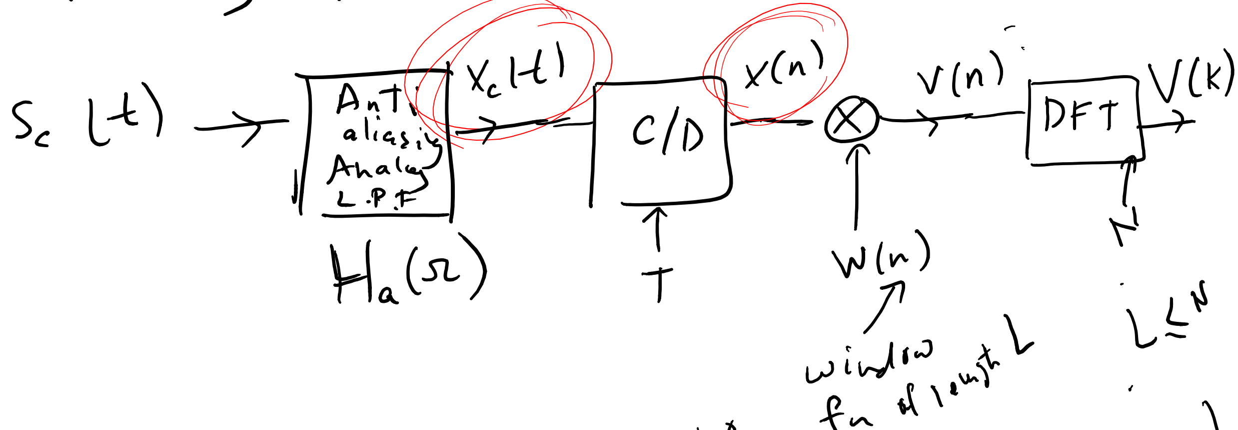
Fourier Analysis of Signals using DFT

Applying DFT to Continuous time signal.



$$X(\omega) = \text{DTFT}\{x(n)\} = \frac{1}{T}\sum_{r=-\infty}^{+\infty} X_c\left(\frac{\omega}{T} + \frac{2\pi r}{T}\right)$$

$$v(n) = w(n)\,x(n)$$

$$V(\omega) = W * X = \frac{1}{2\pi}\int_{-\pi}^{+\pi} X(\theta)\, W(\omega-\theta)\, d\omega$$

$$V(k) = \sum_{n=0}^{N-1} v(n)\, e^{-j\frac{2\pi n k}{N}}$$

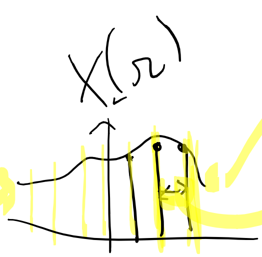
$$V(k) = \left[V(\omega)\right]_{\omega = \frac{2\pi k}{N}}$$

sampling ω every $\frac{2\pi}{N}$

$$\omega = \Omega T$$

Sampling Ω domain every $\frac{2\pi}{NT}$

$V(k)$ $\leftarrow$ $\Omega_K = \frac{2\pi K}{NT}$ $\rightarrow$ Kth sample



Kth sample

EX $X_c(t)$ bandlimited $X_c(\Omega) = 0$ for $|\Omega| > 2\pi \cdot 2500$

- Assume ideal antialiasing
- sample rate Nyquist $\Rightarrow \frac{1}{T} = 5000$ sample/sec.
- Want DFT sample of $V(k)$ To represent $X_c(\Omega)$ at 10 Hz every $2\pi \cdot 10$ rad/s

$$\left.\begin{array}{l} \Delta\Omega = \frac{2\pi}{NT} = 2\pi \cdot 10 = \frac{2\pi}{NT} \\ T = \frac{1}{5000} \end{array}\right\} \Rightarrow$$

$$2\pi \cdot 10 = \frac{2\pi}{N \cdot \frac{1}{5000}} \Rightarrow$$

$$N = 500 \Longrightarrow N' = 512$$

DFT Analysis of Sinusoidal Signals

$$S_c(t) = A_0 \cos(\Omega_0 t + \theta_0) + A_1 \cos(\Omega_1 t + \theta_1)$$

Ideal Sampling

$$X(n) = A_0 \cos(w_0 n + \theta_0) + A_1 \cos(w_1 n + \theta_1)$$

$$w_0 = \Omega_0 T \qquad w_1 = \Omega_1 T$$

$$v(n) = X(n)\, w(n)$$

$$\ldots(w_0 n + \theta_0) + A_1 w(n) \cos(w_1 n + \ldots)$$

$v(n) = x(n)\, w(n)$

$v(n) = A_0\, w(n) \cos(\omega_0 n + \theta_0) + A_1\, w(n) \cos(\omega_1 n + \theta_1)$

$= \frac{A_0}{2} w(n) e^{j\theta_0} e^{j\omega_0 n} + \frac{A_0}{2} w(n) e^{-j\theta_0} e^{-j\omega_0 n}$

$+ \frac{A_1}{2} w(n) e^{j\theta_1} e^{j\omega_1 n} + \frac{A_1}{2} w(n) e^{-j\theta_1} e^{-j\omega_1 n}$

$V(\omega) = \frac{A_0}{2} e^{j\theta_0} W(\omega - \omega_0) + \frac{A_0}{2} e^{-j\theta_0} W(\omega + \omega_0)$

$+ \frac{A_1}{2} e^{j\theta_1} W(\omega - \omega_1) + \frac{A_1}{2} e^{-j\theta_1} W(\omega + \omega_1)$

Ex $\frac{1}{T} = 16\,\text{KHZ}$ $w(n)$ has length

$L = 64$

$A_0 = 1$ $A_1 = 0.75$ $\theta_0 = \theta_1 = 0$

Properties of Windows

Resolution: Width of the main lobe

Leakage: relative amplitude of main lobe to side lobe

Rectangular $w(n) = \begin{cases} 1 & 0 \le n \le M \\ & \text{elsewhere} \end{cases}$

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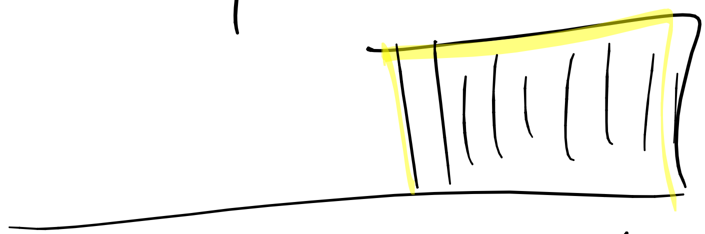
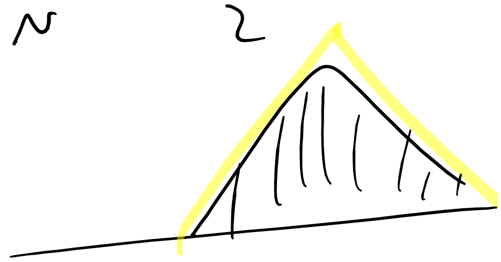
$L = M+1$

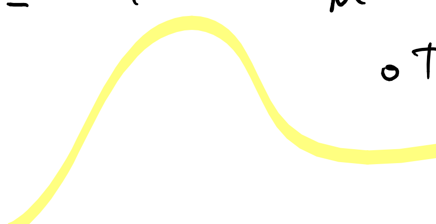


Bartlett $w(n) = \begin{cases} \dfrac{2n}{M} & 0 \le n \le M/2 \\ 2 - \dfrac{2n}{M} & \dfrac{M}{2} \le n \le M \end{cases}$ $L = M+1$



Hamming Window

$w(n) = \begin{cases} 0.54 - 0.46 \cos\left(\dfrac{2\pi n}{M}\right) & 0 \le n \le M \\ 0 & \text{otherwise} \end{cases}$



Blackman Window

$w(n) = \begin{cases} 0.42 - 0.5 \cos\left(\dfrac{2\pi n}{M}\right) + 0.08 \cos\left(\dfrac{4\pi n}{M}\right) & 0 \le n \le M \\ 0 & \text{otherwise} \end{cases}$

Kaiser Window

$w(n) = \begin{cases} \dfrac{I_0\left[\beta \sqrt{1 - \left[\dfrac{n-\alpha}{\alpha}\right]^2}\right]}{I_0(\beta)} & 0 \le n \le M \\ 0 & \text{otherwise} \end{cases}$

$I_0(\beta)$

0 otherwise

$\alpha = \frac{M}{2}$

$M+1 = L$ = length of window

$I_0(.)$ is zeroth order modified Bessel fn of first kind.

$$I_0(x) = 1 + \frac{x^2}{2^2(1!)^2} + \frac{x^4}{2^4(2!)^2} + \frac{x^6}{2^6(3!)^2} + \cdots$$

$I_n(x)$ nth order modified Bessel fn of 1st kind

soln to $$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - (x^2+n^2)y = 0$$

$\beta \longrightarrow$ affects shape of Kaiser window

Trade off between mainlobe width against side lobe amplitude using β.

A_{sl} = ratio in dB of amplitude of main lobe to the amplitude of side lobe

$A_{sl} < 13.26$

$$\beta = \begin{cases} 0 \\ 0.76609(A_{sl} - 13.26)^{0.4} + 0.09834(A_{sl} - 13.26) \end{cases}$$

$13.26 < A_{sl} < 60$

$(A_{sl} = 13.26)$

$13.26 < A_{sl} < 60$

$0.2438(A_{sl} + 6.3) \qquad 60dB \leq A_{sl} \leq 120$

$\Delta m\ell$ = main lobe width

$$L \simeq \frac{24\pi (A_{sl} + 12)}{155 \, \Delta m\ell} + 1$$

$A_{sl} \longrightarrow \beta$

Time dependent Fourier Transform

Short Term Fourier Transform STFT

long window $\longrightarrow$ can resolve Two close frequencies

But no good in detecting point of change.

Freq. Content

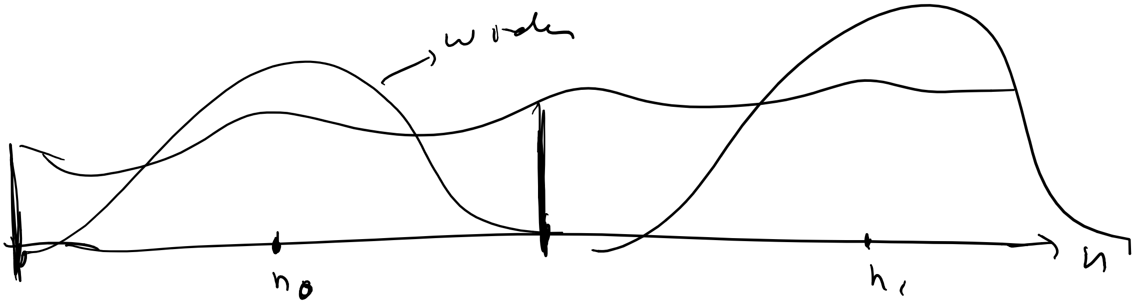
– Real life signal vary over Time –

$\Longrightarrow$ Non-stationary.

$x(n)$. S.T.F.T

$$\underline{X(n,\omega)} = \sum_{m=-\infty}^{+\infty} x(n+m)\, w(m)\, e^{-j\omega m}$$

$$X(n,\omega) = \sum_{m=-\infty}$$



Ex $\quad x_c(t) = \cos(\theta(t)) = \cos(A_0 t^2)$ linear chirp signal

Instanteous freq $= \Omega_i(t) = \frac{d}{dt}(\theta(t))$

$= 2A_0 t$

Time

$$x(n) = x_c(nT) = \cos(A_0 T^2 n^2) = \cos(\alpha_0 n^2)$$

$$\alpha_0 = A_0 T^2$$

Inst. freq $\quad \omega_i(n) = 2\alpha_0 n \longrightarrow$ linear in n

Sampling in Time and in Freq

$$X(n,k) = \sum_{m=0}^{L-1} x(n+m)\, w(m)\, e^{-j\frac{2\pi k m}{N}} \qquad 0 \le k < N$$

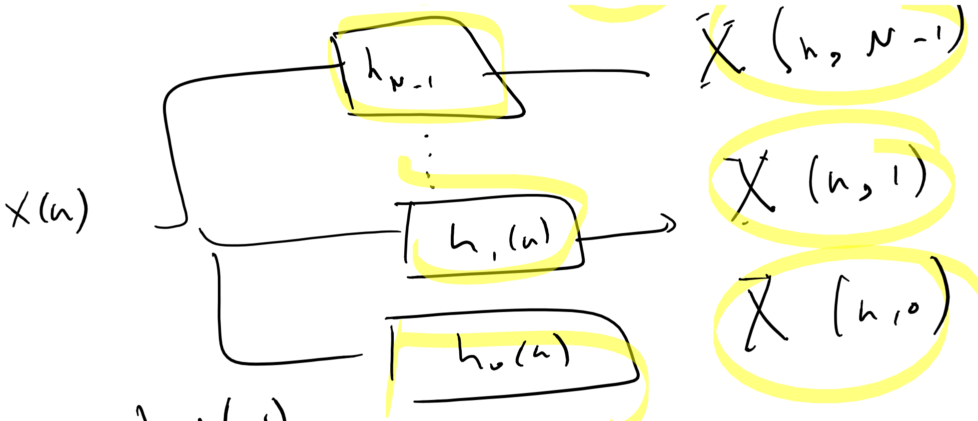
$$= x(n) * h_k(n)$$

$$h_k(n) = w(-n)\, e^{j\frac{2\pi k n}{N}}$$

$$H_k(\omega) = W\left(\frac{2\pi k}{N} - \omega\right)$$

$k = 0, \dots, N-1$

h_{N-1} $\quad X(n, N-1)$



$W(\omega)$

$W\left(\frac{2\pi}{N} - \omega\right)$ . . . .

$W\left(\frac{4\pi}{N} - \omega\right)$