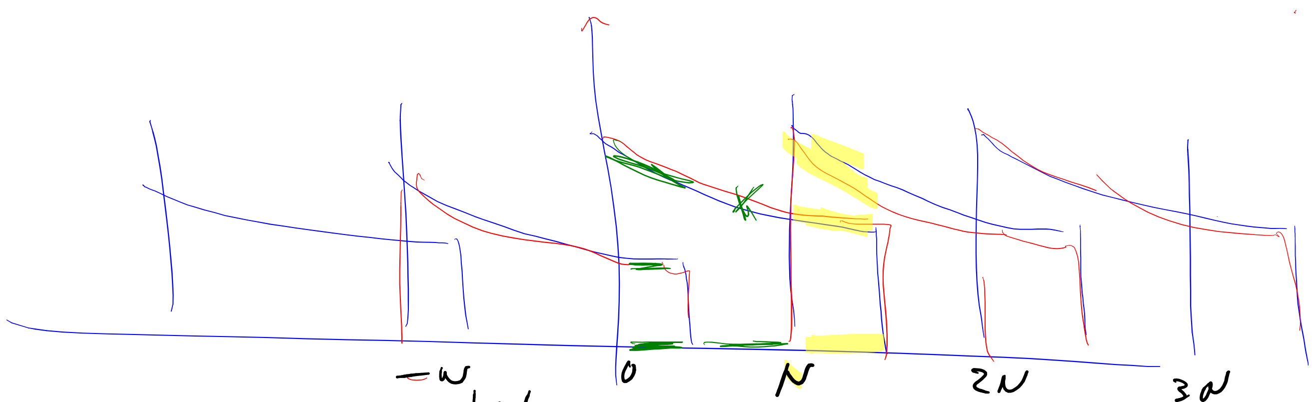
$$x(n) = \begin{cases} \sum_{m=-\infty}^{+\infty} y(m) f(n-m) & 0 \le n < N \\ 0 & \text{otherwise} \end{cases}$$

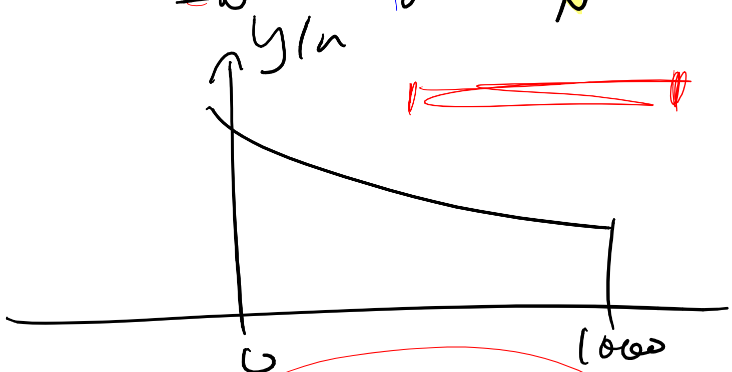
$$\begin{cases} \sum_{r=-\infty}^{+\infty} y(n+rN) & 0 \le n < N \end{cases}$$

$$= \begin{cases} \sum_{r=-\infty}^{+\infty} y(n+rN) & 0 \le n < N \\ \_\_\_\_\_\_ & \text{otherwise} \end{cases}$$



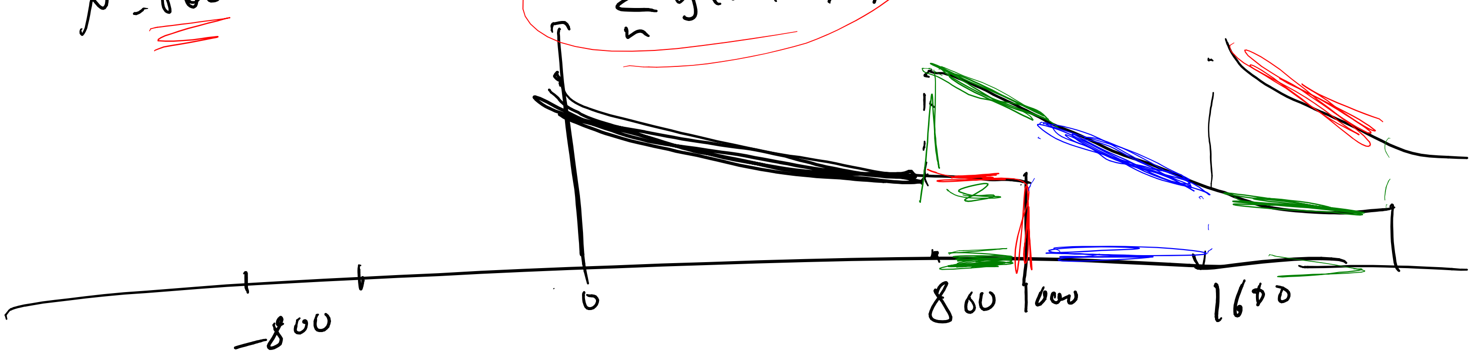






$N = 800$

$$\sum_n y(n+rN)$$



If $N \geq$ duration of $y(n)$
get back $y(n)$ exactly.

If $N <$ duration of $y(n)$
one period of $x(n)$ will be aliased vers of $y(n)$

How To use DFT To do Convolution

$x_1 * x_2 = x_3$

Linear Conv $x_3(n) = \sum_K x_1(k)\, x_2(n-k)$

$x(n) \rightarrow$ LTI $h(n)$ $\rightarrow y(n) = x * h$

- periodic Convolution $\underline{\tilde{x}_1} \circledast \underline{\tilde{x}_2} = \underline{\tilde{x}_3}$.

periodic N N N.

- Circular Convolution

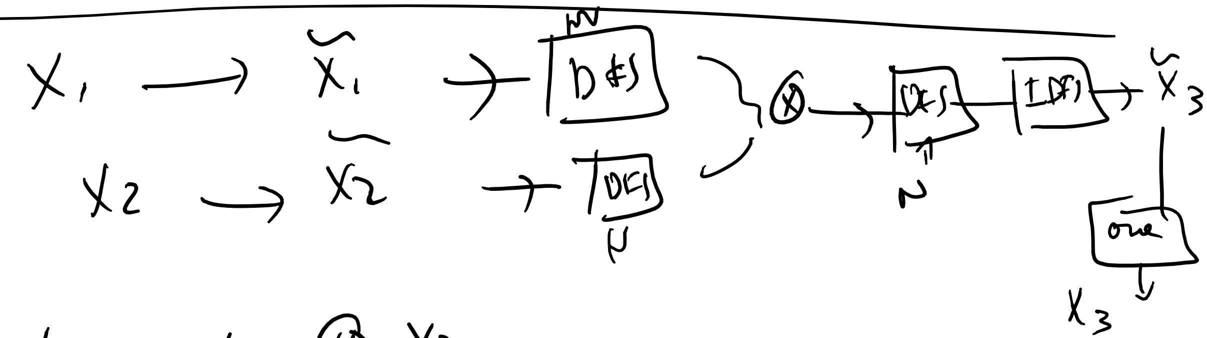
$x_1 \circledast x_2 \triangleq \left[\tilde{x}_1 \circledast \tilde{x}_2\right] R_N(n)$

If Multiply N pt DFT of 2
finite length N pt sequen $\rightarrow$

finite length N pt sequence →

DFT of Then circular Convolution

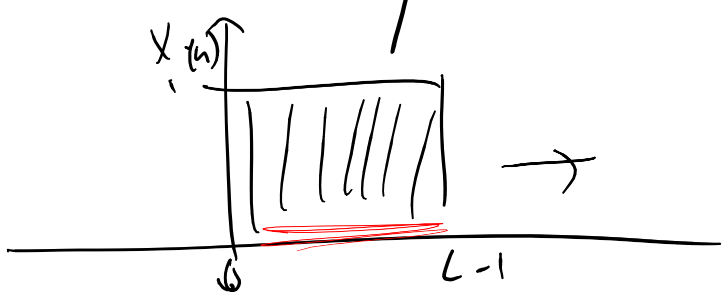
NOT DFT of Their Linear Convolution



$x_3 = x_1 \circledast x_2$

Ex Not: Circular Convolution of

$x_1(n) = x_2(n) = \begin{cases} 1 & 0 \le n \le L-1 \\ 0 & \text{other} \end{cases}$



2 Cases:

① $N = L$

② $N = 2L$

Case ① : $N = L$. Do DFT for circular convolution.

DFT of x_1

contd.

N pt = L pt DFT of x_1

$$X_L(k) = \sum_{n=0}^{L-1} x_1(n) e^{-j\frac{2\pi nk}{N}}$$

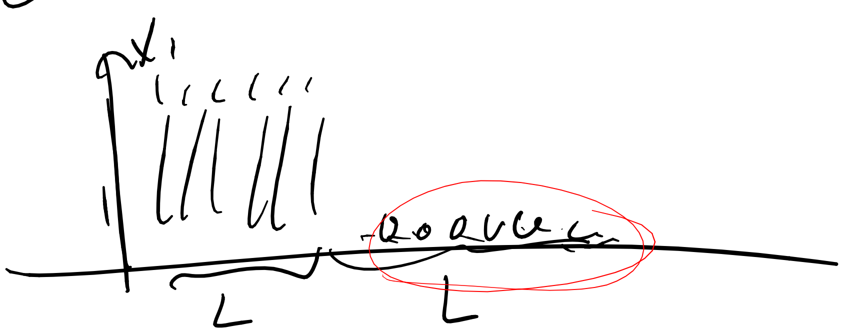
$$= \begin{cases} L & k=0 \\ 0 & \text{otherwise} \end{cases}$$

$X_3(k)$: L pt circular conv of x_1 and x_2

$$= X_L(k) X_L(k) = \begin{cases} L^2 & k=0 \\ 0 & \text{otherwise} \end{cases}$$

L pt IDFT $\{X_3(k)\}$, $\begin{cases} L & 0 \le n \le L-1 \\ 0 & \text{otherwise} \end{cases}$

Case ② $N = 2L$.



2L pt DFT of x_1 and x_2

$$\sum^{L-1} \quad j\frac{2\pi nk}{}$$

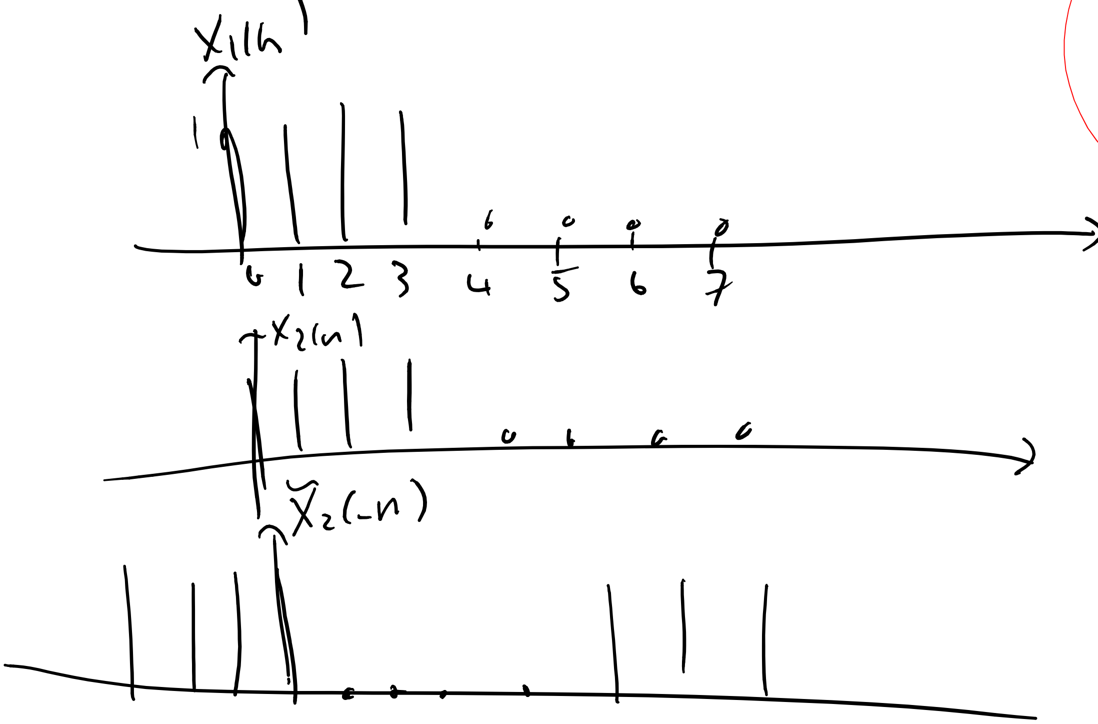
2L pt DFT

$$X_{2L}(k) = \sum_{n=0}^{L-1} x_1(n)\, e^{-j\frac{2\pi nk}{2L}}$$

$$IDFT_{2L}\left\{ X_{2L}(k)\, X_{2L}(k) \right\} = x_1 \circledast x_2$$

2L pts.

Does yield Linear Convolution



$x_1(n) \longrightarrow L$ $\qquad N > L$

$x_2(n) \longrightarrow P$ $\qquad N > P$

goal $\quad x_3 \triangleq x_1 * x_2$

$$X_3(\omega) = \sum x_3(n)\, e^{-j\omega n}$$

$$\text{DTFT}\{x_3\} = X_3(\omega) = \sum_n x_3(n) e^{-j\omega n}$$

$$X_3(\omega) = \underline{X_1(\omega) X_2(\omega)}$$

Sample $X_3(\omega)$ at N equally spaced samples $= Y(k)$

$$\underline{Y(k)} = \left[\underline{X_3(\omega)}\right]_{\omega = \underline{\frac{2\pi k}{N}}}$$

$$\text{IDFT}\{Y(k)\} = \begin{cases} \underline{\displaystyle\sum_{r=-\infty}^{+\infty} x_3(n + rN)} & 0 \le n < N \\ 0 & \text{otherwise} \end{cases}$$

N pt

$$[X_3(\omega)]_{\omega = \frac{2\pi k}{N}} = Y[k] = [X_1(\omega)]_{\omega = \frac{2\pi k}{N}} [X_2(\omega)]_{\omega = \frac{2\pi k}{N}}$$

$$\text{IDFT}[Y(k)] = x_1 \circledast x_2$$

N pt   N pt circular

$$x_1 \underset{N}{\circledast} x_2 = \begin{cases} \displaystyle\sum_{r=-\infty}^{+\infty} x_3(n + rN) & 0 \le n < N \\ 0 & \text{otherwise} \end{cases}$$

To get Linear Convolution of x_1 and x_2

L ↗ (length of x_1), P ↗ (length of x_2)

(1) pad x_1 lots of zeros $\rightarrow N$

pad x_2 .... $\rightarrow N$

$N \geq P + L - 1$

(3) take DFT of both

N pt

multiply

IDFT $\rightarrow N$

N

---

x_1 L pt

x_2 P pt

(1) Pad x_1 with $N-L$ zeros to get N pt.

(2) Pad x_2 with $N-P$ zeros to get N pt.

(3) Take N pt DFT of $x_1 \rightarrow X_1(k)$

" " " " " $x_2 \rightarrow X_2(k)$

multiply $X_1(k)\,X_2(k) \rightarrow N$ pt

(4) Multiply $X_1(k)\,X_2(k) \rightarrow N$ pt

(5) $\underset{N}{\mathrm{IDFT}}\{X_1(k)\,X_2(k)\} \rightarrow$ answer

If $N \geq L + P - 1$

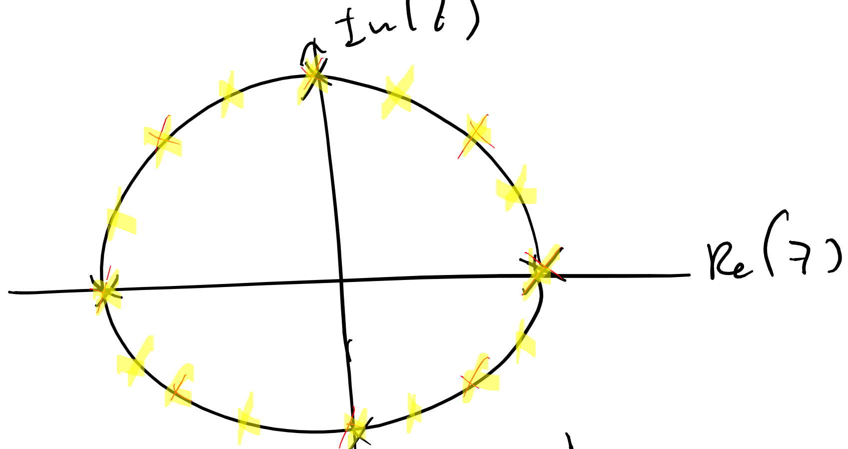
$x_3 = x_1 * x_2 =$ The answer

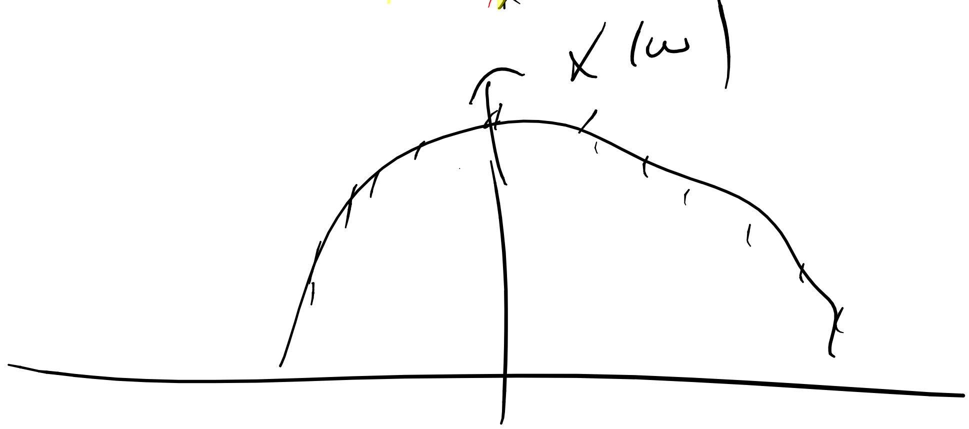
$x(n) = 4$ pt.

4 pt DFT

8 pt DFT

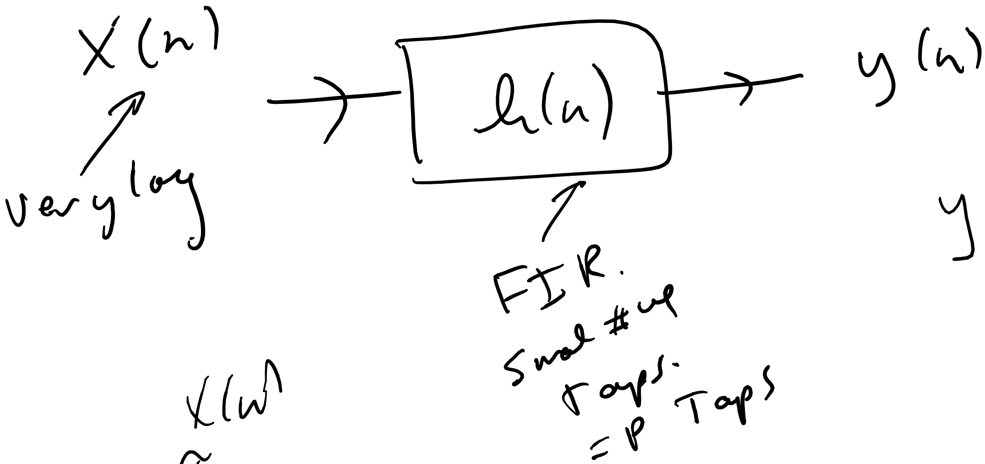
16 pt



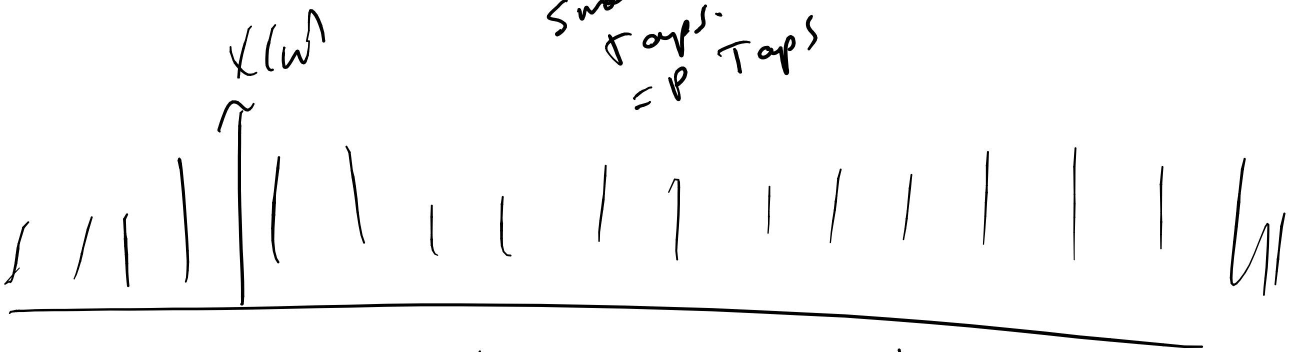


Using DFT for filtering long sequence

long sequence



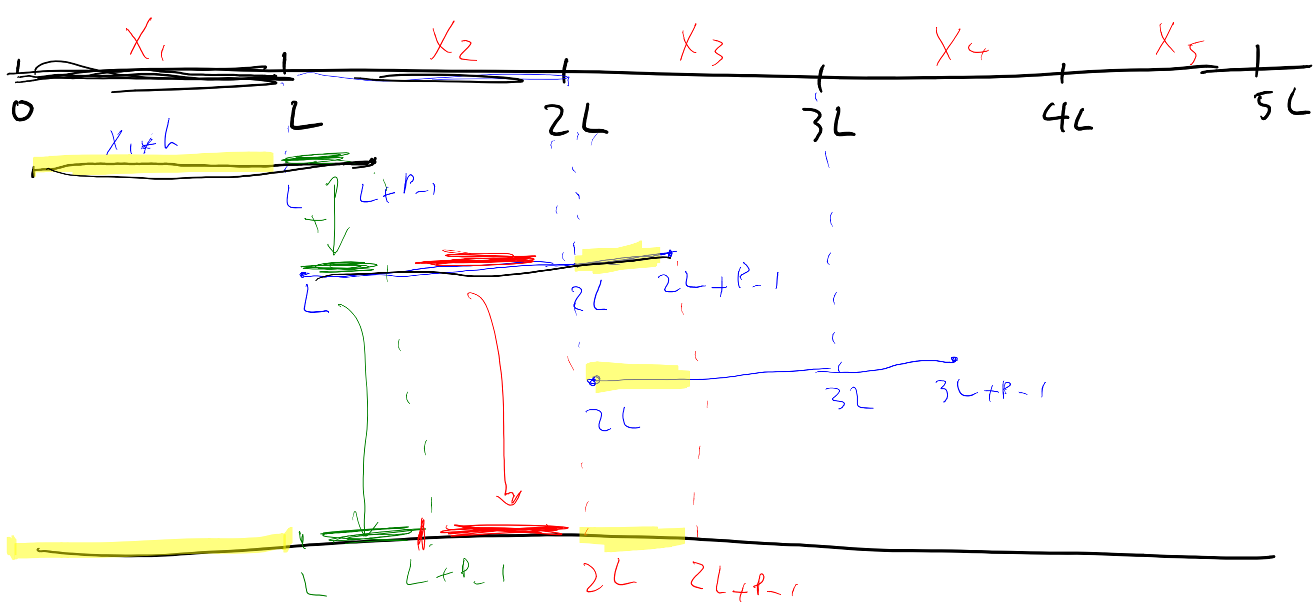
$y = x * h.$



Exploit linearity of convolution

$\rightarrow [x_1(n) + x_2(n)] * h = x_1 * h + x_2 * h$

Overlap Add



Overlap save

$x_1 \longrightarrow L$ pt
$x_2 \longrightarrow P$ pt

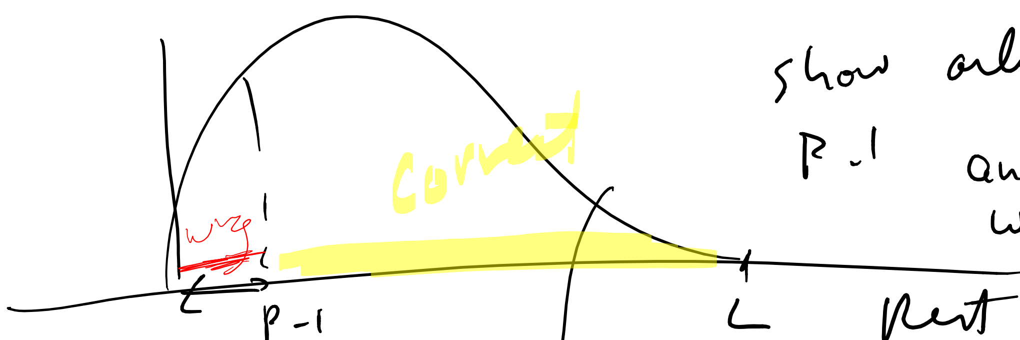
$\Big\}$ $x_1 * x_2 \longrightarrow L+P-1$ pts

Correct: pad with zeros to get $L+P-1$ pt seq.

- multiply these. $L \gg P$
- Inverse DFT. ($L+P-1$ pt.)

Suppose take L pt DFT of x_1 and x_2

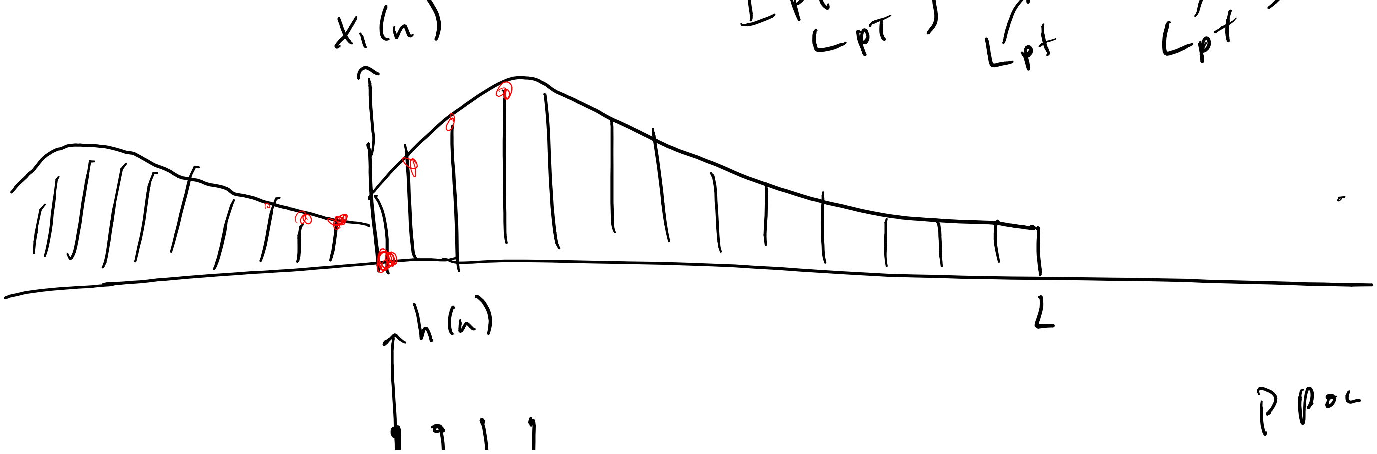
- Multiply these
- take L pt IDFT.



show only the first $P-1$ answers are wrong.

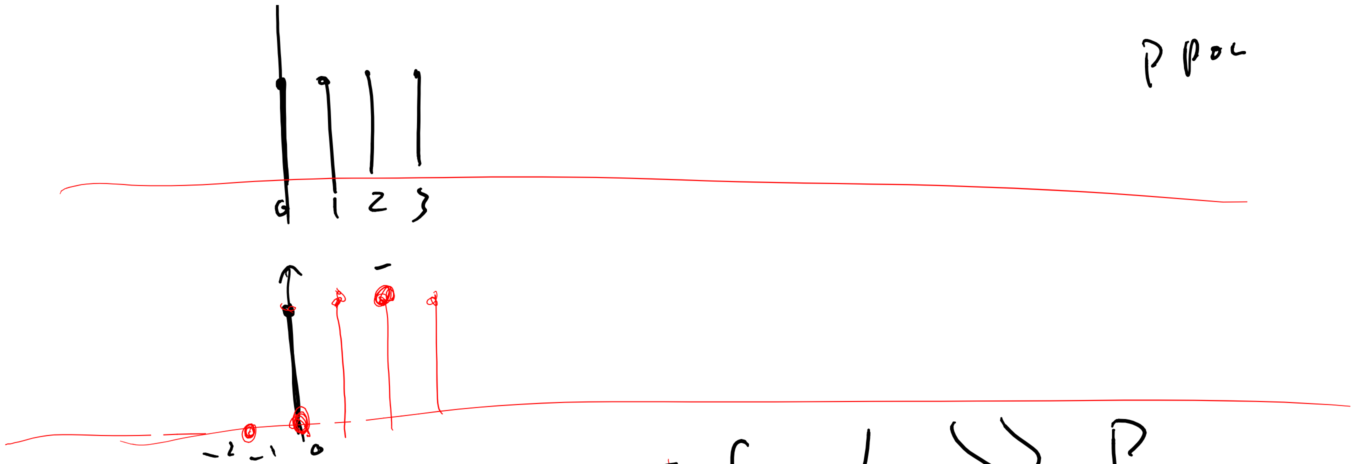
Rest are cor

$\underset{L\,pt}{IDFT} \{ \underbrace{X_1(K)}_{L\,pt} \; \underbrace{H(K)}_{L\,pt} \}$



P po

P pol



$x(0) \leftarrow$ wrong

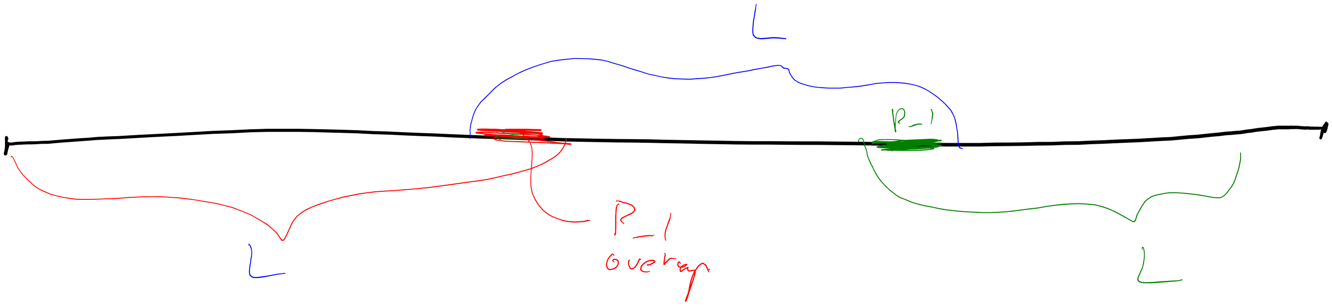
If L >> P

If L pt DFT of Both and multiply

Then only first P_1 point are wrong

Remaining answer L_(P_1) are correct

Overlap Save



① Segment into L point chunks. Overlap by P_1

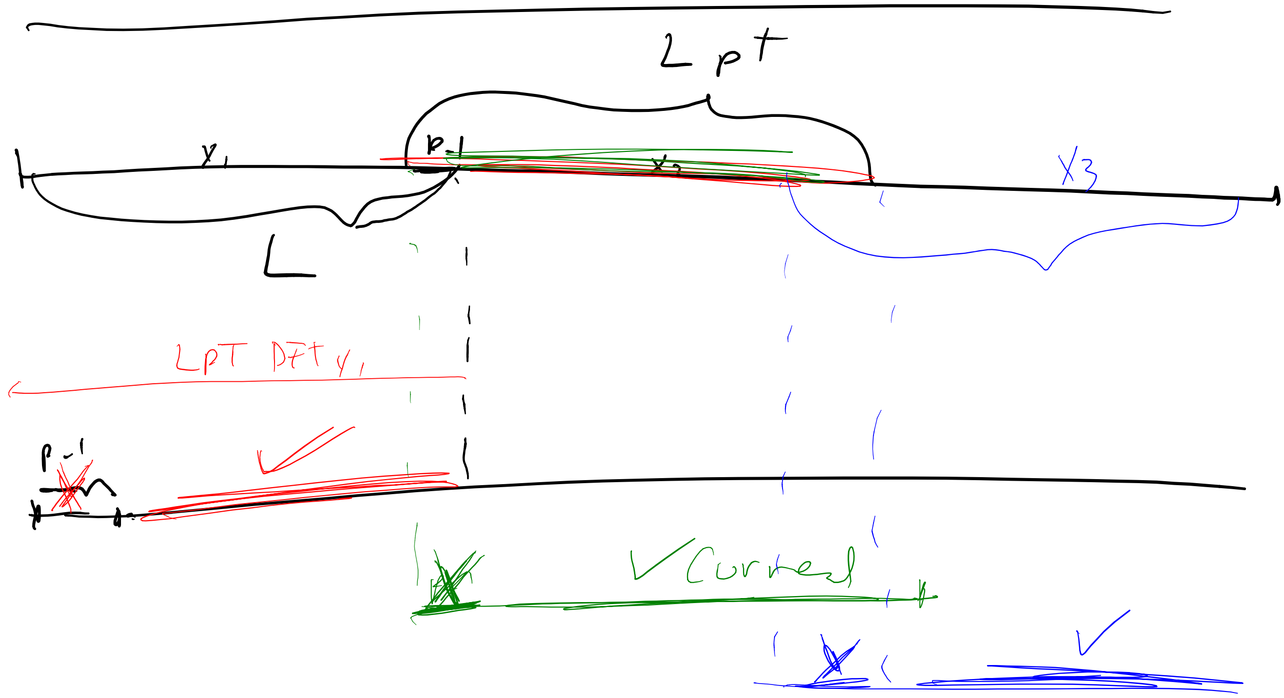
② Take L point circular Convolution of each chunk

Multiply L pt DFT of chunk
L pT DFT of h
→ L pt

IDFT
L point

IDFT L point → L point

(3) Throw away first P-1 points of the answer in part 2, Replace with previous segment



Discrete Cosine Transform

* Compaction property
* Coeff in freq. domain are uncorrelated with each other

→ Compression → Audio, Speech → DCT wavelet; Image ← wavelets; Video ← DCTs

JPEG 2000

JPEG 2000 ← video ← DCT

~~wavelet~~

MPEG 4 → H.264 → H.265

Computer vision, geophysical → Stephane Mallat

Harmonic Analysis → Ingrid Daubechies

Filter banks, Signal processing → Vetterli

Wavelets 2000

Haar, Hartley, Hadamard, Walsh

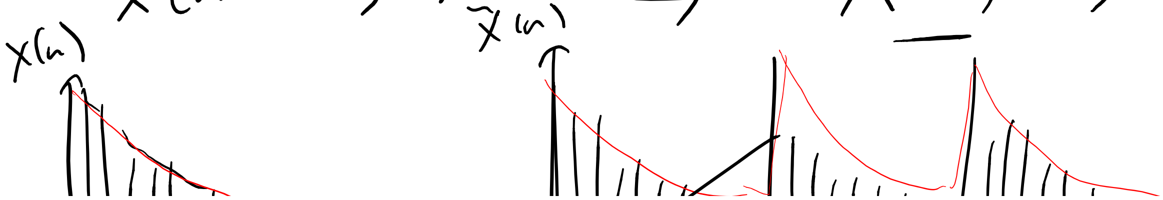
$x(n)$ real → DFT → Complex array of #s

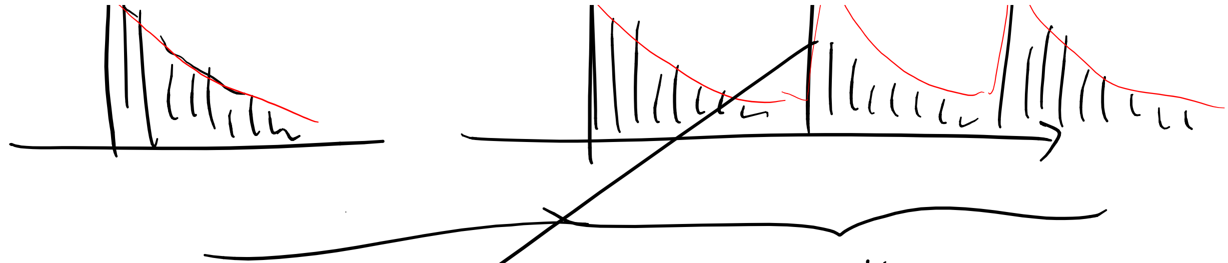
$x(n)$ real → DCT → Real #s

DFT $x(n)$

$$x(n) \longrightarrow \tilde{x}(n) \xrightarrow{\text{DFS}} \tilde{X}(k) \xrightarrow{\text{period one}} X(k)$$

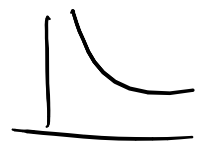
$\uparrow$ DFT



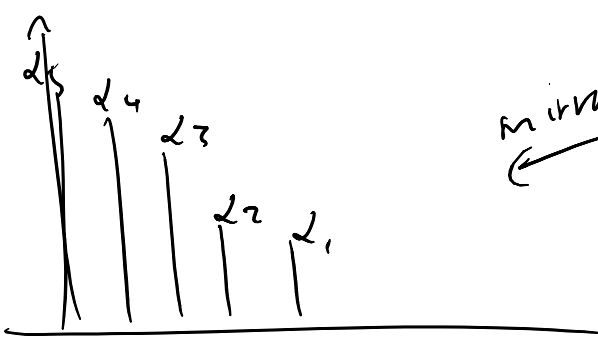


expand $e^{j\frac{2\pi K}{N}}$

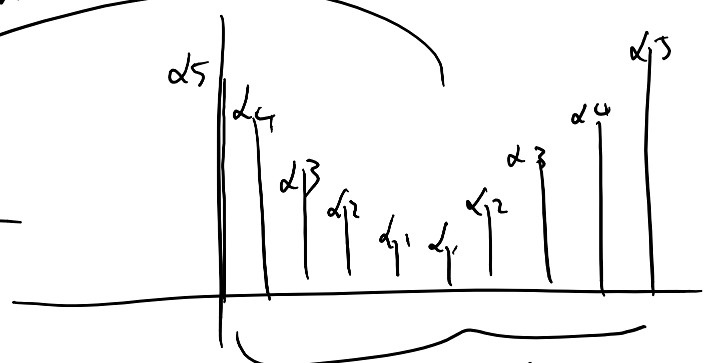
sharp discontinuity results in high frequency component



x(n)



mirror.



DFS of mirrors signal → DCT

4 Kinds of DCT

DCT_2

$$X^{C_2}(K) = 2\sum_{n=0}^{N-1} x(n)\cos\left(\frac{\pi K(2n+1)}{2N}\right)$$

Analysis

Synthesis

$$x(n) = \frac{1}{\ } \sum^{N-1} \beta(K)\, X^{C_2}(k)\, \cos\left(\frac{\pi K(2n+1)}{\ }\right)$$

$$x(n) = \frac{1}{N} \sum_{k=0} \beta(k)\, X(k) \cos\left(\frac{\pi k(2n+1)}{2N}\right)$$

$$\beta(k) = \begin{cases} \frac{1}{2} & k=0 \\ 1 & 1 \le k < N \end{cases}$$

Relate DFT of $x(n)$ to 1D DCT ...

Proposal 1:

1. $x(n)$ N pt seq. Real
2. Take 2N pt DFT $\longrightarrow X(k)$
3. $2\,\mathrm{Re}\left\{ X(k)\, e^{-j\frac{2\pi n k}{2N}} \right\}$

Proposal 2

1. Start with N pt Real seq $x(n)$.
2. Pad it with N zeros $\longrightarrow x_{2N}(n)$
3. Form a periodic seq.
   $\longrightarrow \tilde{x}_{s}(n) = \tilde{x}_{2N}(n) + \tilde{x}_{2N}(-n-1)$
4. Take 2N pt DFT of one period

④ ~~Take 2N pt DFT of one period of $x_2(n)$~~ $\longrightarrow X_2(k)$

⑤ Relate $X_2(k)$ to $X^{c_2}(k)$

Proposal 1 :

Proof :

2N point DFT of $x(n)$ is

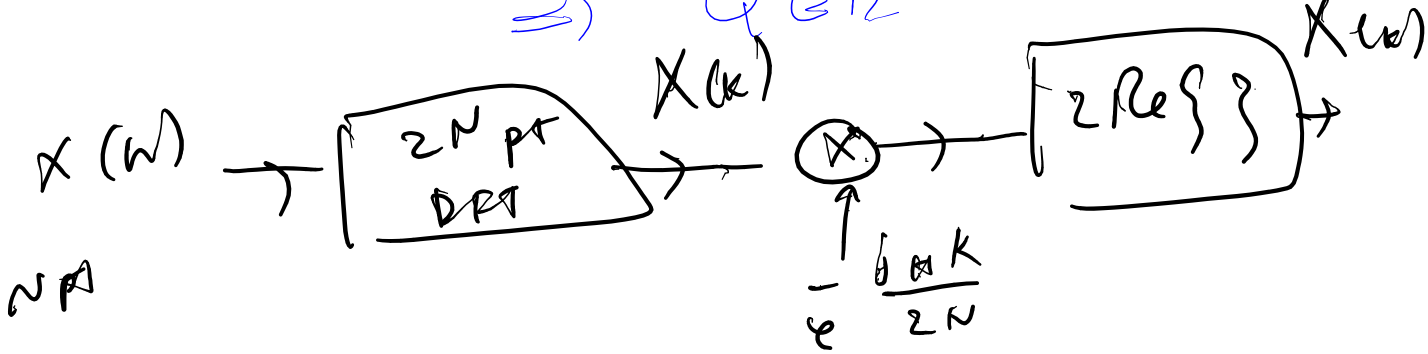
$$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi n k}{2N}}$$

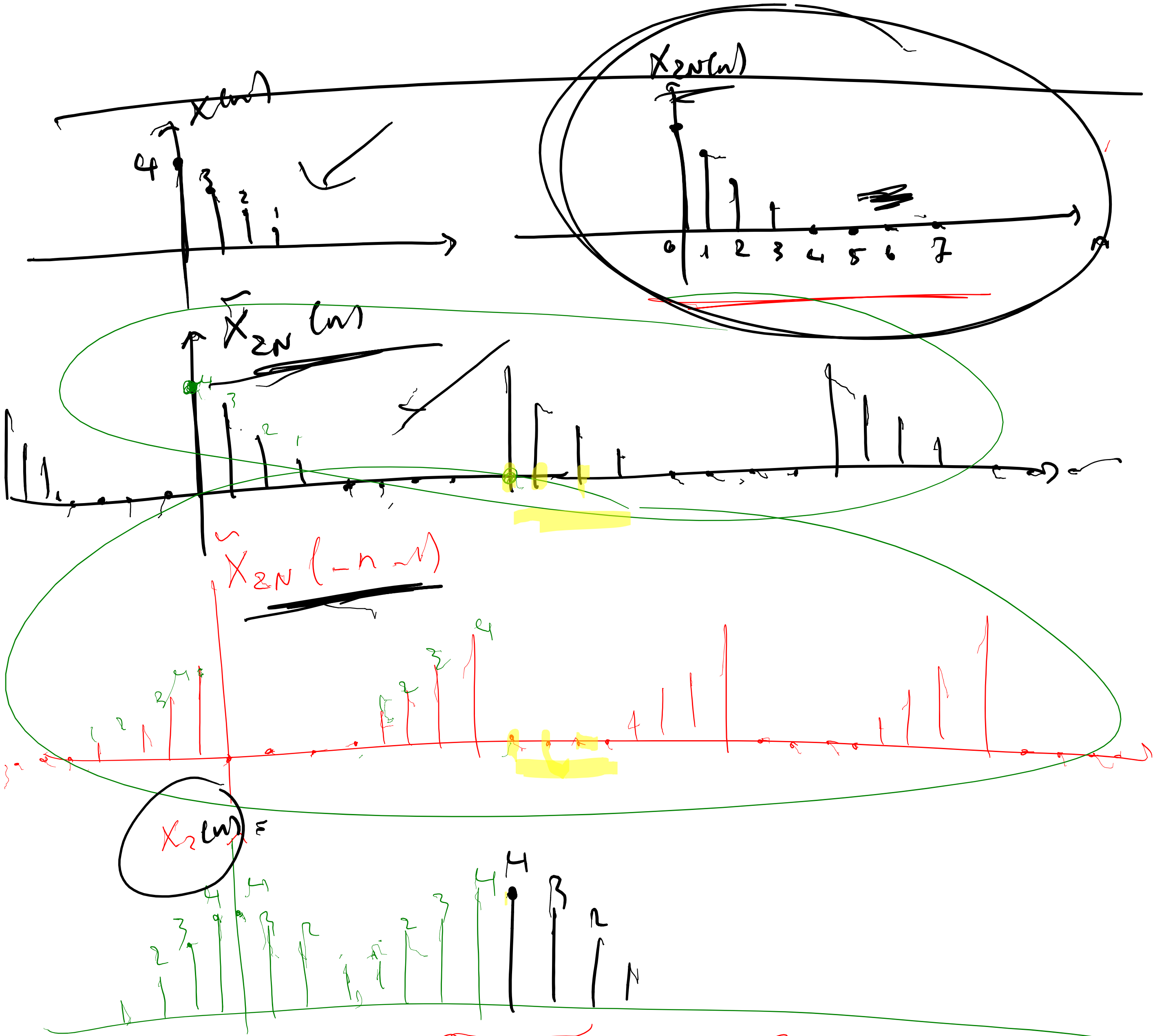
$$\underbrace{X(k)\, e^{-j\frac{\pi k}{2N}}}_{} = \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{\pi k(2n+1)}{2N}}$$

$$2\,\mathrm{Re}\left\{ \downarrow \right\} = 2\sum_{n=0}^{N-1} x(n) \cos \frac{\pi k(2n+1)}{2N}$$

$=$

$\Rightarrow$ QED





2N pt DFT of this $\longrightarrow X_2(k)$

8 pt DFT

How is $X_2(k)$ related to $X^{C_2}(k)$

$X(k) =$ 2 N pt DFT of $x(n)$

$\tilde{X}(-n-1)$

$$\tilde{x}_2(n) = \tilde{x}_{2N}(n) + \tilde{x}_{2N}(-n-1)$$

$$X_2(k) = X(k) + X^*(k)\, e^{j\frac{2\pi k}{2N}}$$

$$= e^{j\frac{\pi k}{2N}} \left[X(k)\, e^{-j\frac{\pi k}{2N}} + X^*(k)\, e^{j\frac{\pi k}{2N}} \right]$$

$$\underline{X_2(k)} = \underline{e^{j\frac{\pi k}{2N}}}\; \underbrace{2\,\mathrm{Re}\left\{ X(k)\, e^{-j\frac{\pi k}{2N}} \right\}}_{\underline{X^{C_2}(k)}}$$

$\leftarrow$ proof 1

$$\boxed{X_2(k)\, e^{-j\frac{\pi k}{2N}} = X^{C_2}(k)} \longleftarrow$$

Fast Fourier Transform

FFT

DFT: $$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi n k}{N}}$$

Direct Computation

Direct Computation

For each k: N complex mult
$N-1$ complex adds $\simeq N$

N values of k

$\Rightarrow$ $N(N-1)$ adds $\simeq O(N^2)$
N^2 mults $\Rightarrow O(N^2)$

FFT $\rightarrow$ $N \log N$

$N = 10^6$ Direct Computation $O(10^{12})$

FFT $\rightarrow 10^6 \log_2 (10^6) \simeq 2 \times 10^7$

5 orders of magnitude

Decimation in Time Frequency

Decimation in Time

$$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi n k}{N}} \qquad 0 \le k < N$$

$$X(k) = \sum x(n)\, e^{-j\frac{2\pi n k}{N}} \; . \; \sum^{N-1} x(n)\, e^{-j\frac{2\pi n k}{N}}$$

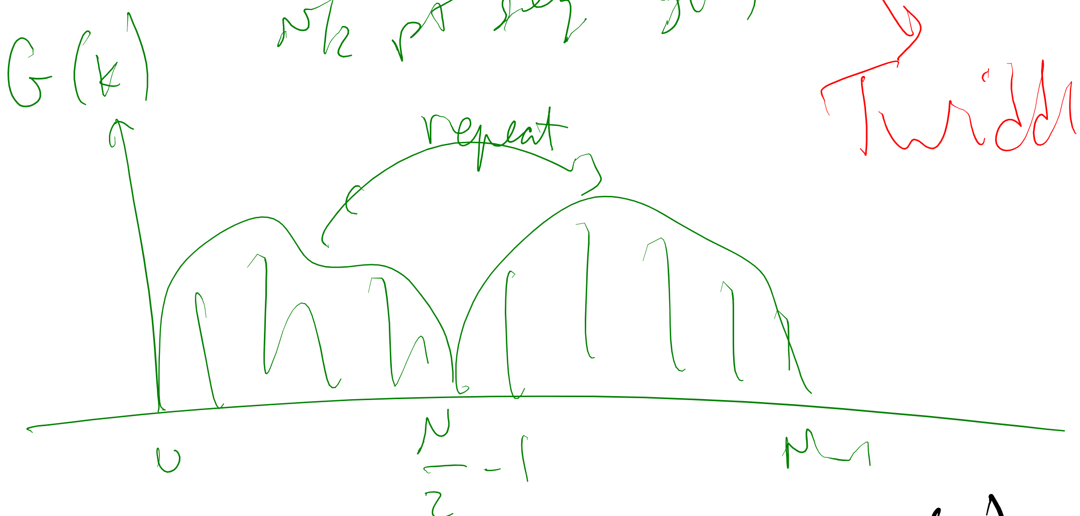
$$X(k) = \sum_{n \text{ even}} x(n)\, e^{-j\frac{2\pi n k}{N}} + \sum_{n \text{ odd}} x(n)\, e^{-j\frac{2\pi n k}{N}}$$

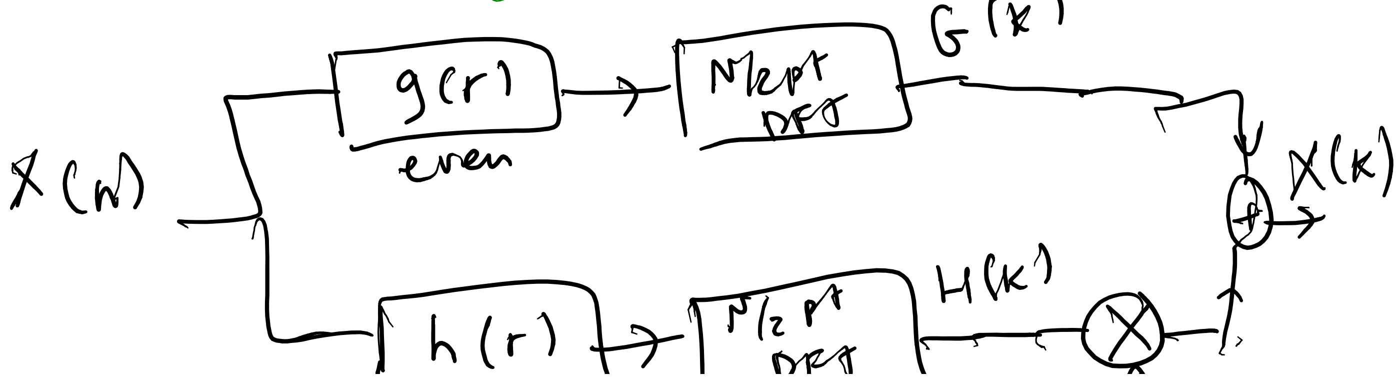
$n = 2r$ $\qquad\qquad$ $n = 2r+1$

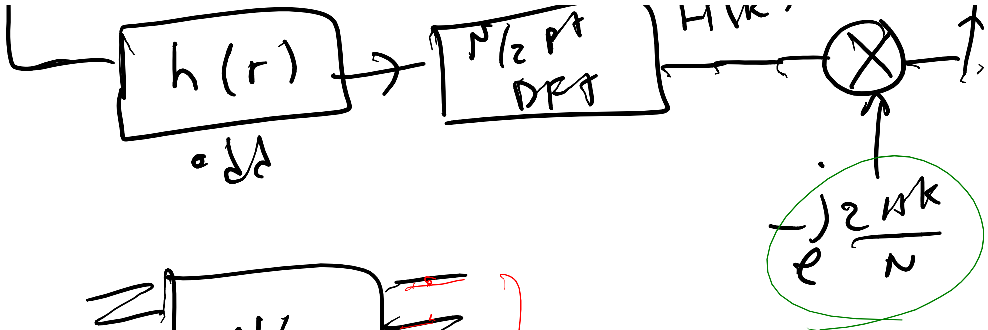
$r: 0 \rightarrow \frac{N}{2} - 1$ $\qquad\qquad$ $r = 0 \rightarrow \frac{N}{2} - 1$

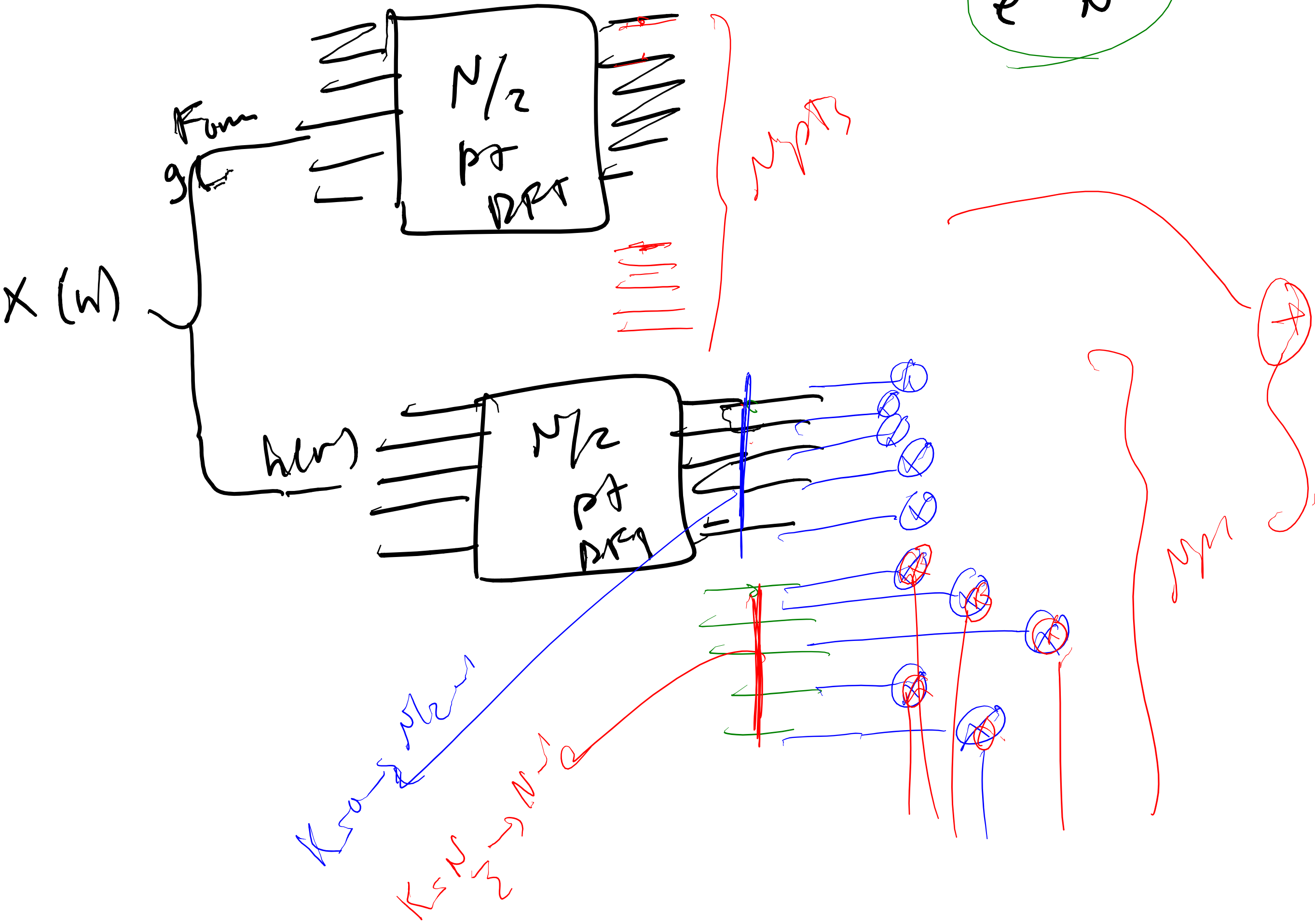
$$= \sum_{r=0}^{\frac{N}{2}-1} \underbrace{x(2r)}_{g(r)}\, e^{-j\frac{2\pi 2rk}{N}} + \sum_{r=0}^{N/2-1} \underbrace{x(2r+1)}_{h(r)}\, e^{-j\frac{2\pi k(2r+1)}{N}}$$

$$X(k) = \underbrace{\sum_{r=0}^{N/2-1} g(r)\, e^{-j\frac{2\pi k r}{N/2}}}_{N/2 \text{ pt DFT of } N/2 \text{ pt seq } g(r)} + \underbrace{e^{-j\frac{2\pi k}{N}}}_{\text{Twiddle}} \underbrace{\sum_{r=0}^{N/2-1} h(r)\, e^{-j\frac{2\pi k r}{N/2}}}_{N/2 \text{ pt DFT of } N/2 \text{ pt seq } h(r)}$$

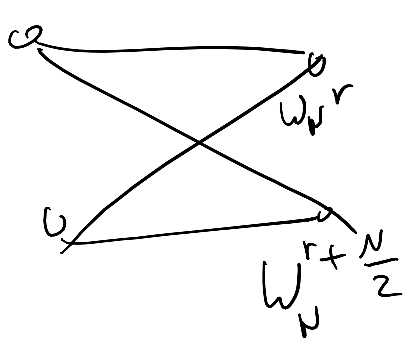




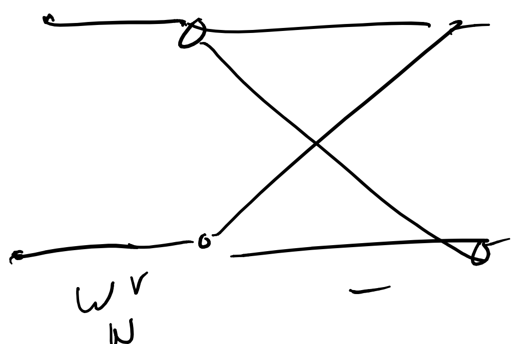




W_N^K Twiddle factor



with



W_N – W_N –

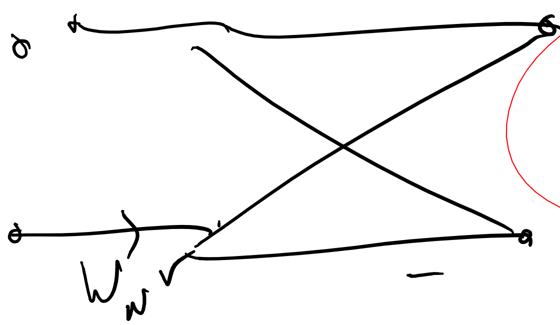
2 mults 2 adds   1 mult 2 adds

Operation Count

- # of stages $\log_2 N$
- Each stage: N inputs, N outputs

  $N/2$ butterflies

  Each butterfly has 2 inputs, 2 outputs



1 mult, 2 adds

$\frac{N}{2}$ mults, N adds

Total: $\frac{N}{2}\log_2 N$ mults, $N\log_2 N$ adds

Decimation in Frequency FFT

$$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi n k}{N}} \qquad k \rightarrow 2r$$

(1) k even $\quad k = 2r \qquad -j\frac{2\pi n r}{N/2}$

$N-1$

① k even

$$X(2r) = \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi n r}{N/2}}$$

$$= \sum_{n=0}^{N/2-1} x(n)\, e^{-j\frac{2\pi n r}{N/2}} + \sum_{n=\frac{N}{2}}^{N-1} x(n)\, e^{-j\frac{2\pi n r}{N/2}}$$

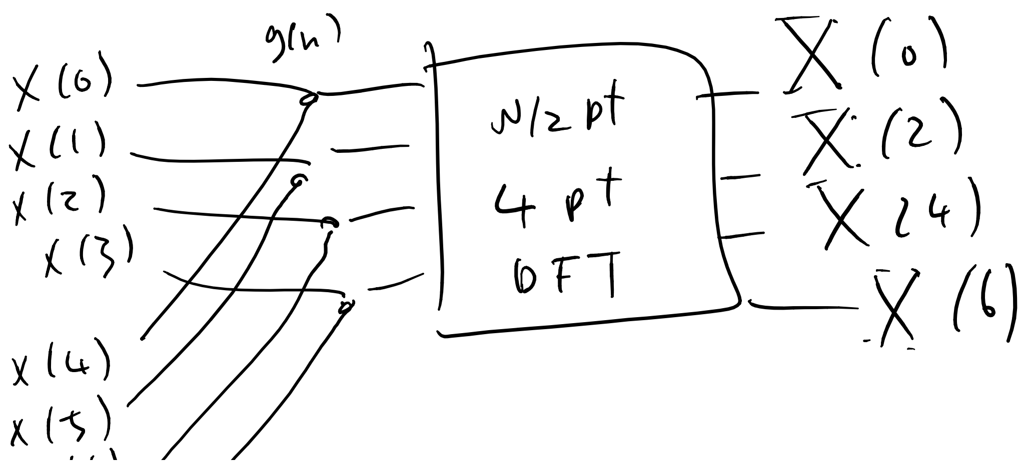
$n = m + \frac{N}{2}$

$$= \underbrace{\qquad}_{\downarrow} + \sum_{m=0}^{N/2-1} x\left(m+\frac{N}{2}\right) e^{-j\frac{2\pi r}{N/2}\left(m+\frac{N}{2}\right)}$$

$$= \sum_{n=0}^{N/2-1} x(n)\, e^{-j\frac{2\pi n r}{N/2}} + \sum_{n=0}^{N/2-1} x\left(n+\frac{N}{2}\right) e^{-j\frac{2\pi r n}{N/2}}$$

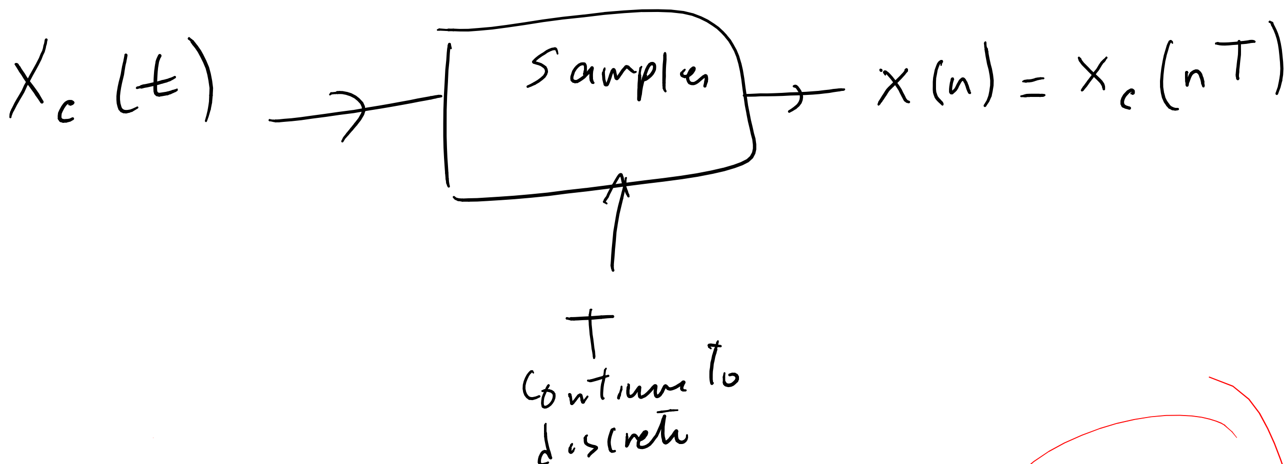
$$X(2r) = \sum_{n=0}^{N/2-1} \underbrace{\left[x(n) + x\left(n+\frac{N}{2}\right)\right]}_{g(n)} e^{-j\frac{2\pi r n}{N/2}} \qquad 0 \le r < N/2$$

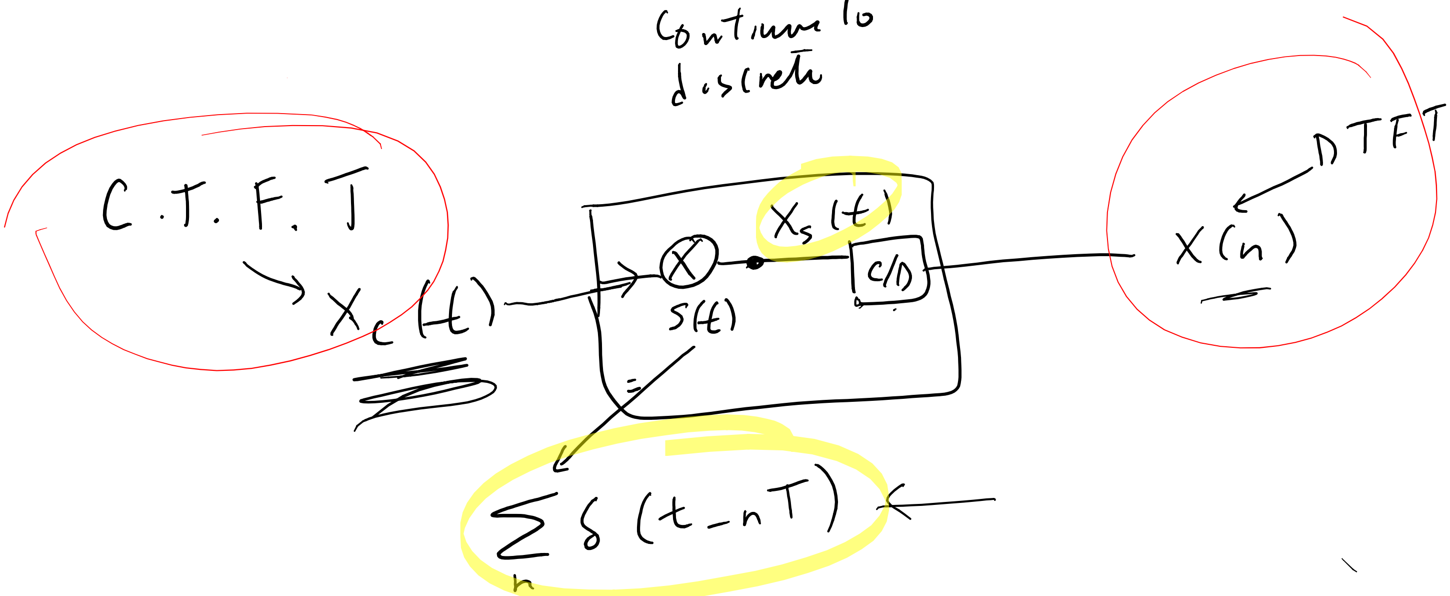
→ N/2 pt DFT of g(n)



$x(4)$
$x(5)$
$x(6)$
$x(7)$

Sampling . Chap 4 O&S





C.T.F.T $x_c(t)$

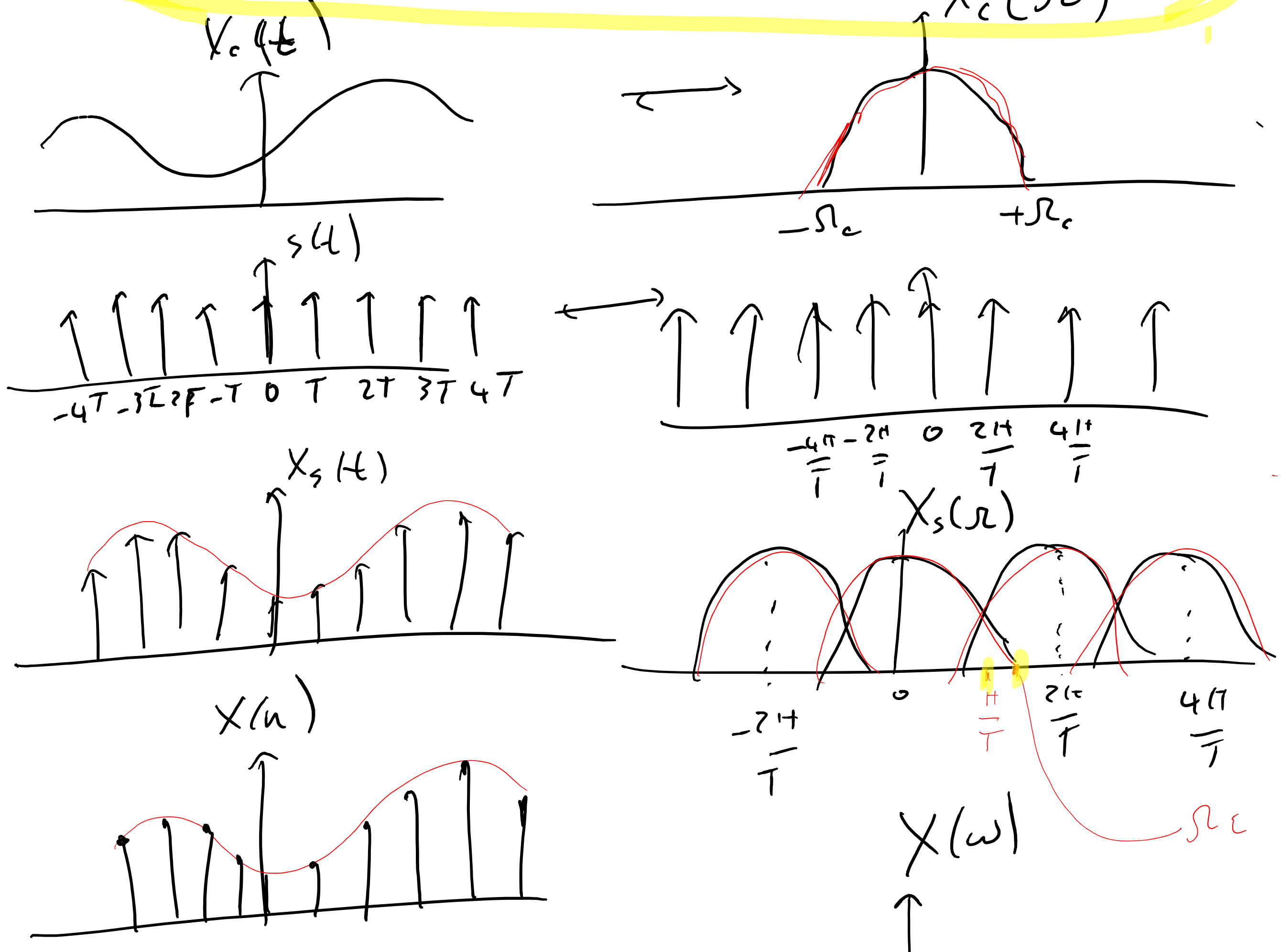
$$X_c(\Omega) = C.TFT\{x_c(t)\} = \int_{-\infty}^{+\infty} x_c(t)\, e^{-j\Omega t}\, dt$$

$X_c(j\Omega) = \ldots \int \ldots$ $-\infty$

$$X(\omega) = \text{D.T.F.T.}\{x(n)\} = \sum_{n=-\infty}^{+\infty} x(n)\, e^{-j\omega n}$$

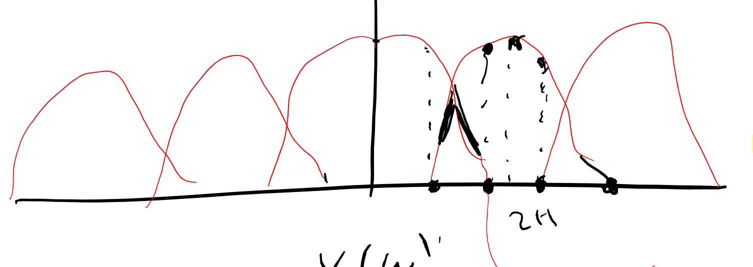
Can be shown:

$$X(\omega) = \frac{1}{T} \sum_{k=-\infty}^{+\infty} X_c\left(\frac{\omega - 2\pi k}{T}\right)$$



If $\Omega_c < \frac{\pi}{T} \rightarrow$ no aliasing

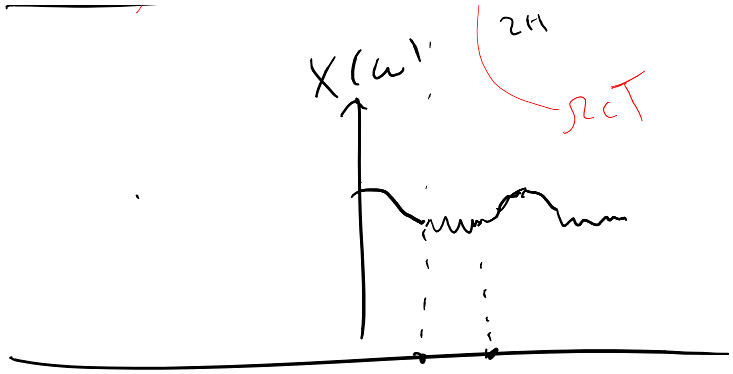
$\Rightarrow$ perfectly recover

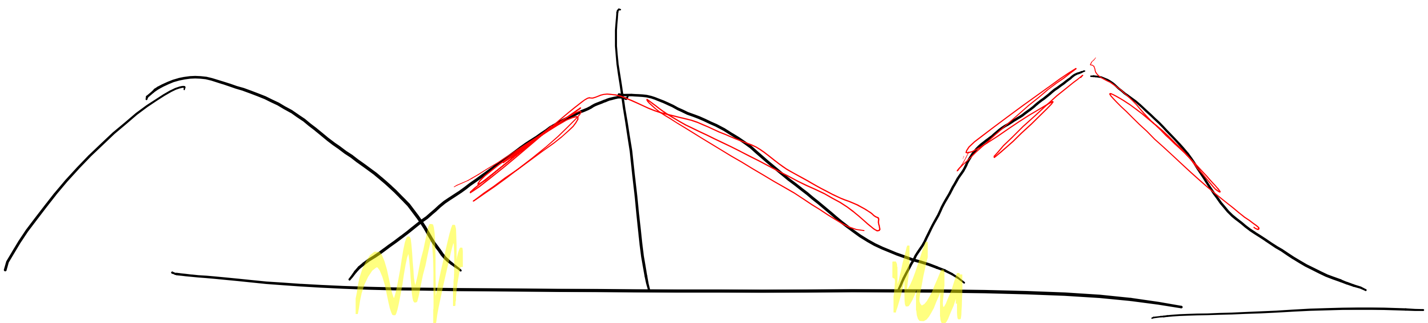


$X(\omega)$

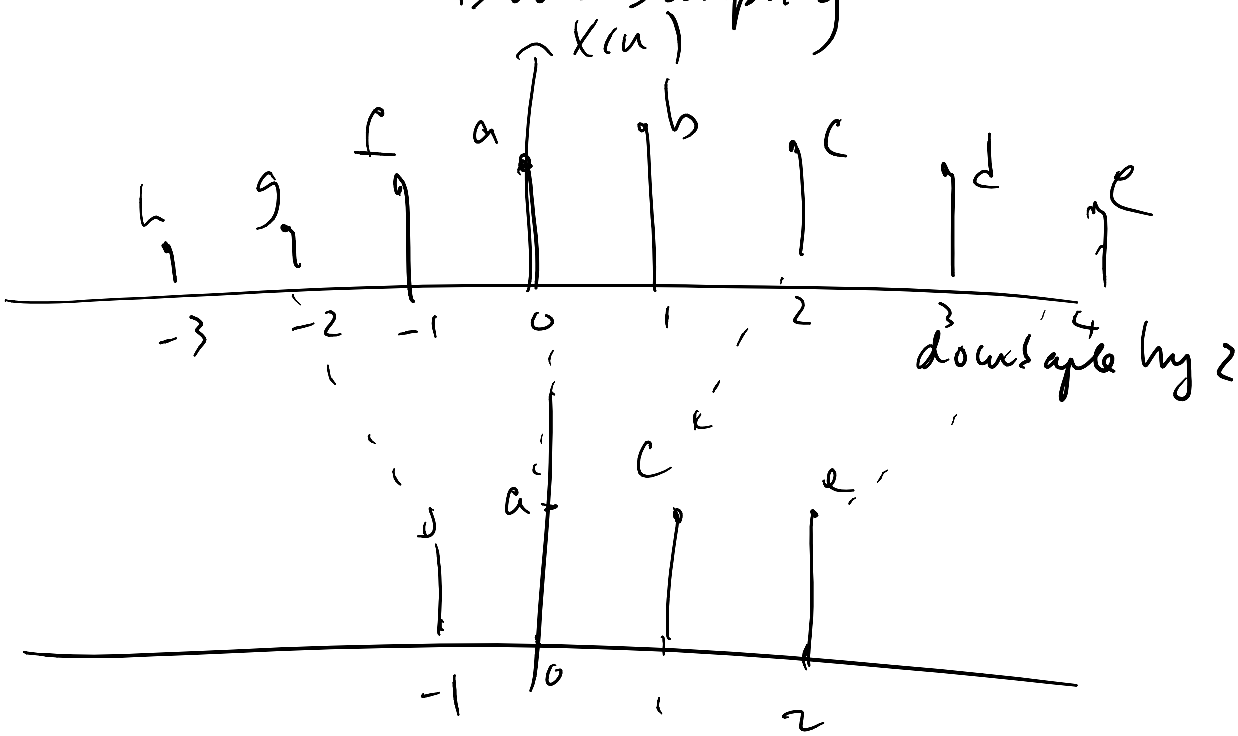
$\Rightarrow$ perfectly recover $x_c(t)$ from $x(n)$

- If $\Omega_c > \frac{\pi}{T}$ $\rightarrow$ aliasing





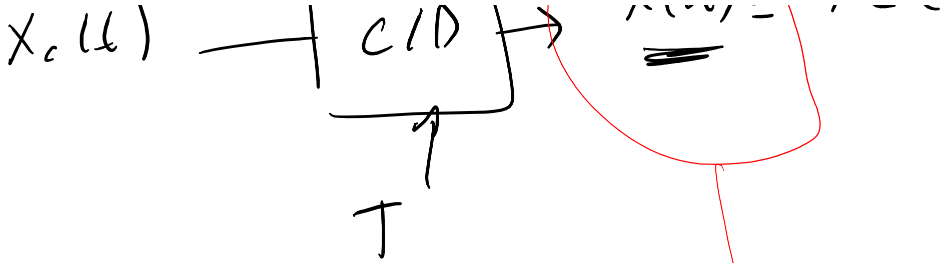
Downsampling



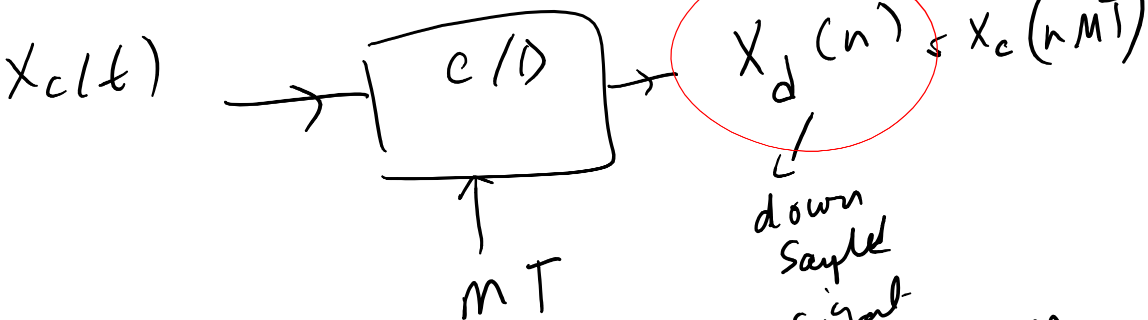
$x_c(t)$

① $x_c(t) \rightarrow \boxed{C/D} \rightarrow \underline{x(n)} = x_c(nT)$

①



②



down sampled signal by factor M

$$X(\omega) = DTFT\{x(n)\} = \frac{1}{T}\sum_{k=-\infty}^{+\infty} X_c\left(\frac{\omega}{T} - \frac{2\pi k}{T}\right)$$

$$X_d(\omega) = DTFT\{x_d(n)\} = \frac{1}{MT}\sum_{r=-\infty}^{+\infty} X_c\left(\frac{\omega}{MT} - \frac{2\pi r}{MT}\right)$$

change of variable $\quad r = i + kM$

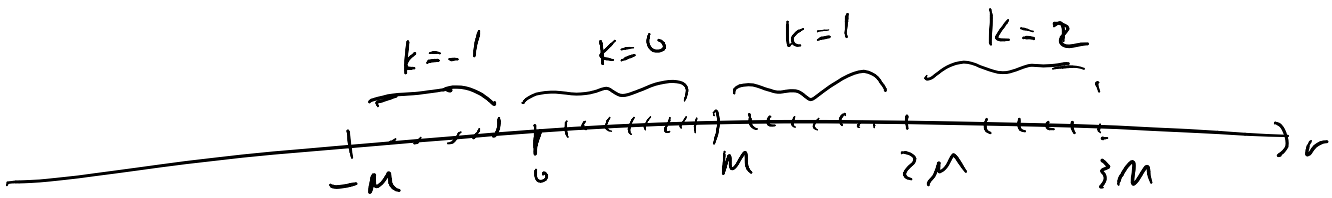
$+\infty < k < +\infty \qquad 0 \to M-1 \qquad +\infty$

$0 \leq i \leq M-1$

$\left.\begin{array}{l} k=0 \\ i: 0 \to M-1 \end{array}\right\} \longrightarrow r: \; 0 \to M-1$

$\left.\begin{array}{l} k=1 \\ i=0 \to M-1 \end{array}\right\} \longrightarrow r: \; M \to 2M-1$

$\left.\begin{array}{l} k=2 \\ i=0 \to M-1 \end{array}\right\} \longrightarrow r: \; 2M \to 3M-1$



$$X_d(\omega) = \frac{1}{M}\sum_{i=0}^{M-1}\frac{1}{T}\sum_{k=-\infty}^{+\infty} X_c\left(\frac{\omega}{MT} - \frac{2\pi k}{T} - \frac{2\pi i}{MT}\right)$$

$$X_d(\omega) = \frac{1}{M}\sum_{i=0}^{M-1}\underbrace{\frac{1}{T}\sum_{k=-\infty}^{+\infty} X_c\left(\frac{\omega-2\pi i}{MT} - \frac{2\pi k}{T}\right)}_{[X(\omega)]}$$

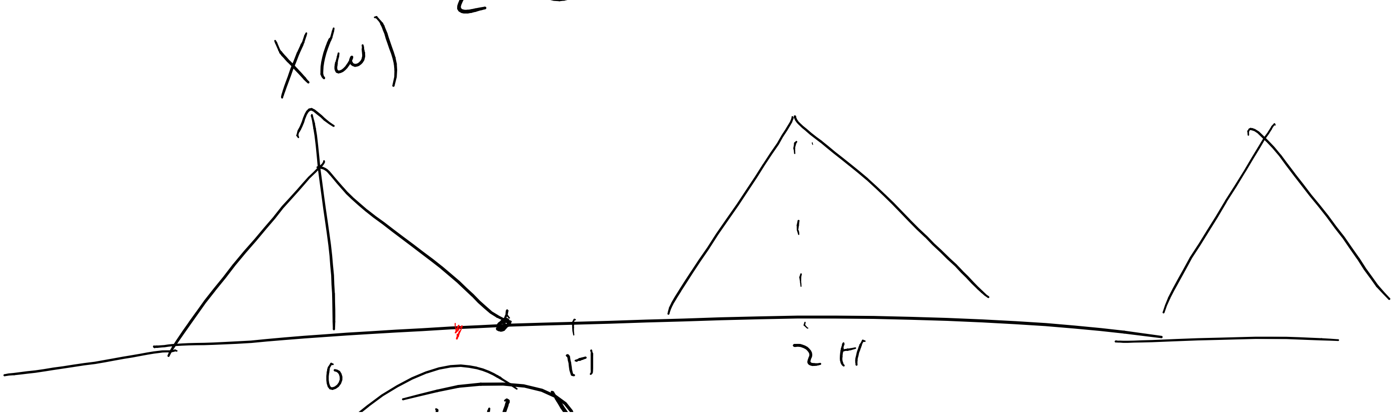
$$\omega \leftarrow \frac{\omega-2\pi i}{M}$$

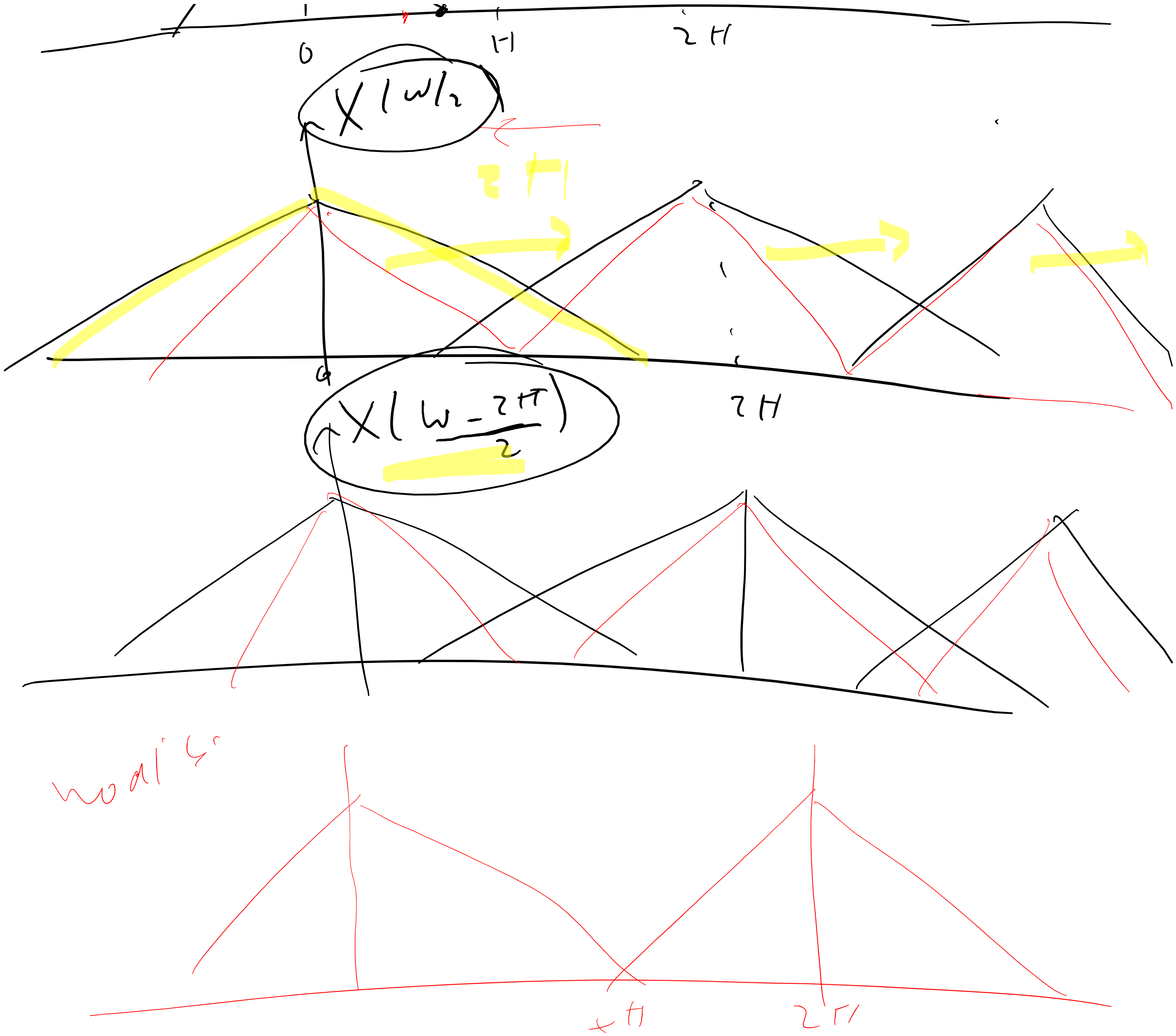
$$\boxed{X_d(\omega) = \frac{1}{M}\sum_{i=0}^{M-1} X\left(\frac{\omega-2\pi i}{M}\right)}$$

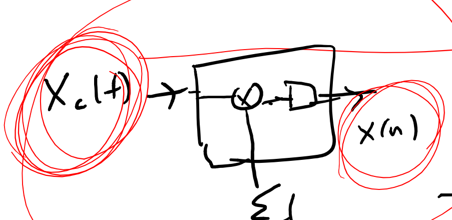
$M=2$

$$X_d(\omega) = \frac{1}{2}\sum_{i=0}^{1} X\left(\frac{\omega}{2} - \frac{2\pi i}{2}\right)$$

$$= \frac{1}{2}\left[X\left(\frac{\omega}{2}\right) + X\left(\frac{\omega}{2}-\pi\right)\right]$$







Nyquist-Shannon Thm

Let $x_c(t)$ be a bandlimited signal with C.T.F.T $X_c(\Omega) = 0$ for $|\Omega| > \Omega_N$

Then $x_c(t)$ is uniquely determined by its samples $x(n) = x_c(t)$ $n = 0, \pm 1, \pm 2, \pm 3 \ldots$

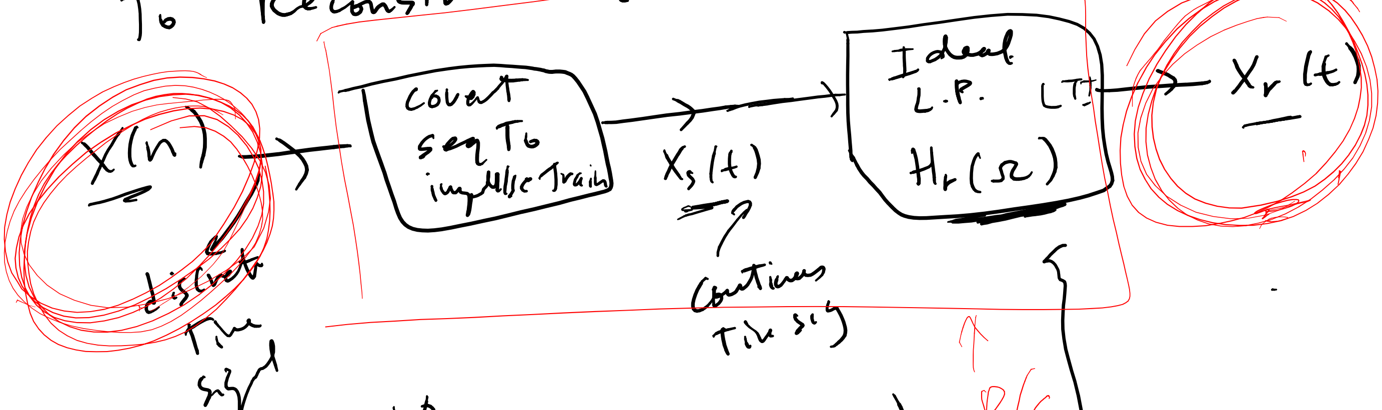
sampler $\quad x(n) = x_c(t) \quad n = 0, \pm 1, \pm 2, \pm 3 \dots$

If $\quad \Omega_s \triangleq \frac{2\pi}{T} \geq 2\,\Omega_N$

$$X_d(\omega) = \frac{1}{T} \sum_{k=-\infty}^{+\infty} X_c\left(\frac{\omega}{T} - \frac{2\pi k}{T}\right)$$

DTFT of $x(n)$ $\quad$ scaling $\quad$ periodic

To Reconstruct $x_c(t)$



$$x_s(t) = \sum_{n=-\infty}^{+\infty} x(n)\, \delta(t - nT)$$

$$x_r(t) = \sum_{n=-\infty}^{+\infty} x(n)\, h_r(t - nT)$$

impulse Resp

$$X_r(\Omega) = \sum_n x(n)\; \text{C.T.F.T.}\left\{ h_r(t - nT) \right\}$$

$$X_r(\Omega) = \sum_{n=-\infty}^{+\infty} x(n)\, H_r(\Omega)\, e^{-j\Omega T n}$$

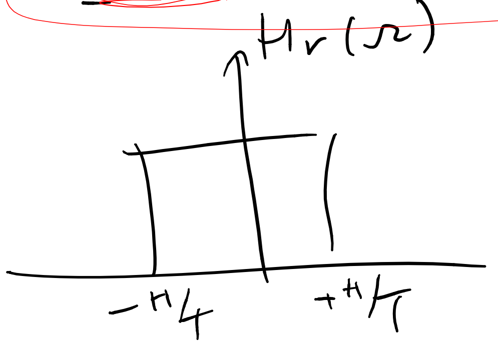
$$= H_r(\Omega) \sum_{n=-\infty}^{+\infty} x(n)\, e^{-j\Omega T n}$$

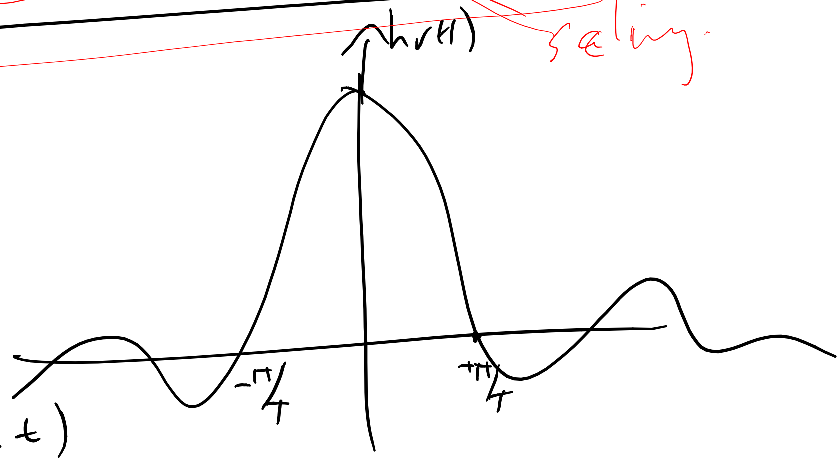
$$n=-\infty$$

$$\xleftarrow{\text{DTFT}} \left[X_d(\omega)\right]_{\omega=\Omega T}$$

$$\boxed{X_r(\Omega) = H_r(\Omega)\, X_d(\Omega T)}$$

scaling





$$h_r(t) = \frac{\sin\left(\frac{\pi}{T}t\right)}{\frac{\pi}{T}t}$$

$$x_r(t) = \sum_{n=-\infty}^{\infty} x(n) \frac{\sin\left[\frac{\pi(t-nT)}{T}\right]}{\frac{\pi(t-nT)}{T}}$$