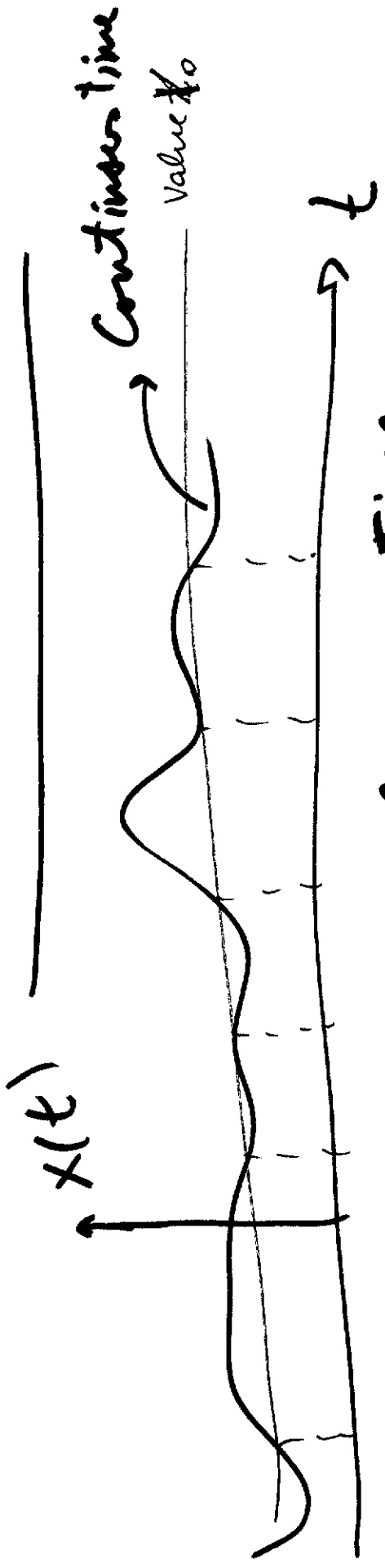
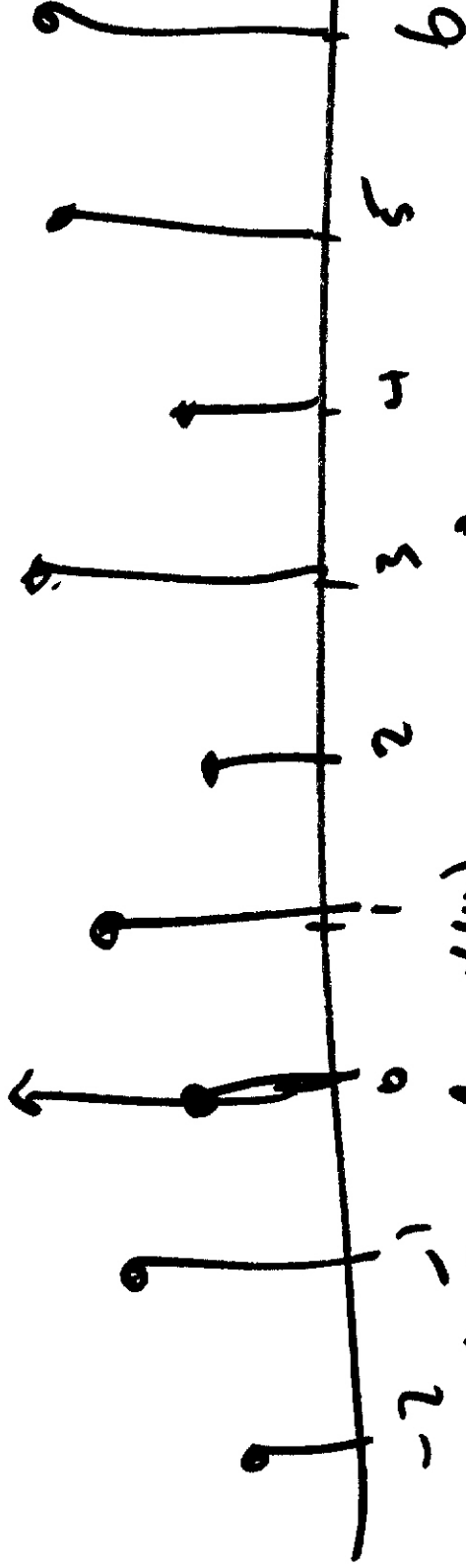


Digital Signal Processing (DSP)

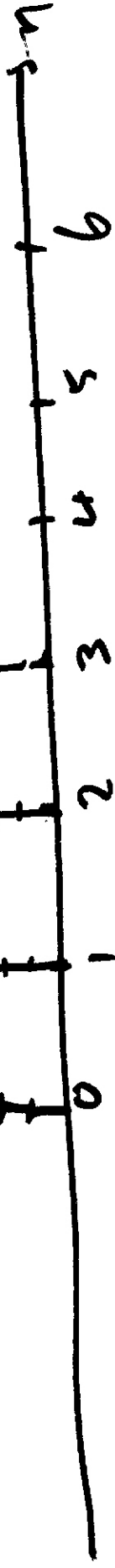


Discretize C.T. signal in Time



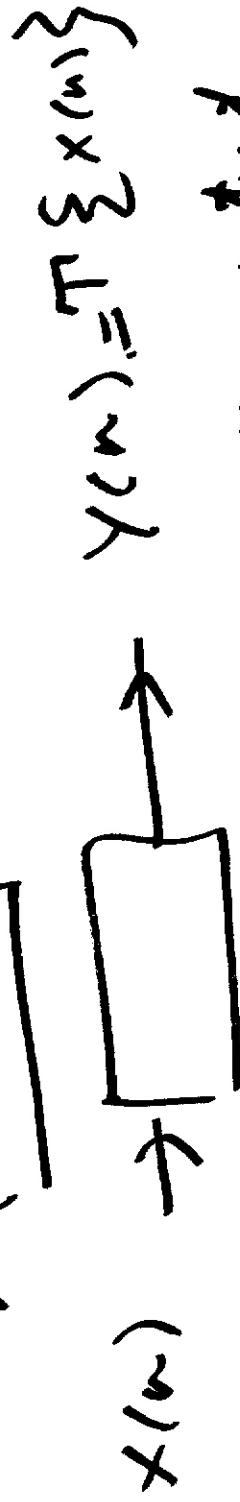
Digital signal $x(n)$

8 bits / 16 bits



System

Processes a signal.



unique input result in a unique output.

Linear Systems:

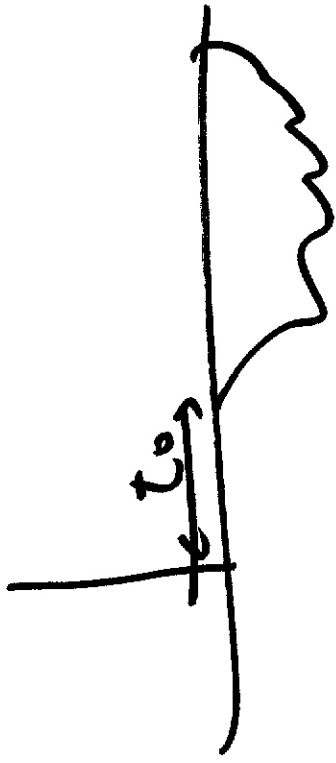
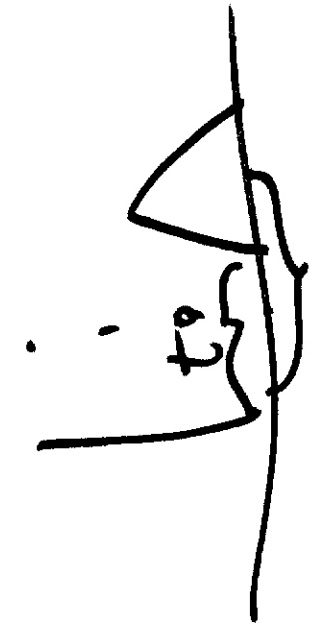
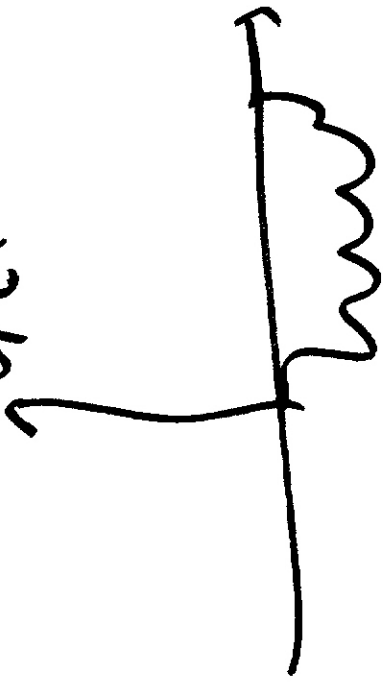
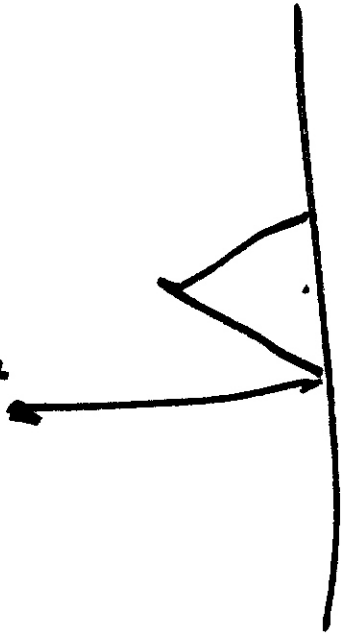
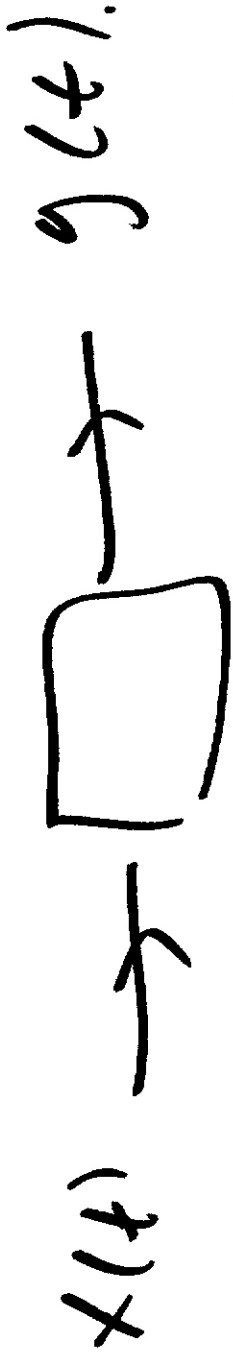


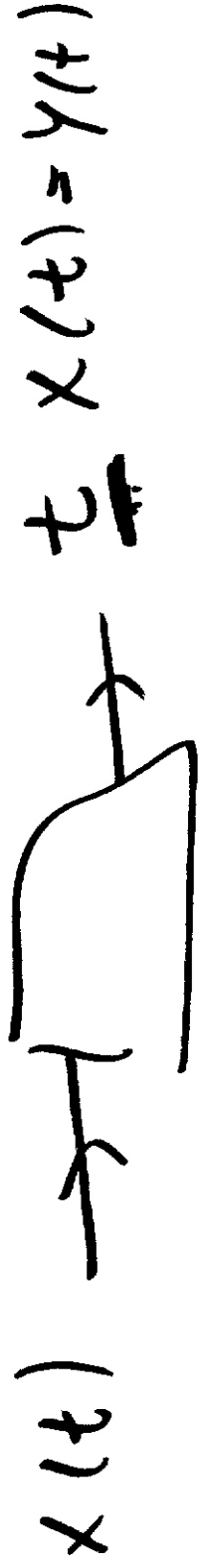
Linear:

$$T \{ a x(n) \} = a T \{ x(n) \}$$

$$x_1(n) \rightarrow y_1(n) \quad ; \quad x_2(n) \rightarrow y_2(n) \quad ; \quad x_1(n) + x_2(n) \rightarrow y_1(n) + y_2(n)$$

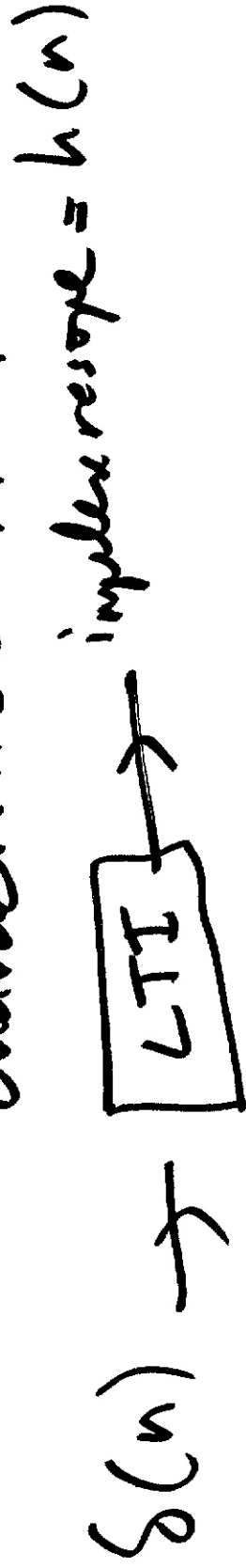
Time Invariance





Linearity + T.I. = LTI.

LTI \rightarrow Impulse response completely characterizes it.



$$y(n) = \sum_k h(k) x(n-k)$$

(convolution)

Ex 1) D of finite duration

finitely many non zero values.

FIR = Finite Impulse Response

② ∞ ly many non zero values.

IIR = Infinite Impulse Response


NOT

Ratio of polynomials

Rational Transfer function = $\frac{P(z)}{Q(z)}$ → Diff. Equation To implement this.

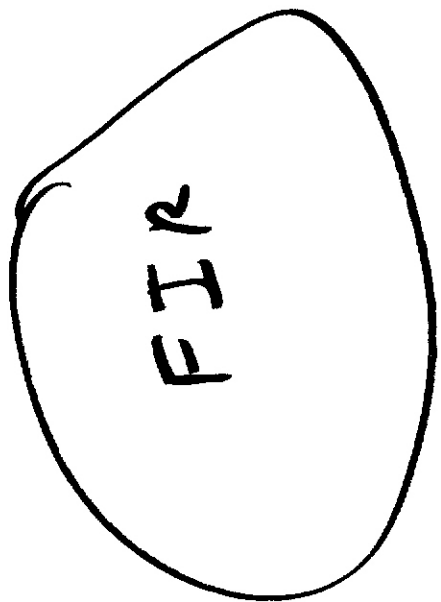
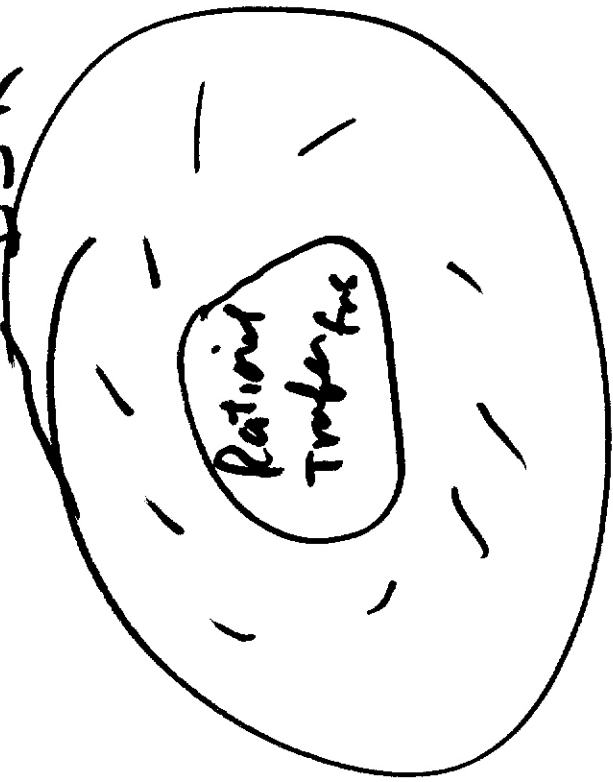
$$H(z) = \frac{Y(z)}{X(z)} = \frac{P(z)}{Q(z)}$$

Polynomial in z

a Rational Transfer function (in z) 

IIR.

IIR



LTI

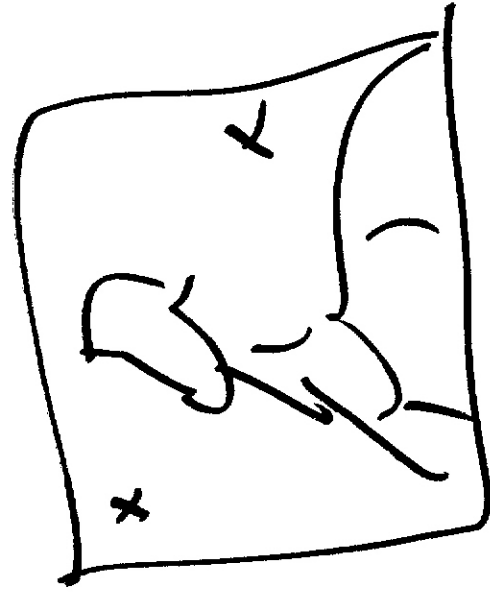
Causality:

LTI is causal

$\delta(n)$



$h(n) = 0 \quad n < 0$



imag.

Stability

BIBO = Bounded input
Bounded output.

• LTI system
BIBO stable

absolutely
summable
 $\sum_{n=-\infty}^{+\infty} |h(n)| < \infty$



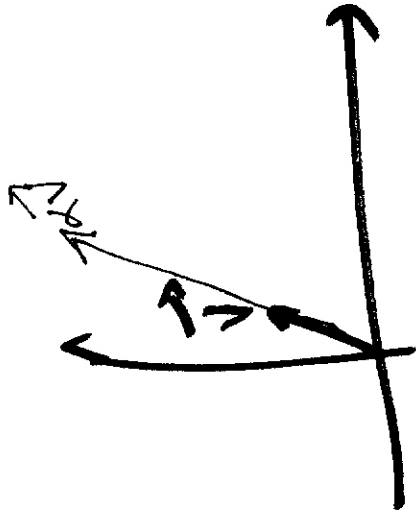
Eigen function

Eigen-value / eigen vector.

$A, \vec{v} :$

$$A\vec{v} = \alpha \vec{v}$$

$$A = [:::]$$



If \vec{v} is eigenvector

o matrix $A,$

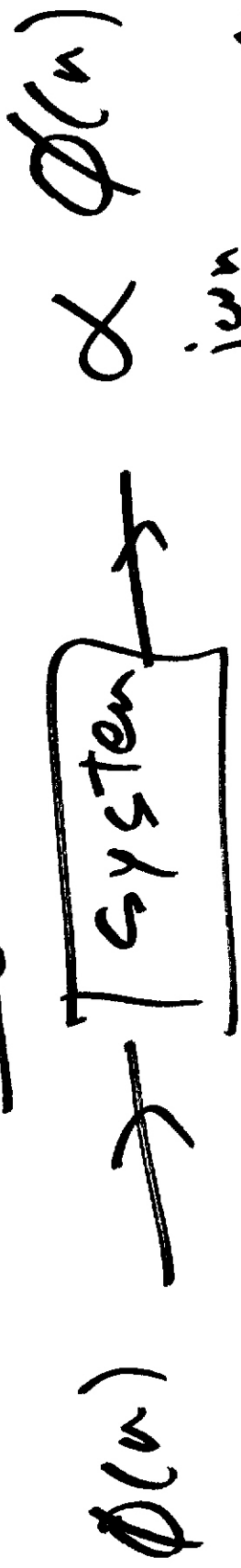
then applying

matrix to $\vec{v},$

does not change direction of it, only the magnitude.

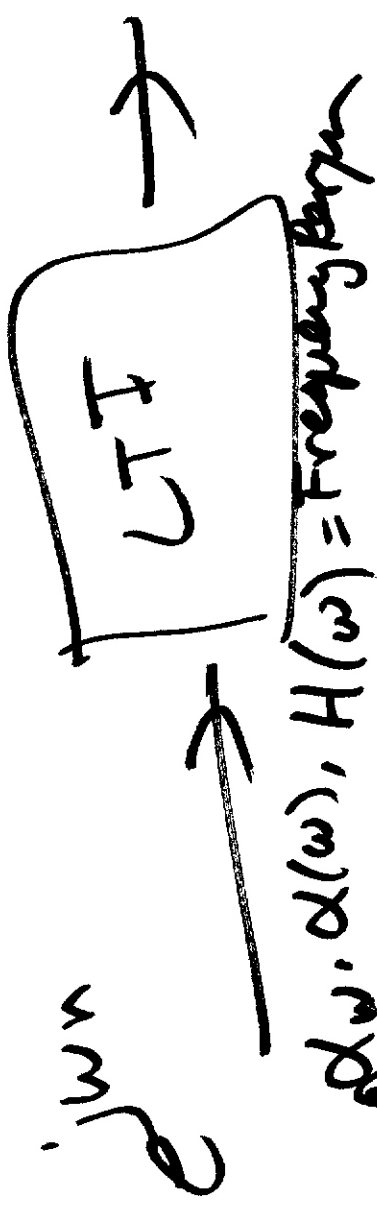
Eige

Eigenfunction

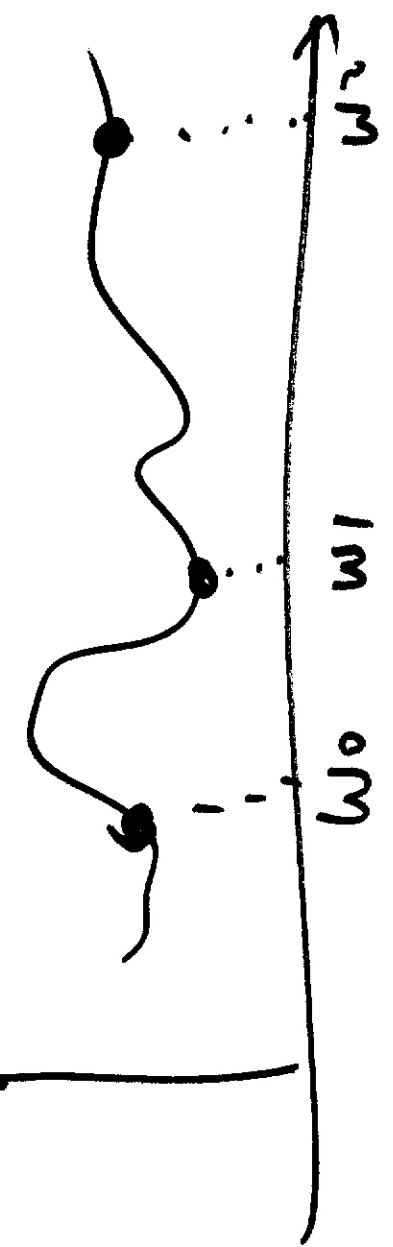


$e^{j\omega n}$ = complex exponential.

Show for LTI systems. $e^{j\omega n}$ is eigenfunction.



α function of ω .
 $\alpha(\omega) = H(\omega)$





$$y(n) = \sum_k e^{j\omega n} h(n-k) = \sum_m e^{j\omega(n-m)} h(m)$$

$$= e^{j\omega n} \sum_m e^{-j\omega m} h(m)$$

$$\alpha_{\omega} = \alpha(\omega) = H(\omega)$$

DTFT of $h(n)$

Frequency
Response for
LTI system

$$\text{DTFT} \{ h(n) \} = H(\omega) = \sum_n h(n) e^{-j\omega n}$$

$x(n)$:

$$\text{DTFT } \sum_{n} x(n) e^{-j\omega n} = \sum_{n} x(n) e^{-j\omega n}$$

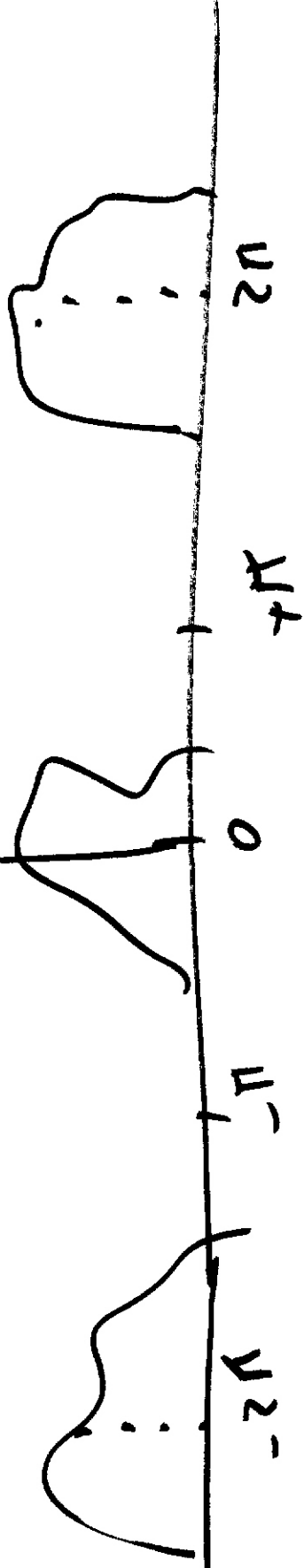
$n = \text{integer}$ $\omega = \text{continuous}$

$$\int_{-\pi}^{+\pi} X(\omega) e^{j\omega n} d\omega$$

$$x(n) = \frac{1}{2\pi}$$

\rightarrow periodic on 2π period

$X(\omega)$





$$y = x * h$$

$$Y(\omega) = X(\omega) H(\omega)$$

Parsavals Thm :

$$\int_{-\pi}^{\pi} X(\omega) Y(\omega) d\omega$$

$$\sum_n x(n) y^*(n) = \frac{1}{2\pi}$$

special case $x(n) = y(n)$

$$\int_{-\pi}^{\pi} |X(\omega)|^2 d\omega$$

$$\sum_n |x(n)|^2 = \frac{1}{2\pi}$$



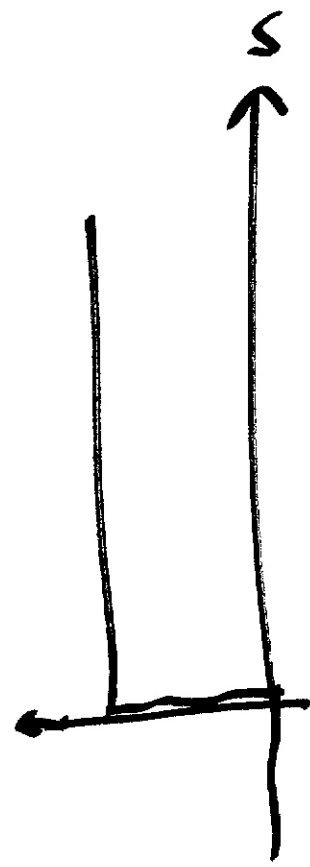
Ex

$$\delta(n) \rightarrow$$

$$x(n) = \left(\frac{1}{2}\right)^n u(n) \rightarrow \frac{1}{1 - \frac{1}{2} e^{-j\omega}}$$

$$\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n e^{-j\omega n} = X(\omega)$$

$$u(n) \quad 1 + x + x^2 + \dots = \frac{1}{1-x} \quad |x| < 1$$



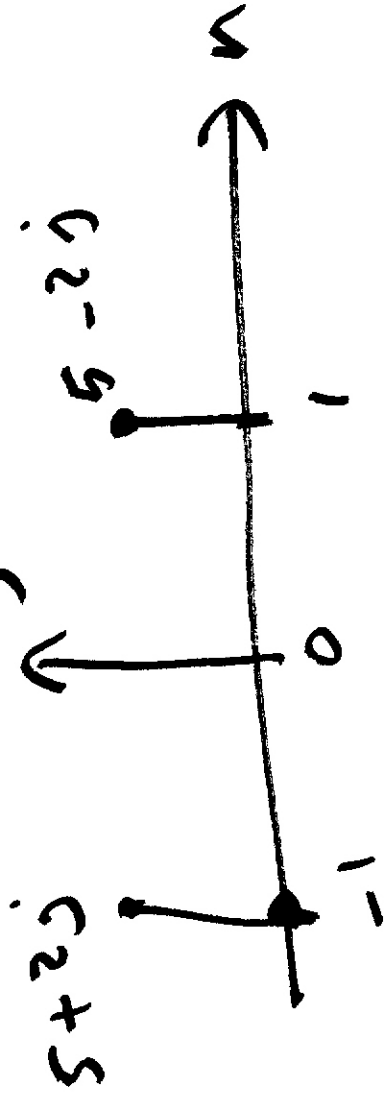
Ex : $z^n u(n) = x(n)$

$\sum_{n=-\infty}^{+\infty} x(n) e^{-j\omega n}$ might or might not converge.

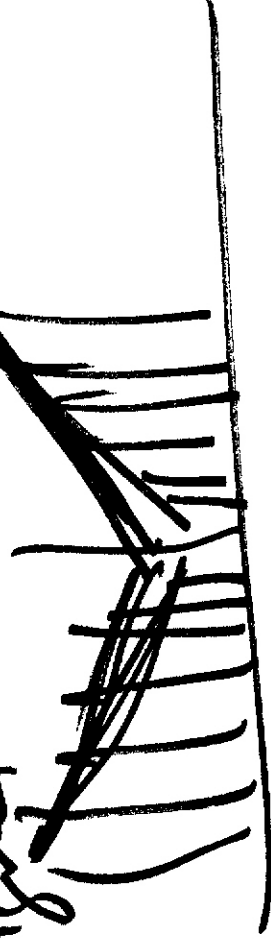
↳ DTFT for $z^n u(n)$ doesn't exist.

Symmetry Properties

Conjugate symmetric $y(n)$ $Y(n) = Y^*(-n)$

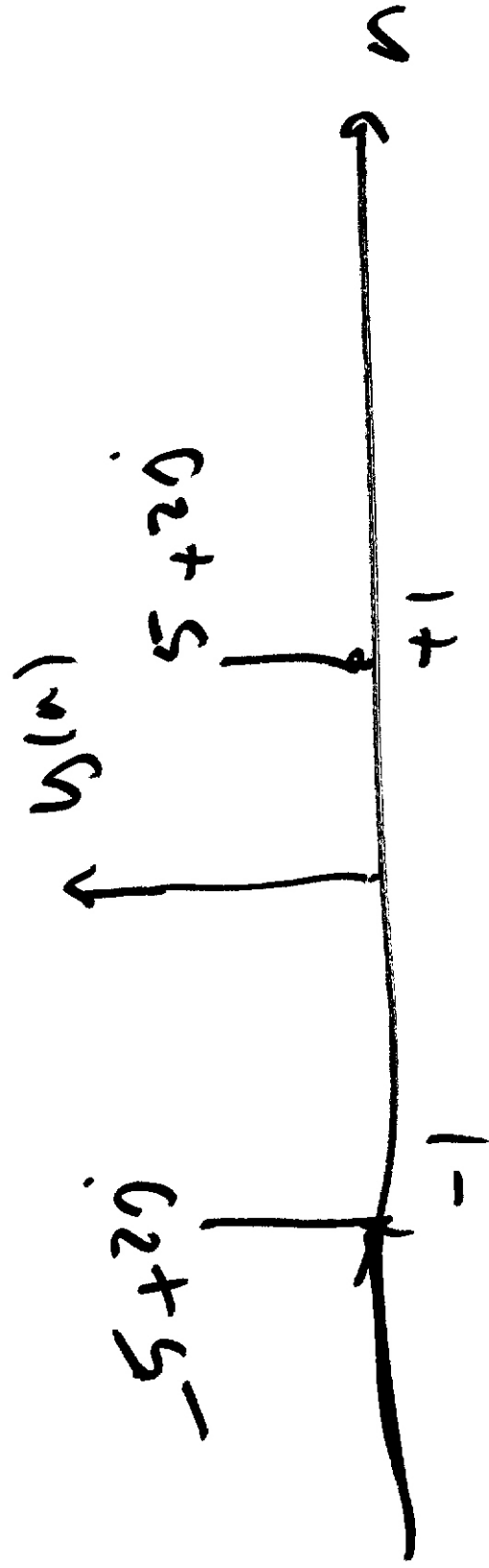


for real signals



Conjugate anti-symmetric:

$$Y(n) = -Y^*(-n)$$



Any signal $x(n)$ can be decomposed into
 sum of conjugate symmetric &
 conjugate anti-symmetric part.

$$\begin{aligned}
 X_e(n) &\triangleq \text{conjugate symmetric part of } x(n) \\
 &= \frac{1}{2} [x(n) + x^*(-n)] \quad \rightarrow \text{CS} \\
 &= \frac{1}{2} [x^*(-n) + x(n)]
 \end{aligned}$$

$X_0(n) \triangleq$ conjugate antisymmetric

part of $x(n)$ $*$ $x(-n)$

$$= \frac{1}{2} [x(n) - x^*(-n)]$$

~~$\frac{1}{2} [x(n) + x^*(-n)]$~~

$$X_e(n) + X_o(n) = x(n)$$

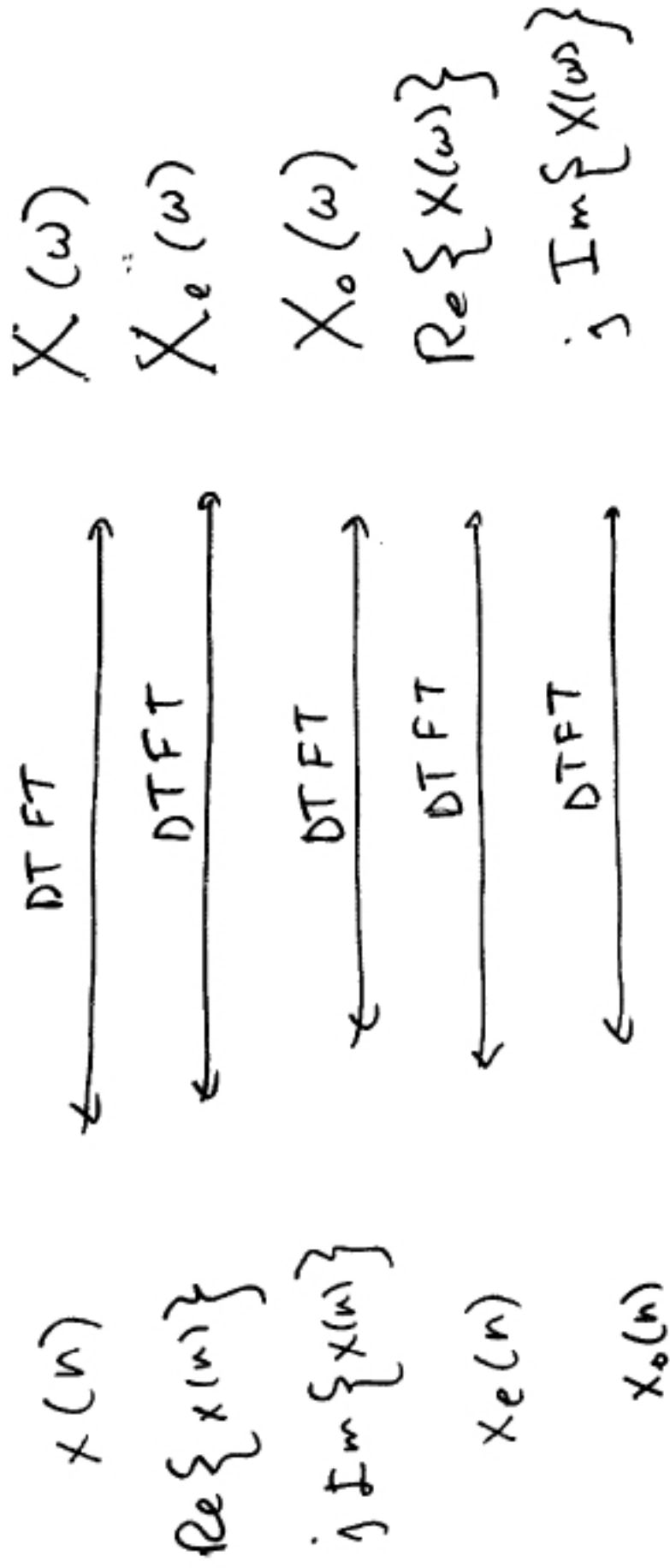
\downarrow
CS

$$X(\omega) = X_e(\omega) + X_o(\omega)$$

$$X_e(\omega) \stackrel{A}{=} CS = \frac{1}{2} [X(\omega) + X^*(-\omega)]$$

$$X_o(\omega) \stackrel{D}{=} CAS = \frac{1}{2} [X(\omega) - X^*(-\omega)]$$

Symmetry Properties



if $x(n)$ is real $\Rightarrow x(n) = \text{Re}\{x(n)\}$
 $X(\omega) = X_e^*(\omega) = \frac{1}{2} [X(\omega) + X^*(-\omega)]$
 $\Rightarrow X(\omega) = X^*(-\omega)$

If $x(t)$ real \Rightarrow F.T. is Conjugate Symmetric.

$$\text{Re} \{ X(\omega) \} = \text{Re} \{ X^*(-\omega) \}$$

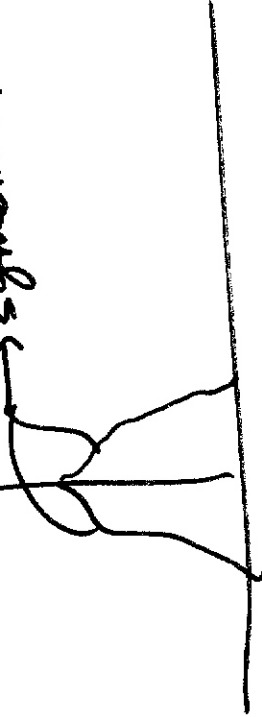
$$\Rightarrow \textcircled{1} X_R(\omega) = X_R(-\omega)$$

real part

\Rightarrow Real part of $X(\omega)$ is even.

$$X_R(\omega)$$

\rightarrow symmetric.

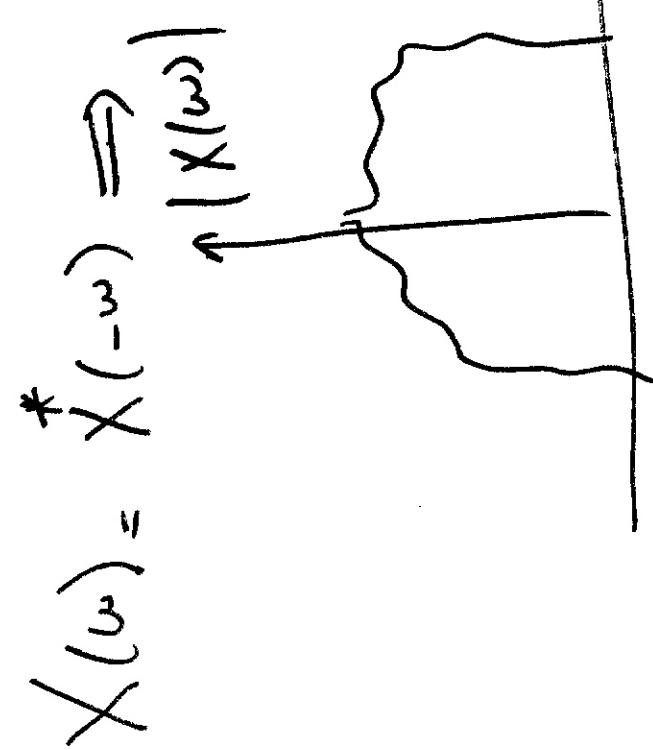
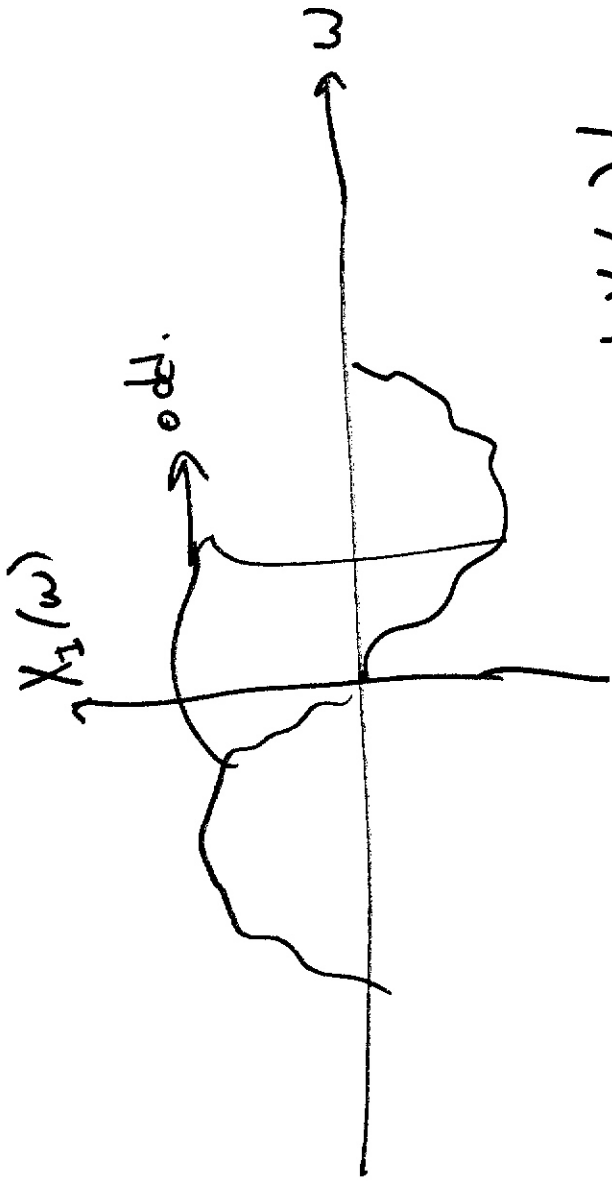


$$X_I(-\omega) = -X_I(\omega)$$

$$\Rightarrow \textcircled{2} X_I(\omega) = -X_I(-\omega)$$

imaginary is odd

Imaginary part of $X(\omega)$



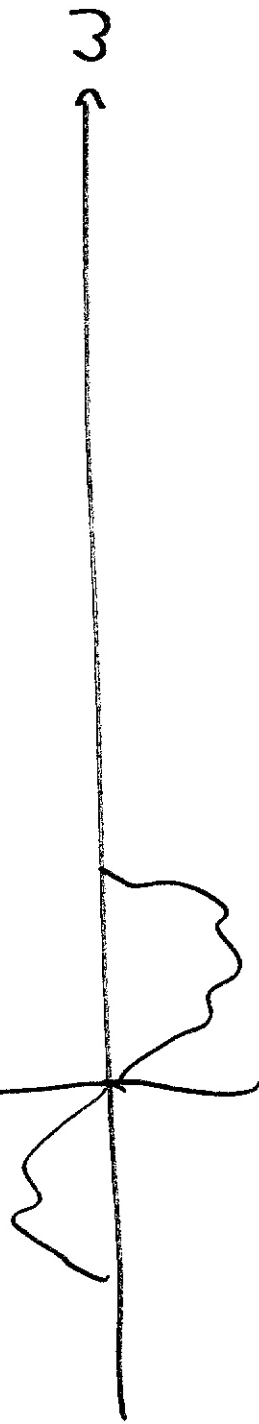
$$X(\omega) = X^*(-\omega) \Rightarrow |X(\omega)| = |X(-\omega)|$$

magn. of $X(\omega)$ is even.
for $x(n)$ real.

③

$$X(\omega) = X^*(-\omega) \implies \textcircled{4} \quad \cancel{X}(\omega) = -\cancel{X}(-\omega)$$

$$\cancel{X}(\omega)$$



If $X(\omega)$ real \implies

- ① $X_R(\omega)$ is symmetric
- ② $X_I(\omega)$ is anti-symmetric
- ③ $|X(\omega)|$ is symmetric
- ④ $\cancel{X}(\omega)$ is anti-symmetric

Z. Transform

Motivation: F.T. doesn't exist for some signal.

$$x(n) = 2^n u(n)$$

$$\text{or } x(n) = a^n u(n)$$

$$|a| > 1$$

F.T. doesn't exist.

converges

$$\mathcal{Z}\{x(n)\} = X(z) \triangleq \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

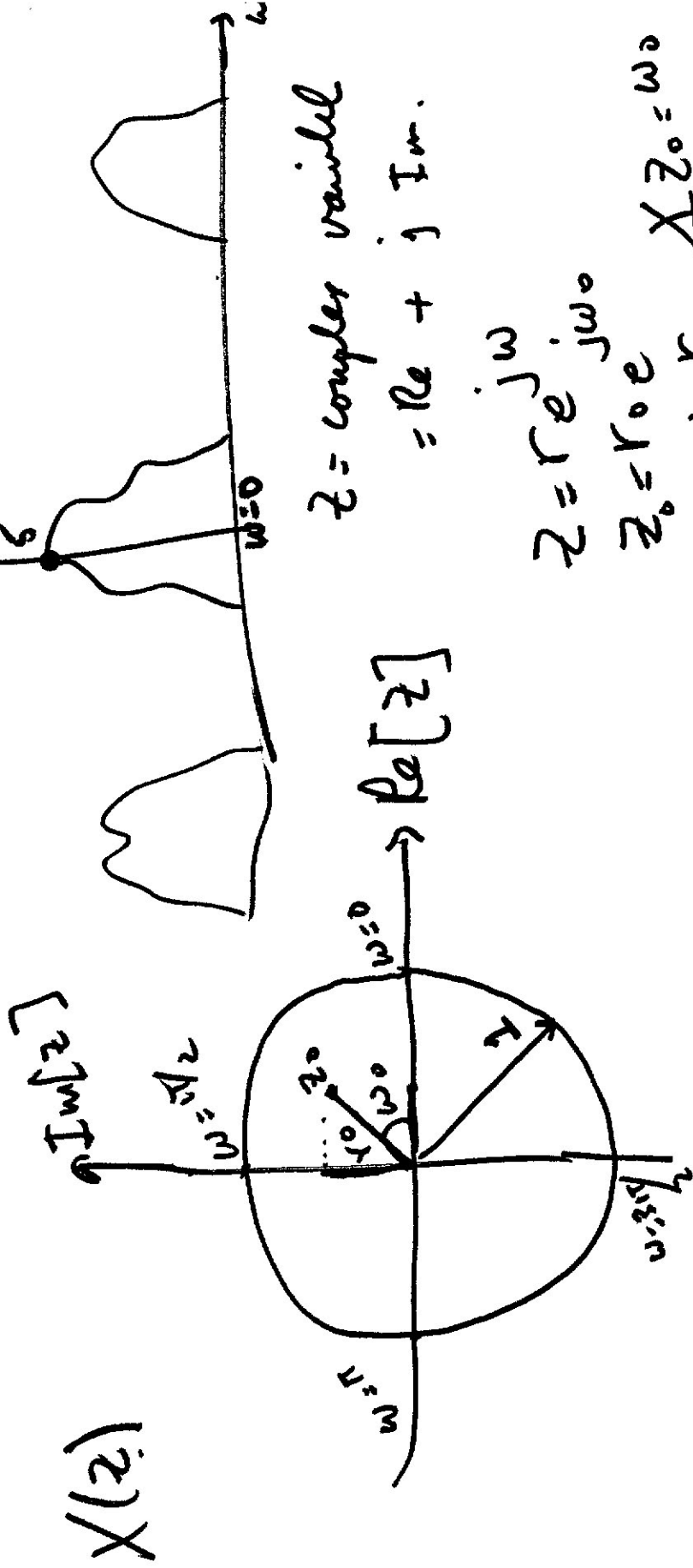
$$x(n) \longleftrightarrow$$

$$X(\omega) = \text{D.T.F.T.}\{x(n)\} = \sum_n x(n) e^{-j\omega n}$$

$$[X(z)]_{z=e^{j\omega}} = X(\omega) \rightarrow \text{Real variable.}$$

Z.T. evaluated at unit circle corresponds to

D.T.F.T.



$$z = \text{complex variable} = Re + j Im.$$

$$z = r e^{j\omega}$$

$$z_0 = r_0 e^{j\omega_0} \quad \& \quad z_0 = \omega_0$$

$$\Rightarrow |z_0| = r_0$$

Does $X(z)$ converge? For what values of z does $X(z)$ converge?

$$\sum_{n=-\infty}^{+\infty} x(n) z^{-n}$$

$$X(z) = \sum_{n=-\infty}^{+\infty} x(n) (re^{+j\omega})^{-n} e^{-j\omega n}$$

$$X(z) = \sum_{n=-\infty}^{+\infty} (x(n) r^{-n}) e^{-j\omega n}$$

D.T.F.T. of $x(n)r^{-n}$

(converges)

Remember: P.T.F.T. of $x(n)r^{-n}$ exists (converges) if $x(n)r^{-n}$ is abs. summable.

i.e. $\sum_n |x(n)r^{-n}| < \infty$

If $\sum_n |x(n) r^{-n}| < \infty \implies$ Z.T. exists & converges.

\implies Conclusion: Convergence only

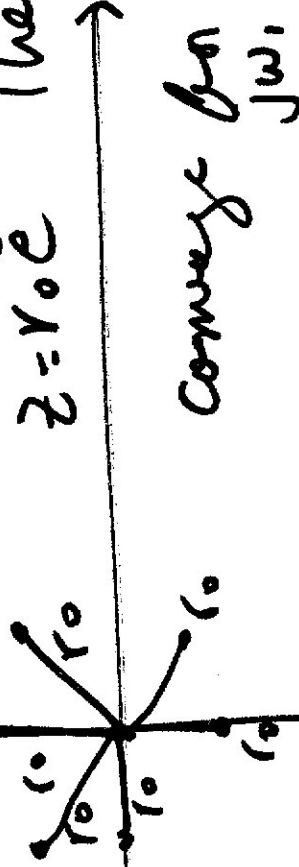
depends on r & NOT ω .

\implies ROC = always radially symmetric

if $x(z)$ converges for

$z = r_0 e^{j\omega_0}$ Then it also

converges for



$z = r_0 e^{j\omega_1}$
 $z = r_0 e^{j\omega_2}$

\implies NOT angle dependent.

ROC = Region of convergence.

Region in z domain for which

$\sum_{n=-\infty}^{\infty} x(n) z^{-n}$ converges.

ROC possibilities:

- inside of some circle.
 - outside of some circle.
 - Between 2 circles \rightarrow ring.
 - A point i.e. origin.
-

Region of Convergence for Z.T.

Claim $X(z)$. by itself does not

specify uniquely a sequence..

$$X(z) + \underline{\text{ROC}} \longleftrightarrow X(n)$$

$$X(z) + \text{ROC 1} \longleftrightarrow X_1(n)$$

$$X(z) + \text{ROC 2} \longleftrightarrow X_2(n)$$

Example: $X(z) = \frac{1}{1-az^{-1}}$

Region of Convergence for Z.T.

Claim $X(z)$. by itself does not

specify uniquely a sequence..

$$X(z) + \underline{\text{ROC}}$$

$x(n)$

$$X(z) + \text{ROC 1}$$

$x_1(n)$

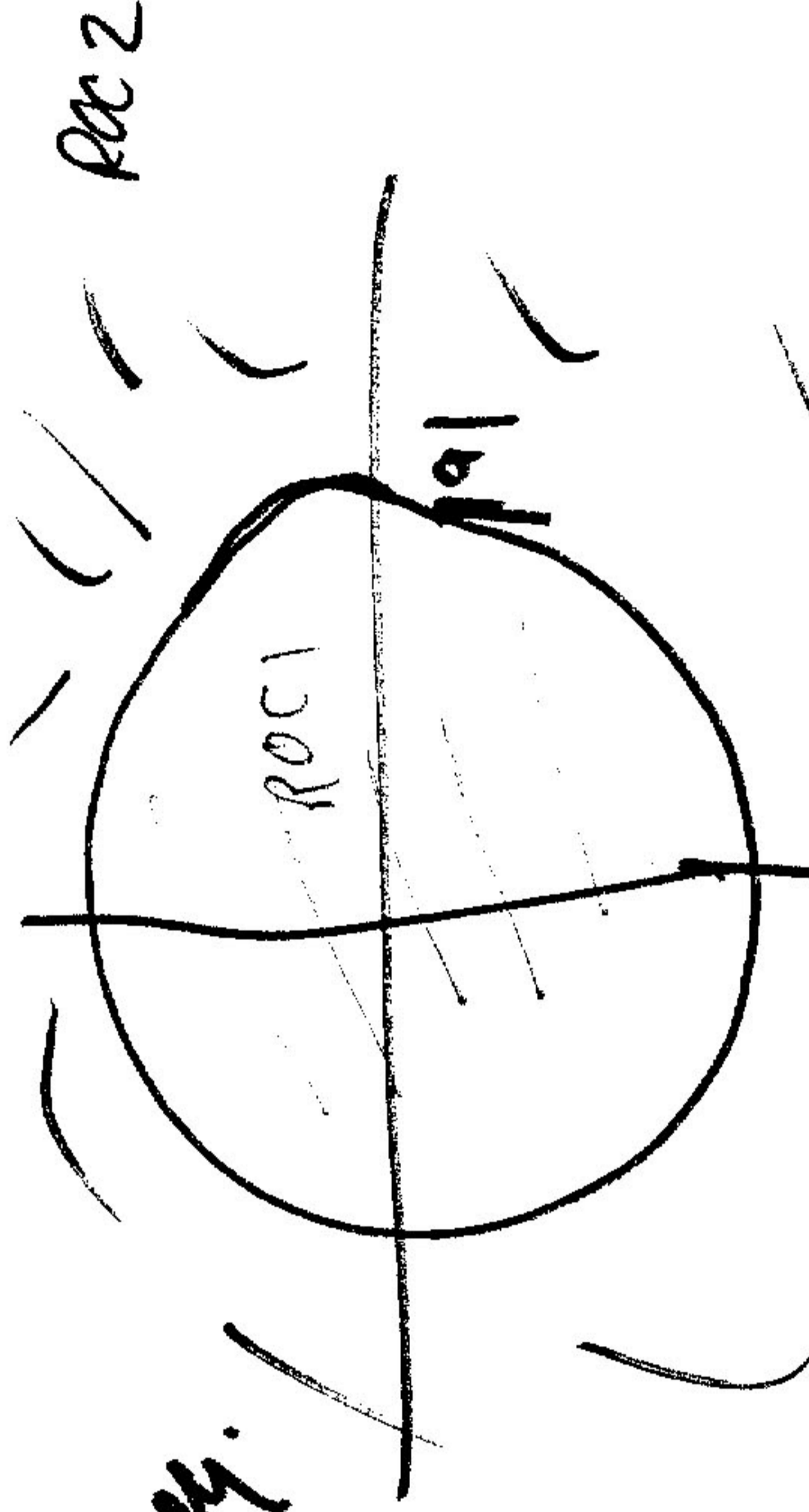
$$X(z) + \text{ROC 2}$$

$x_2(n)$

Example: $X(z) = \frac{1}{1 - az^{-1}}$

ROC 1: ^{left} Right handed seq.

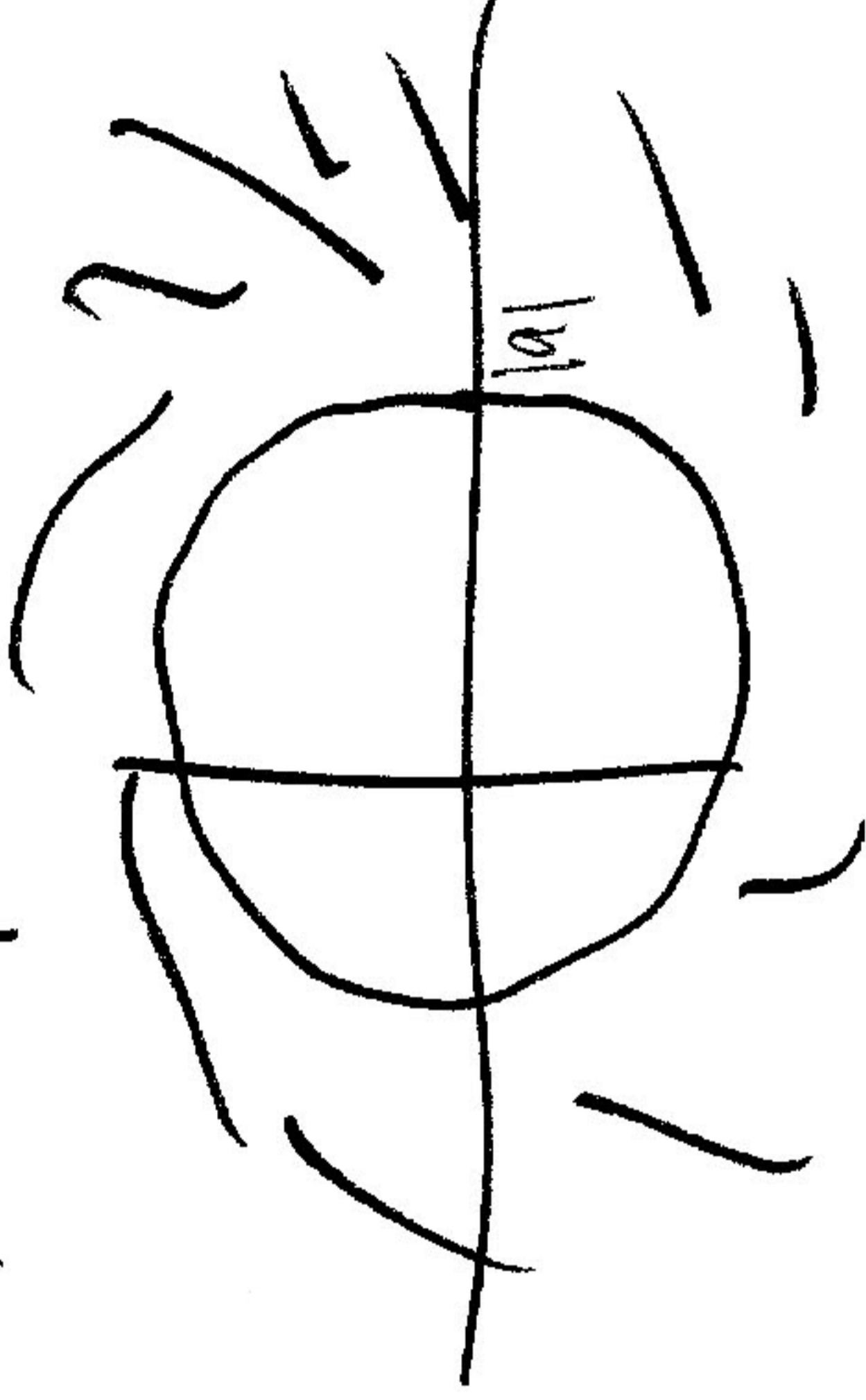
ROC 2: ~~Left~~ Right handed seq.



$x(n) = a^n u(n) \leftarrow$ Right handed seq.

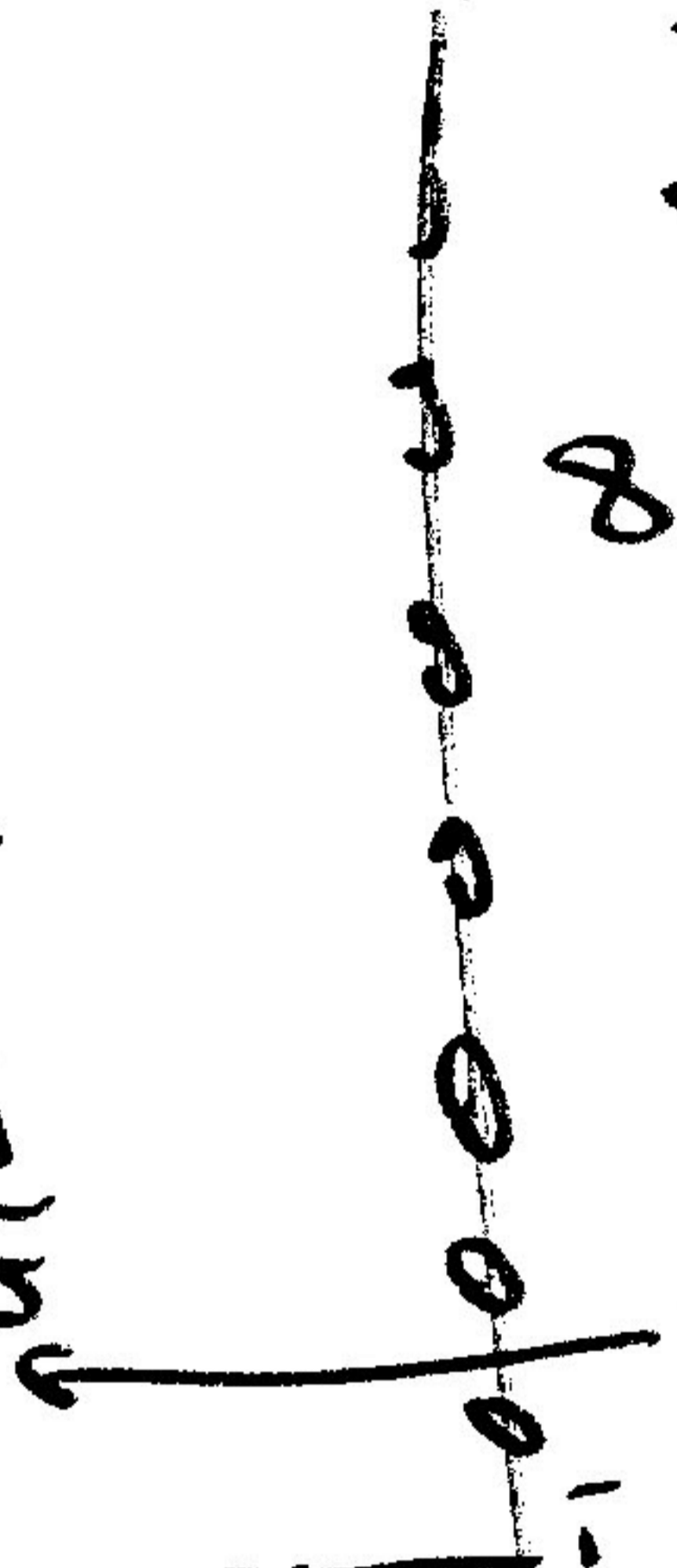
$$X(z) = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} (a z^{-1})^n = \frac{1}{1 - a z^{-1}}$$

converges if $|a z^{-1}| < 1 \Rightarrow |z| > |a|$



left handed

$$x(n) = -a^n u(-n-1)$$



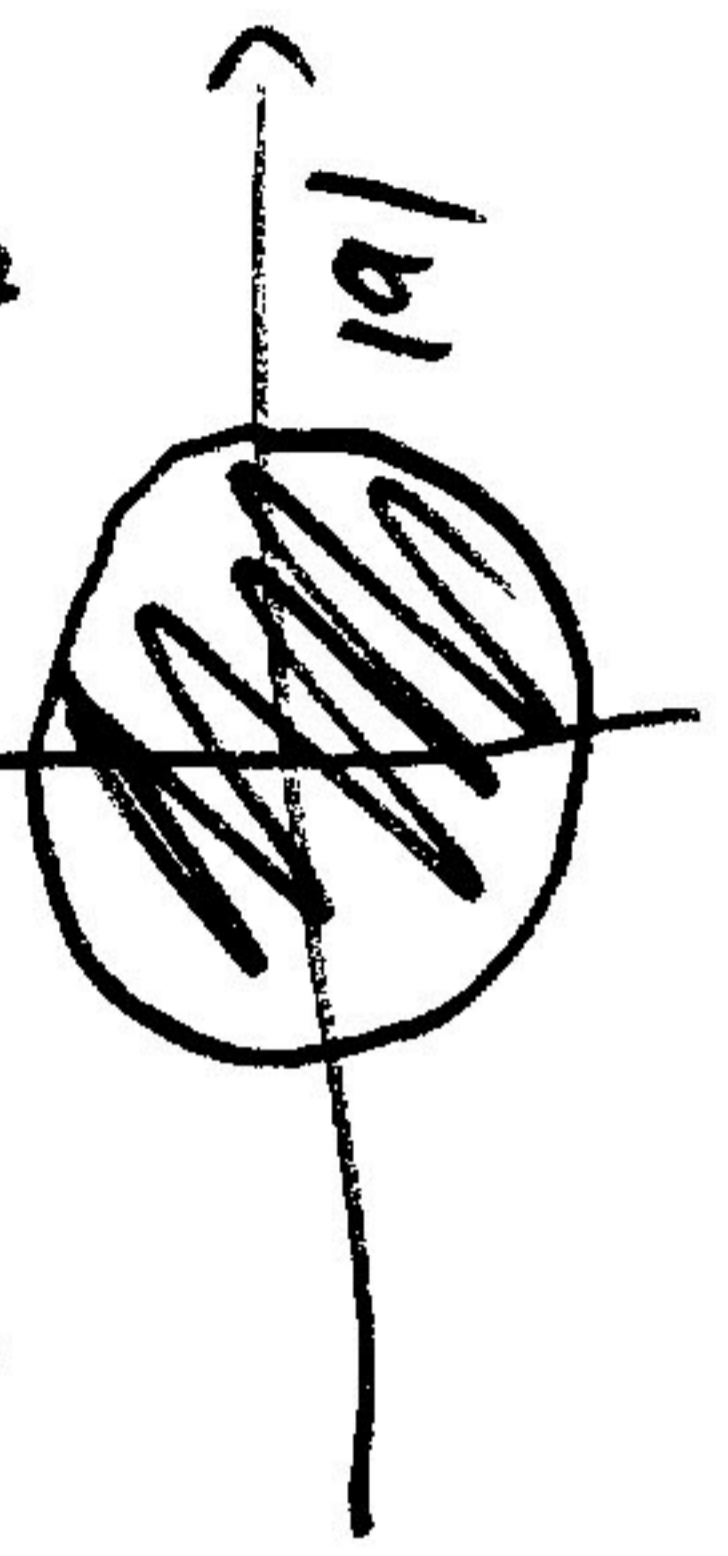
$$X(z) = \sum_{n=-\infty}^{-1} a^n z^{-n} = 1 -$$

$$\sum_{m=0}^{\infty} (a^{-1}z)^m$$

change of variable
 $m = -n$

Converges if $|a^{-1}z| < 1 \Rightarrow |z| < |a|$ ROC

$$X(z) = \frac{1}{1 - az}$$

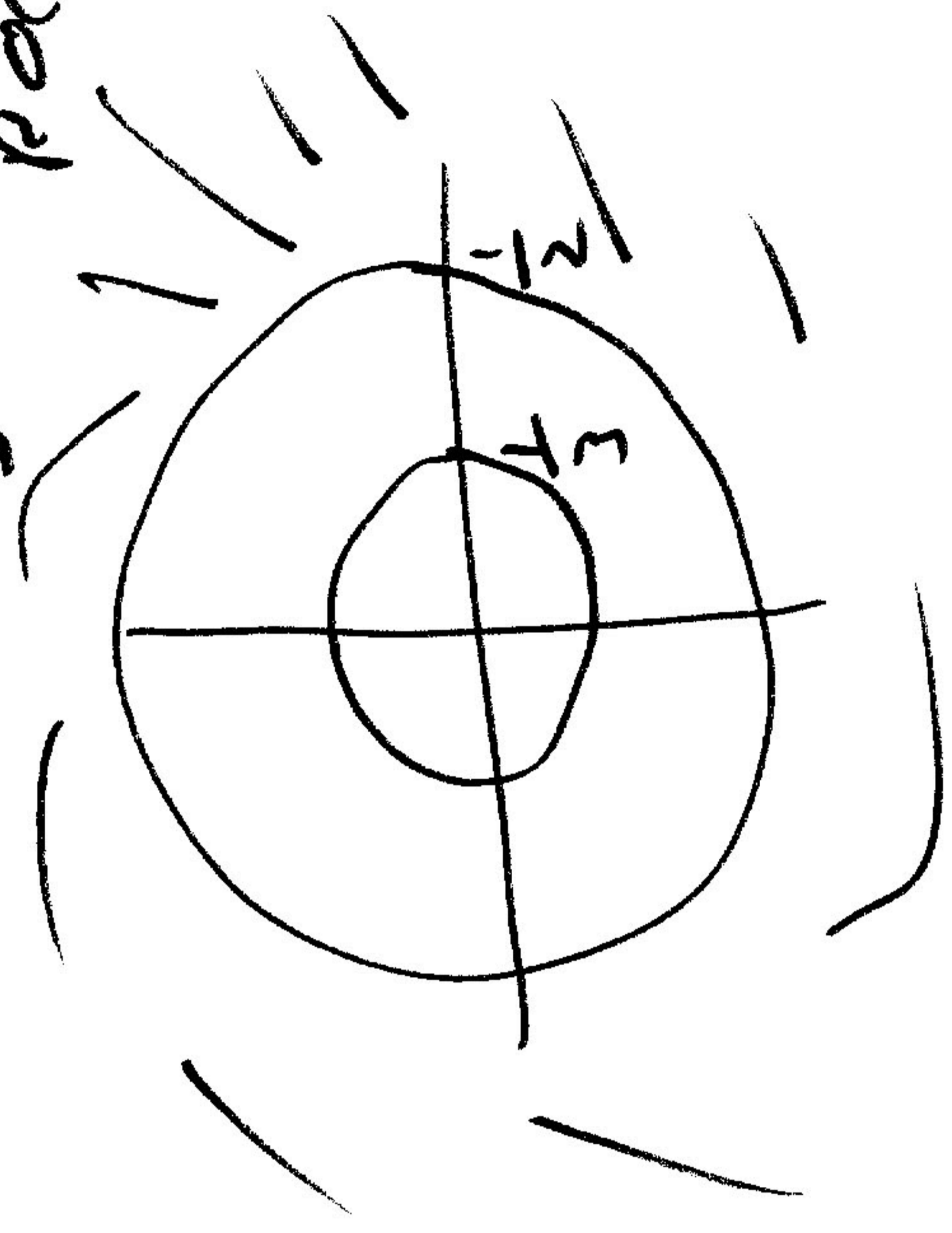


Ex $X(z) = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n u(n) + \sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n u(n)$

converge: $|z| > \frac{1}{2}$

converge: $|z| > \frac{1}{3}$

ROC:



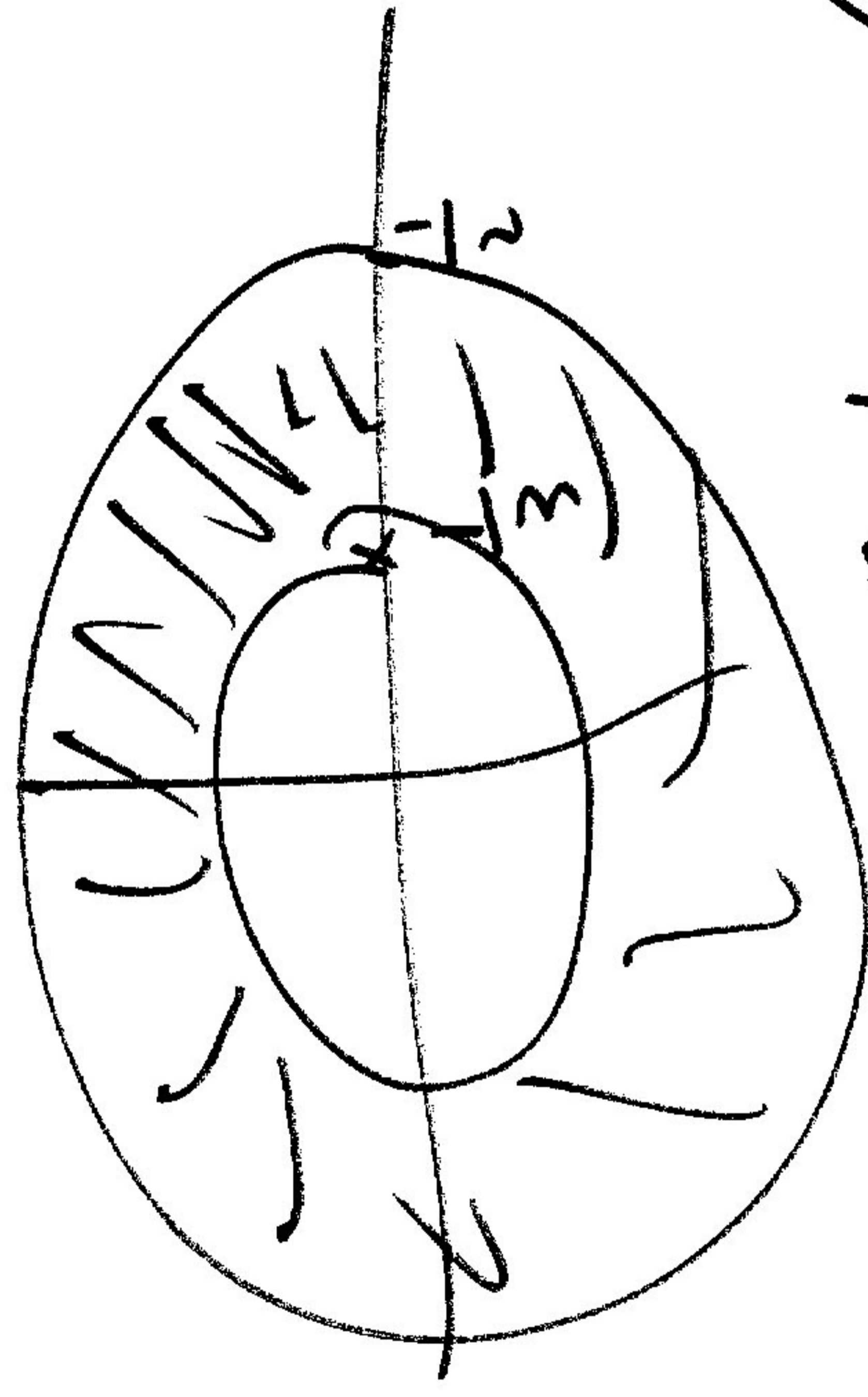
ROC: $|z| > \frac{1}{2}$

$$\overline{\text{Ex}} \quad x(n) = \left(-\frac{1}{3}\right)^n u(n) - \left(\frac{1}{2}\right)^n u(n-1)$$

$$\underbrace{\left(-\frac{1}{3}\right)^n u(n)}_{|z| < \frac{1}{3}}$$

$$\underbrace{\left(\frac{1}{2}\right)^n u(n-1)}_{|z| < \frac{1}{2}}$$

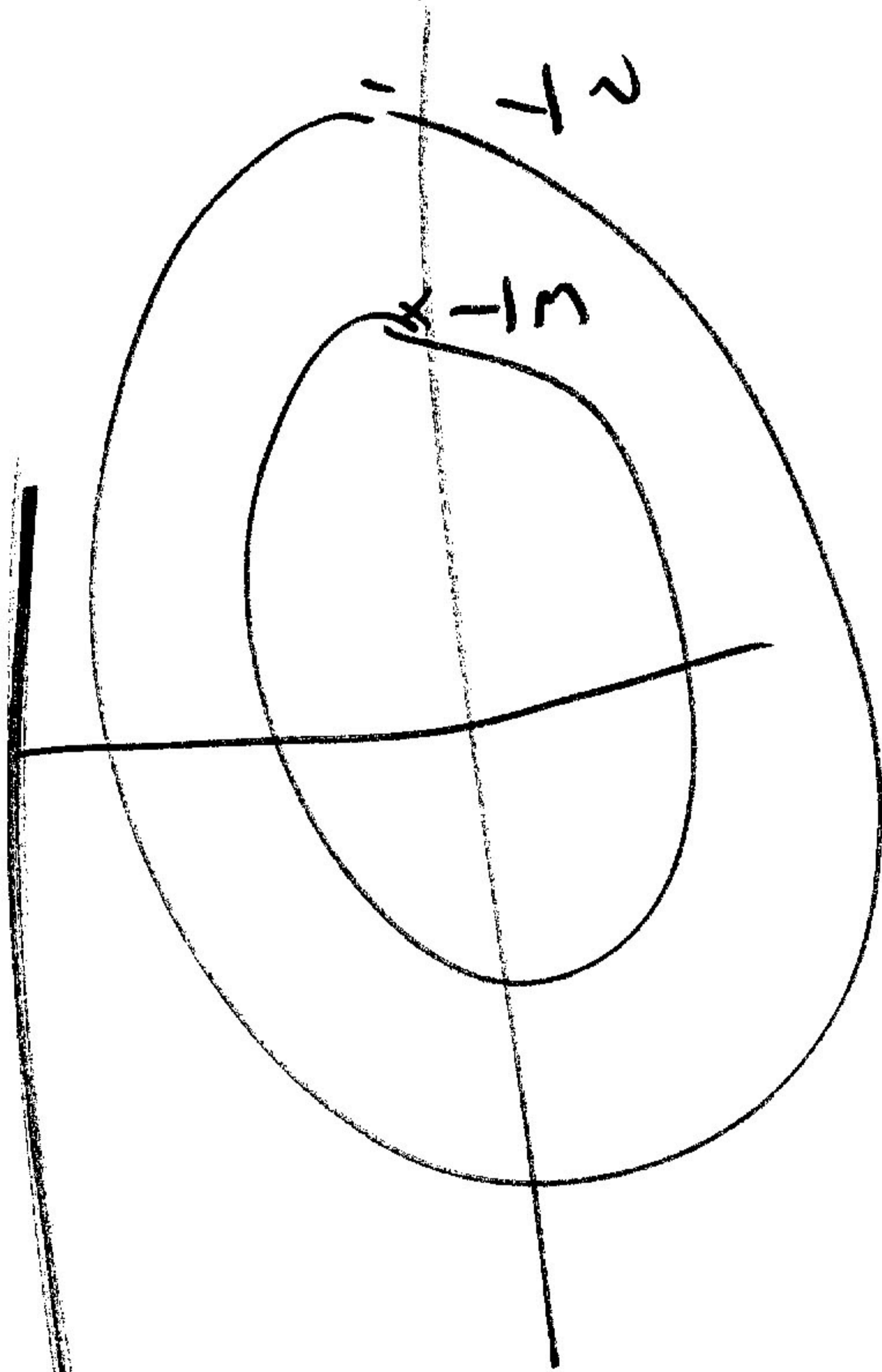
ROC is a ring.



Both handed Seq.

Ex
$$x(n) = \begin{cases} (-\frac{1}{2})^n u(n) & |z| < \frac{1}{2} \\ (\frac{1}{3})^n u(-n-1) & |z| > \frac{1}{3} \end{cases}$$

Z.T. does not exist.



Observations RHS \implies ROC outside of some circle.

LHS \implies ROC is inside of some circle.

BHS \implies Ring or donut exist.

Ex $x(n)$... Finite length seq.

$$X(z) = \sum_{n=0}^N x(n) z^{-n} = x(0)z^0 + x(1)z^{-1} + x(2)z^{-2} + \dots + x(-1)z + x(-2)z^2 + \dots$$

Converge everywhere

except maybe at $z=0$, $z \rightarrow \infty$

Ex $x(n) = \begin{cases} a^n & 0 \leq n \leq N-1 \\ 0 & \text{otherwise.} \end{cases}$

$$X(z) = \sum_{n=0}^{N-1} z^{-n} x(n) = \sum_{n=0}^{N-1} (az^{-1})^n$$

$$X(z) = \frac{1 - (az^{-1})^N}{1 - az^{-1}}$$

$$= 1 + az^{-1} + a^2 z^{-2} + \dots + a^{N-1} z^{-(N-1)}$$

$$= 1 + \frac{a}{z} + \frac{a^2}{z^2} + \dots + \frac{a^{N-1}}{z^{N-1}}$$

ROC: everywhere except for $z=0$.

$$\underline{\text{Ex}} \quad x(n) = \begin{cases} a^n & -N \leq n \leq 0 \\ 0 & \text{elsewhere} \end{cases}$$

$$-N \leq n \leq 0$$

elsewhere.

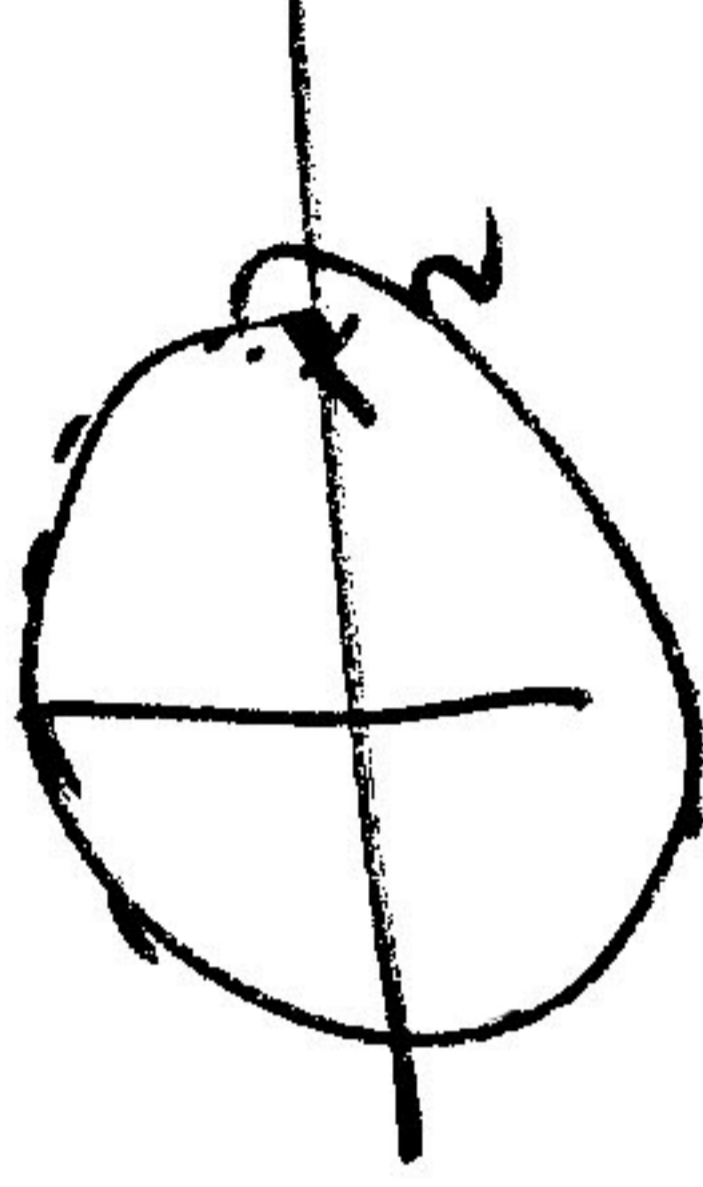
$$X(z) = \sum_{n=0}^{-N} a^n z^{-n} = 1 + a^{-1} z + a^{-2} z^2 + \dots + \dots$$

ROC: converges everywhere except $z = -a$

Properties of ROC

1. ROC is a fn of r and not ω
2. F.T. exists if ROC of the Z.T. includes unit circle
3. ROC cannot include a pole

$$\frac{1}{1 - \frac{1}{2}z}$$



4. Finite length seq ROC: everywhere except at $z=0$ or $z \rightarrow \infty$.

Show : If seq. RHS \Rightarrow ROC outside of some circle.

$X(n)$ RHS : no zero for some n_0 and larger indices than n_0



ROC outside of some circle: \Leftrightarrow it also converges for r_0 , if $r < r_0$ converges for $r > r_0$

Assume $X(z)$ converges for some r_0 , show that it also converges for $r > r_0$



$$\sum_{h=0}^{+\infty} |x(n) r_0^{-n}| < \infty \quad \text{True}$$

Invoke the RHS assumption

$$\sum_{n=0}^{\infty} |x(n) r_0^{-n}| < \infty \quad \text{True.}$$

Show: $\boxed{r > r_0}$

$$\sum_{n=0}^{\infty} |x(n) r^{-n}| < \infty$$

$$|x(n) r^{-n}| = |x(n) r_0^{-n} \left(\frac{r}{r_0}\right)^{-n}|$$

$$= |x(n) r_0^{-n}| \left(\frac{r_0}{r}\right)^{-n}$$

Show

$$r > r_0$$

$$\sum_{n=0}^{\infty} |x(n) r_0^{-n}| \left| \left(\frac{r}{r_0} \right)^{-n} \right| < \infty$$

$$\left| \frac{r}{r_0} \right| < 1 \Leftrightarrow \left| \frac{r_0}{r} \right| < 1$$

$$r > r_0 \Rightarrow$$

RHS



$\Rightarrow QED$