

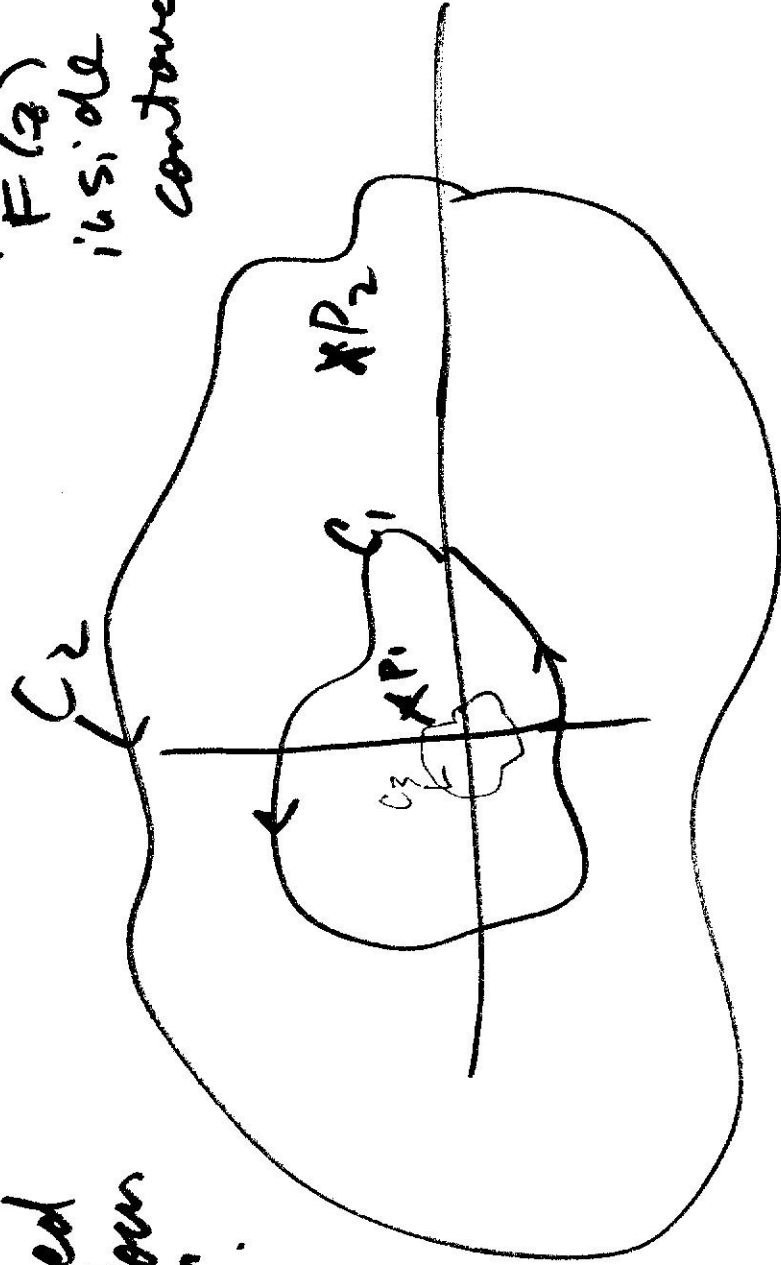
Cauchy Residue Theorem

Residues of $F(z)$ at each of poles of $F(z)$ inside contour C

$$\oint_C F(z) dz = \sum_{\text{poles}} \text{Residues of } F(z)$$

\oint_C

closed contour C .



2019

What is Residue?

If $F(z) = \frac{\phi(z)}{(z-z_0)^n}$ Then,

$$\text{Residue } [F(z)]_{z=z_0} \triangleq \frac{1}{(n-1)!} \left[\frac{d^{n-1} \phi(z)}{dz^{n-1}} \right]_{z=z_0}$$

Ex. $F(z) = \frac{1}{(z-2)^2}$

What is residue of $F(z)$ at $z=2$?

$$\phi(z) = 1 \quad s=2 \quad z_0=2$$
$$\text{Residue } [F(z)]_{z=2} = \frac{1}{1} \left[\frac{d}{dz} (1) \right]_{z=2} = 0$$

$$\text{Ex } F(z) = \frac{z^3}{(z-2)}$$

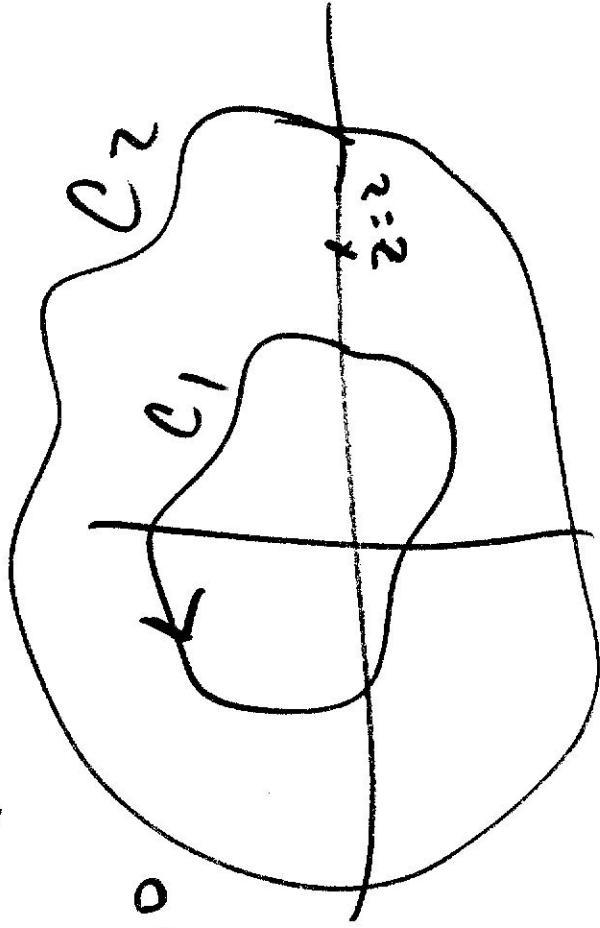
$$\phi(z) = z^3 \quad s=1 \quad z_0=2$$

$$\text{Residue } [F(z)]_{z=z_0} = \frac{1}{1} [\phi(z)]_{z=z_0} = 8$$

$$\text{CRT: } \frac{1}{2\pi j}$$

$$\oint_{C_1} F(z) dz = 0$$

$$\frac{1}{2\pi j} \oint_{C_2} F(z) dz = 8$$



$$\text{Ex } \underline{F(z) = \frac{1}{(z-2)(z-3)}}$$

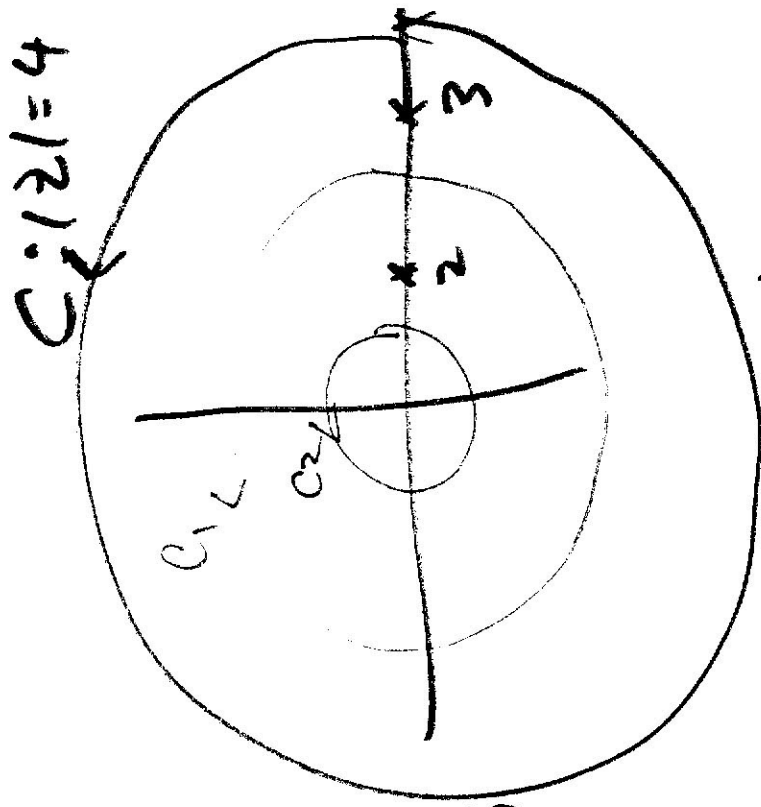
Compute:

$$\oint_{2\pi j} F(z) dz = 1 + (-1) = 0$$

$$\text{For } z=2 \quad F(z) = \frac{\phi(z)}{(z-2)}$$

$$\left[\text{Res } F(z) \right]_{z=2} = \frac{1}{(1-1)}$$

$$\text{For } z=3 \quad \phi(z) = \frac{1}{z-2}$$



$$\phi(z) = \frac{1}{z-3}$$

$$\left[\phi(z) \right]_{z=2} = \left[\frac{1}{z-3} \right]_{z=2}$$

$$[\text{Res } F(z)]_{z=3} = \frac{1}{(1-1)!} \left[\frac{1}{z-2} \right]_{z=3} = 1$$

USE CRT TO SHOW INVERSE.
Z.T.

$$x(n) = \frac{1}{2\pi j}$$

\oint

$$X(z) \frac{z^{n-1}}{z}$$

zP

C: closed contour in
RCC of z^{n-1}
 $X(z) z^{n-1}$

Observation:

$$\int_{\gamma} \frac{1}{z^n} dz$$

$$= \int_{\gamma} \frac{1}{z^{n+1}} dz$$

Proof: $n > 0$

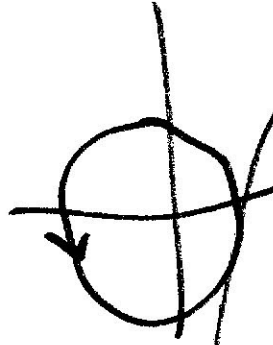
$$\Rightarrow \int_{\gamma} \frac{1}{z^n} dz$$

$$\textcircled{2} \quad n = -2$$

$$\oint_{\gamma} z^n dz = \begin{cases} 1 & n = -1 \\ 0 & \text{otherwise} \end{cases}$$

otherwise:

γ : any closed contour encircling origin n times



z^n has no poles inside this contour

$s=2, z_0=0$ $\phi(z) \geq 1$
 2nd order pole at the origin

Residue:

$$= \left[(z) \phi \left(\frac{z}{p} \right) \right]_{z=0} = \frac{1}{(s-1)!}$$

$$\textcircled{3} \quad n=1 \quad z^1 \quad \textcircled{3} \quad \frac{z^1}{z^2} \quad \textcircled{3}$$

$$s=2 \quad \text{Resid} \quad \frac{1}{(1-1)!} \quad \text{Resid} \quad \left[(z) \phi \right]_{z=0}$$

Inverse z.T.

$$X(z) = \sum_n x(n) z^{-n}$$

$$\text{Inverse z.T.} = \frac{1}{2\pi j} \oint_C \left(\sum_k x(k) z^{-k} \right) z^{n-1} dz$$

$$= \sum_k x(k) \frac{1}{2\pi j} \oint_C \underbrace{z^{-k} z^{n-1}}_{z^{-k+n-1}}$$

$$\delta(-k+n-1) = \delta(n-k)$$

$$= \sum_k x(k) \delta(n-k)$$

$$= x(n) \quad \text{QED}$$

C.R.T. To Compute Inverse Z.T.

\sum^x

$$X(z) =$$

$$\frac{1}{(1 - a\bar{z}^{-1})(1 - b\bar{z}^{-1})}$$

ROC:

$$|z| > |a|$$

$$|b| > |a|$$

$$X(n) = \frac{1}{2\pi j} \oint_C \frac{z^{n+1}}{(z-a)(z-b)} dz$$



Two cases: Pole structure is different for different values of n
 $n < -1 \Rightarrow$ poles at origin.

Consider

- ① $n \geq -1$
- ② $n < -1$

① \Rightarrow 2 poles inside contour a, b .

$$X(n) = \text{Res} \left[F(z) \right]_{z=a} + \text{Res} \left[F(z) \right]_{z=b}$$

$$X(n) = \underbrace{\left[\frac{z^{n+1}}{z-b} \right]_{z=a}}_{z_0=a, S=1} + \underbrace{\left[\frac{z^{n+1}}{(z-a)} \right]_{z=b}}_{z_0=b, S=1}$$

$$X(n) = \frac{a^{n+1}}{a-b} + \frac{b^{n+1}}{b-a}$$

②

$$n < -1$$

$$n = -2 \implies$$

$$z=0, \quad z=a, \quad z=b.$$

$$n = -3 \implies$$

double pole at $z=0$,
 $z=a, \quad z=b$

$$n = -4 \implies$$

triple pole at $z=0$
 $z=a, \quad z=b.$

$$\text{Consider } n = -2 \quad F(z) = \frac{1}{z(z-a)(z-b)}$$

$$f(n) = \text{Res at } z=0 + \text{Res } z=a + \text{Res for } z=b.$$

$$= \left[\frac{1}{(z-a)(z-b)} \right]_{z=0} + \left[\frac{1}{z(z-b)} \right]_{z=a} + \left[\frac{1}{z(z-a)} \right]_{z=b}$$

$$= \frac{1}{ab} + \frac{1}{a(a-b)} + \frac{1}{b(b-a)} = 0$$

$$n = -3, -4, \dots \implies f(n) = 0$$

Combining Case ① + ②:

$$x(n) = u(n+1) \left(\frac{a^{n+1}}{a-b} + \frac{b^{n+1}}{b-a} \right)$$

$$x(n) = u(n) \left(\frac{a^{n+1}}{a-b} + \frac{b^{n+1}}{b-a} \right)$$

↳ ~~Left~~ Right handed Seq.

Partial Fraction Expansion

$$X(z) = \frac{1}{(1-az^{-1})(1-bz^{-1})} \xrightarrow{\text{polynomials in } z} \frac{P(z)}{Q(z)}$$

$$X(z) = \frac{A}{(1-az^{-1})} + \frac{B}{(1-bz^{-1})} \quad \downarrow \text{common denominator}$$

$$X(z) = \frac{A(1-bz^{-1}) + B(1-az^{-1})}{(1-az^{-1})(1-bz^{-1})} =$$

$$A(1-bz^{-1}) + B(1-az^{-1}) = 1$$

True for all powers of z

Equate coeffs of z^{-1} on Right & left:

~~$$A(1-a^{-1}) + B(1-a^{-1}z^{-1}) = 1$$~~

$$-Ab - Ba = 0 \quad \Rightarrow$$

Equate coeff of z^0

$$A + B = 1$$

$$A = \frac{a}{a-b}$$

$$B = \frac{b}{b-a}$$

$$X(z) = \frac{a(b-b)}{1-az^{-1}} + \frac{b(b-a)}{1-bz^{-1}} \quad |z| > b$$

$$\downarrow \quad \downarrow \quad \dots \quad u(n) + \dots \quad x(n)$$

Same answer as before.

Initial Value Theorem

If $x(n) = 0$ for $n < 0$ then

$$x(0) = \lim_{z \rightarrow \infty} X(z) \quad \text{RHS.}$$

$$\text{RHS} = \lim_{z \rightarrow \infty} \left[\sum_{n=-\infty}^{\infty} x(n) z^{-n} \right]$$

$$= \lim_{z \rightarrow \infty} \left[\sum_{n=0}^{\infty} x(n) z^{-n} \right]$$

$$= \lim_{z \rightarrow \infty} \left[x(0) + x(1)z^{-1} + x(2)z^{-2} + \dots \right]$$

$$= x[0]$$

Bag of Tricks for Inverse z.T.

① Cauchy Residue Th.

② Partial Fract. Exp.

③ Direct way: inspection.

④ Table look up. → Look at table 3.1 on p. 104

⑤ Use properties

⑥ Series expansion

$$X(z) = 5 + 20z^{-6}$$

$$X(z) = \sum_n x(n)z^{-n} = x(0) + x(1)z^{-1} + x(2)z^{-2} + \dots + x(-1)z + x(-2)z^2 + \dots$$

$$X(n) = \begin{cases} 5 & n=0 \\ 20 & n=6 \end{cases}$$

Series Exp

$$\log(1 + a\bar{z}^{-1}) = X(z)$$

$$\text{ROC } |a\bar{z}^{-1}| < 1$$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

Apply Series Exp:

$$|x| < 1$$

$$\log(1 + a\bar{z}^{-1}) = a\bar{z}^{-1} - \frac{(a\bar{z}^{-1})^2}{2} + \frac{(a\bar{z}^{-1})^3}{3} - \dots$$

$$\frac{(a\bar{z}^{-1})^4}{4} + \dots$$

$$X(z) = \sum_n x(n)z^{-n} = \dots + x(-2)z^2 + x(-1)z + x(0)z^0 + x(1)z^{-1} + x(2)z^{-2} + \dots$$

$$x(n) = \begin{cases} 0 & n=0 \\ a & n=1 \\ -\frac{a}{2} & n=2 \\ \frac{a}{3} & n=3 \end{cases}$$

$$n \leq 0$$

$$n=1$$

$$n=2$$

$$n=3$$

$$x(n) = u(n-1) \frac{a}{n} (-1)^{n+1}$$

$$\log(1+a z^{-1})$$

Convergence Issues

$$X(\omega) = \text{D.T. F.T. } \{ x(n) \}$$

$$= \sum_{n=-\infty}^{+\infty} x(n) e^{-j\omega n}$$

Q: Does it converge?

If yes, what does it converge to?
 $\longrightarrow Y(\omega).$

Q: If it converged to $Y(\omega)$, plug in $Y(\omega)$ into Inverse F.T. formula do. I get $x(n)$ back?

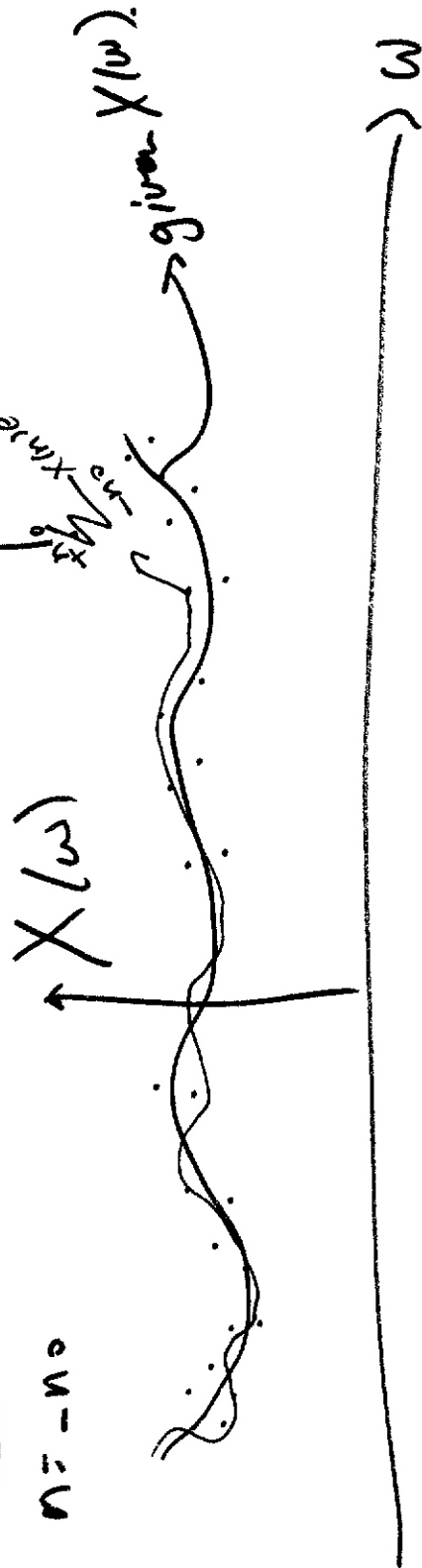
$$\int Y(\omega) e^{j\omega n} d\omega \stackrel{?}{=} x(n)$$

Def uniform convergence

$\sum_n x(n)e^{-j\omega n}$
converges uniformly if \exists a continuous function
of ω , called $X(\omega)$ such that.

$$\forall \epsilon > 0, \exists n_0 \text{ such that}$$

$$\left| \sum_{n=-n_0}^{+n_0} x(n)e^{-j\omega n} - X(\omega) \right| < \epsilon$$

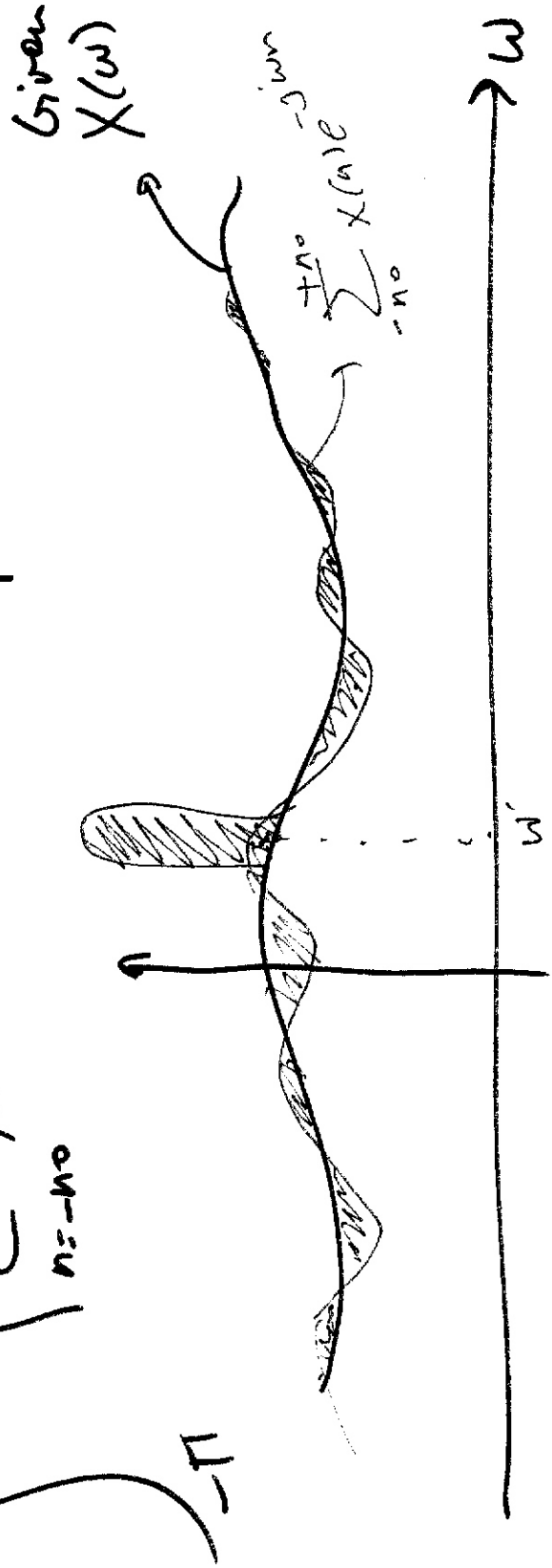


mean Square Error Convergence (MSE).

$\sum_n x(n) e^{-j\omega n}$ converges in mse sense to
a known, given fn, $X(\omega)$ if

$\forall \epsilon, \exists n_0$ s.t.h.

$$\int_{-\pi}^{+\pi} \left| \sum_{n=-n_0}^{+n_0} x(n) e^{-j\omega n} - X(\omega) \right|^2 d\omega < \epsilon$$



① If say $x(n)$ is abs summable, then $\sum_n |x(n)| < A$

$$\sum_n x(n) e^{-j\omega n}$$

converges uniformly \Rightarrow D.T.F.T. exists.

and DTFT is a continuous function.

Furthermore, whatever $\sum_n x(n) e^{-j\omega n}$ converges

to, if I plug into (call it $X(\omega)$), $\int x(n) e^{j\omega n} d\omega$

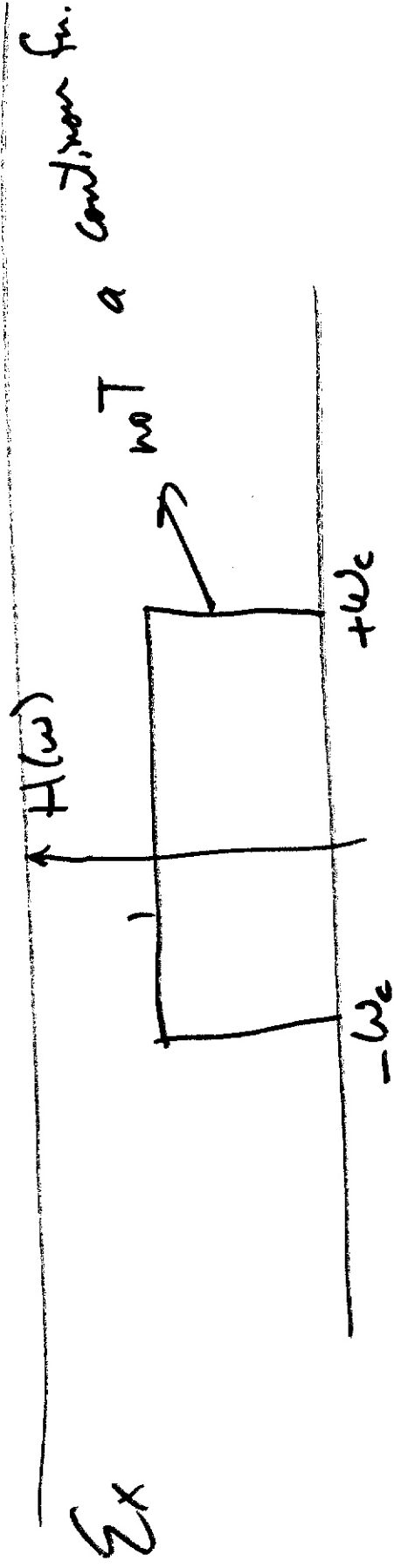
I get back $x(n)$.

Ex.: $\left(\frac{1}{2}\right)^n u(n) = x(n) \rightarrow \frac{1}{1 - \frac{1}{2} e^{-j\omega}}$

$2^n u(n) = x(n) \rightarrow$ F.T. doesn't exist.

(2) If $x(n)$ is square summable,

$\sum_n x^2(n) < \infty \Rightarrow$ converges in msc sense.



$$\int_{-\pi}^{+\pi} H(w) e^{jwn} dw = \frac{\sin wc n}{\pi n} = d_n(n).$$

$d_n(n)$ is not absolutely summable

$\sum_n \frac{1}{n} \rightarrow$ diverges; so

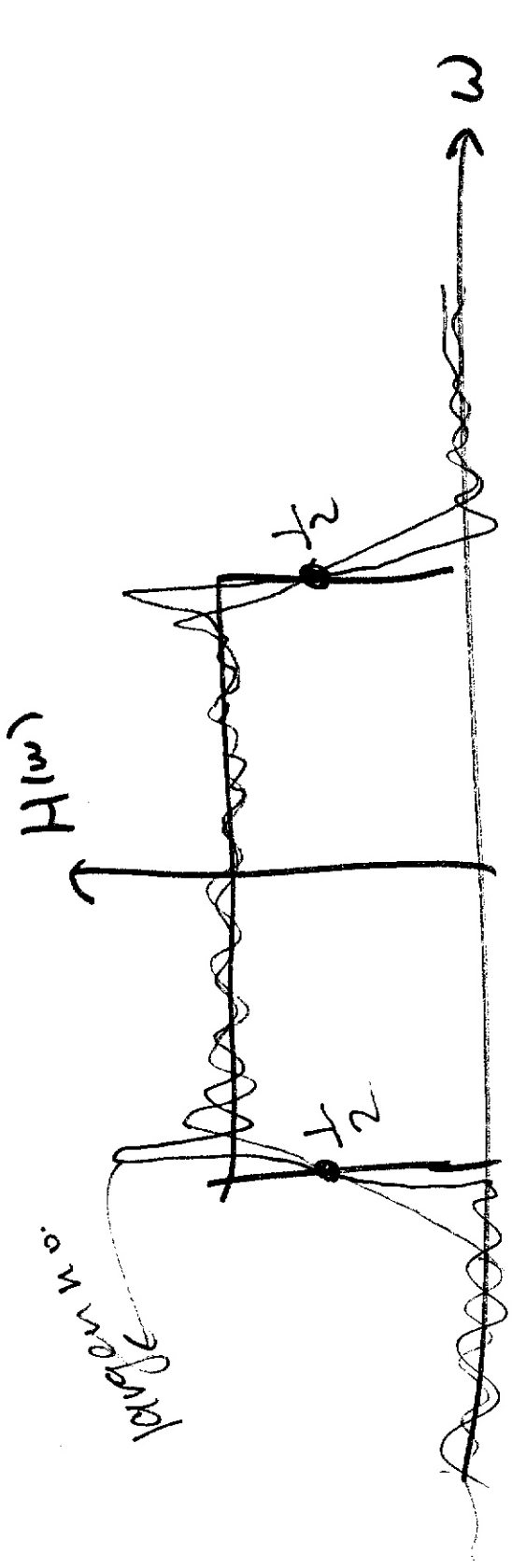
is $d_n(n)$ square summable $\sum_n \frac{\sin^2 wc n}{\pi^2 n^2}$? $\sum_n \frac{1}{n^2}$?

Since $\sum_n \frac{1}{n^2}$ converges

$\Rightarrow \ln(n)$ is square summable \Rightarrow

~~\sum~~ $\sum \ln(n) e^{-j\omega n}$ converges in a
mse sense.

Can show: $\sum_{-n_0}^{+n_0} \ln(n) e^{-j\omega n} = \frac{1}{2}$
at ω_c $\forall n_0$.



$$\sum_{n_0} h(n) e^{-j\omega n}$$