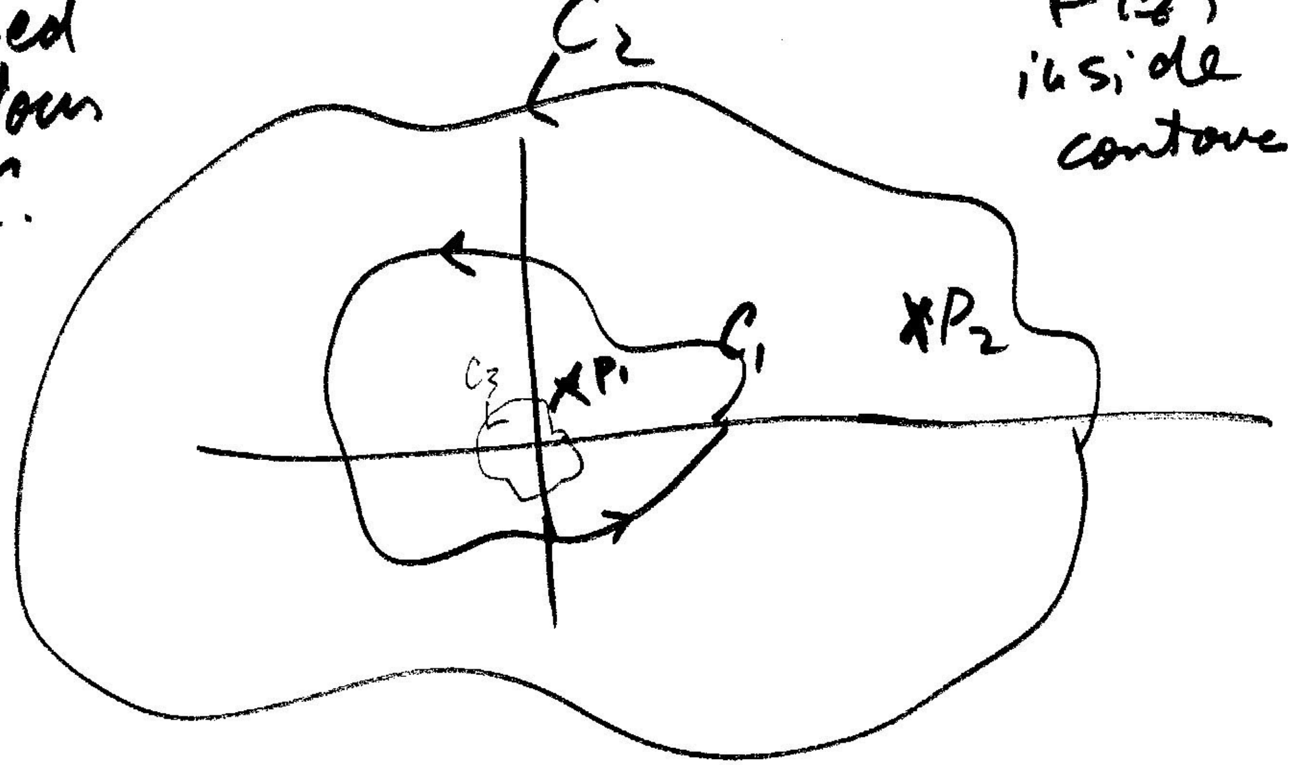
Cauchy Residue Theorem

$$\frac{1}{2\pi j}\oint F(z)\,dz = \sum_{\text{poles}} \text{Residues of } F(z) \text{ at each pole of } F(z) \text{ inside contour } C$$

closed contour C.



what is Residue?

If $F(z) = \dfrac{\phi(z)}{(z-z_0)^s}$ Then.

$$\text{Residue}\,\Big[F(z)\Big]_{z=z_0} \triangleq \frac{1}{(s-1)!}\left[\frac{d^{s-1}\phi(z)}{dz^{s-1}}\right]_{z=z_0}$$

Ex. $F(z) = \dfrac{1}{(z-2)^2}$

what is residue of $F(z)$ at $z=2$?

$\phi(z)=1 \quad s=2 \quad z_0=2$

$$\text{Residue}\,\Big[F(z)\Big]_{z=2} = \frac{1}{1}\left[\frac{d}{dz}(1)\right] = 0$$

Ex $F(z) = \dfrac{z^3}{(z-2)}$

$\phi(z) = z^3 \qquad s = 1 \qquad z_0 = 2$

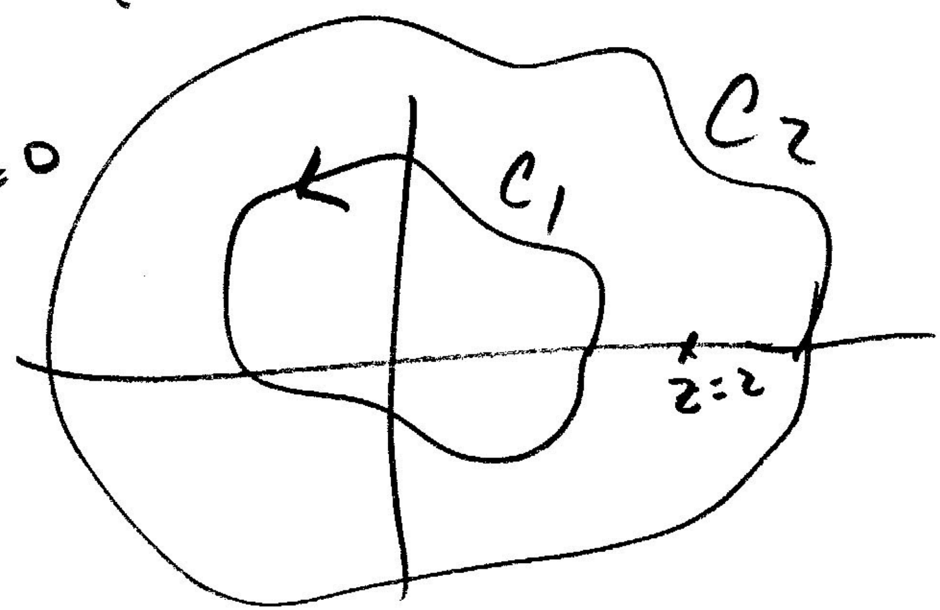
$$\text{Residue}\left[F(z)\right]_{z=z_0} = \frac{1}{1}\left[\phi(z)\right]_{z=z_0}$$

$$= \left[z^3\right]_{z=2} = \underline{8}$$

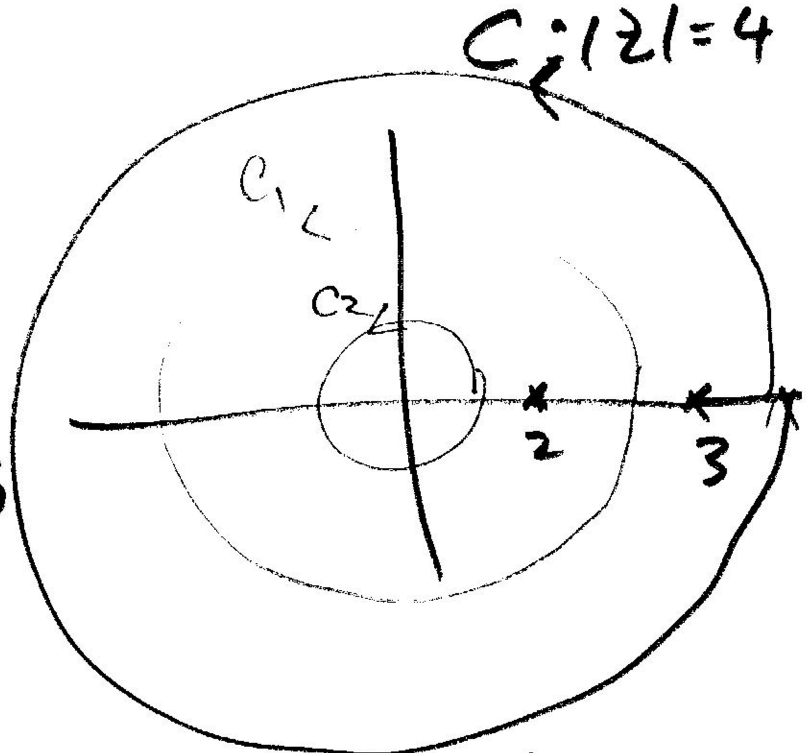
CRT:

$$\frac{1}{2\pi j}\oint_{C_1} F(z)\,dz = 0$$

$$\frac{1}{2\pi j}\oint_{C_2} F(z)\,dz = 8$$



Ex $F(z) = \dfrac{1}{(z-2)(z-3)}$



Compute:

$$\frac{1}{2\pi j} \oint_C F(z)\, dz = 1 + (-1) = 0$$

For $z = 2$ $\quad F(z) = \dfrac{\phi(z)}{(z-2)}$ $\qquad \phi(z) = \dfrac{1}{z-3}$

$$\left[\operatorname{Res} F(z)\right]_{z=2} = \frac{1}{(1-1)!}\left[\phi(z)\right]_{z=2} = \left[\frac{1}{z-3}\right]_{z=2} = -1$$

For $z = 3$ $\quad \phi(z) = \dfrac{1}{z-2}$

$$\left[\operatorname{Res} F(z)\right]_{z=3} = \frac{1}{(1-1)!}\left[\frac{1}{z-2}\right]_{z=3} = 1$$

USE CRT to show Inverse. Z.T.

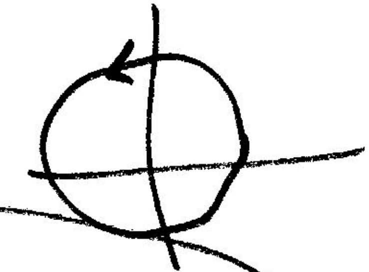
$$x(n) = \frac{1}{2\pi j} \oint_C X(z)\, z^{n-1}\, dz$$

C: closed contour in ROC of $X(z)\, z^{n-1}$

Observation :

$$\frac{1}{2\pi j} \oint_C z^n \, dz = \begin{cases} 1 & n = -1 \\ 0 & \text{otherwise} \end{cases}$$

C: any closed contour encircling origin



$= \delta(n+1)$

Proof : ① $n \geq 0$ $\quad$ z^n has no poles inside this contour

$$\Rightarrow \frac{1}{2\pi j} \oint z^n \, dz = 0$$

② $n = -2$ $\quad \frac{1}{2\pi j} \oint z^{-2} \, dz$

$s=2$, $\quad z_0 = 0 \quad \phi(z) = 1$

2nd order pole at the origin

Residue: $\dfrac{1}{(s-1)!}\left[\dfrac{d}{dz}\phi(z)\right] = 0$

3. $n = -1 \qquad \dfrac{1}{2\pi j}\oint z^{-1}\,dz$

$s = 1$

Resid $\dfrac{1}{(1-1)!}\left[\phi(z)\right]_{z=0} = 1$

Inverse Z.T.

$$X(z) = \sum_n x(n)\, z^{-n}$$

$$\text{Inven Z.T.} = \frac{1}{2\pi j} \oint_C \left(\sum_K x(K)\, z^{-K} \right) z^{n-1}\, dz$$

$$= \sum_K x(K) \underbrace{\frac{1}{2\pi j} \oint \underbrace{z^{-K}\, z^{n-1}}_{z^{-K+n-1}}\, dz}_{\delta(-K+n-1+1 \;=\; \delta(n-K)}$$

$$= \sum x(K)\, \delta(n-K)$$

$$= x(n) \Longrightarrow \text{QED}$$

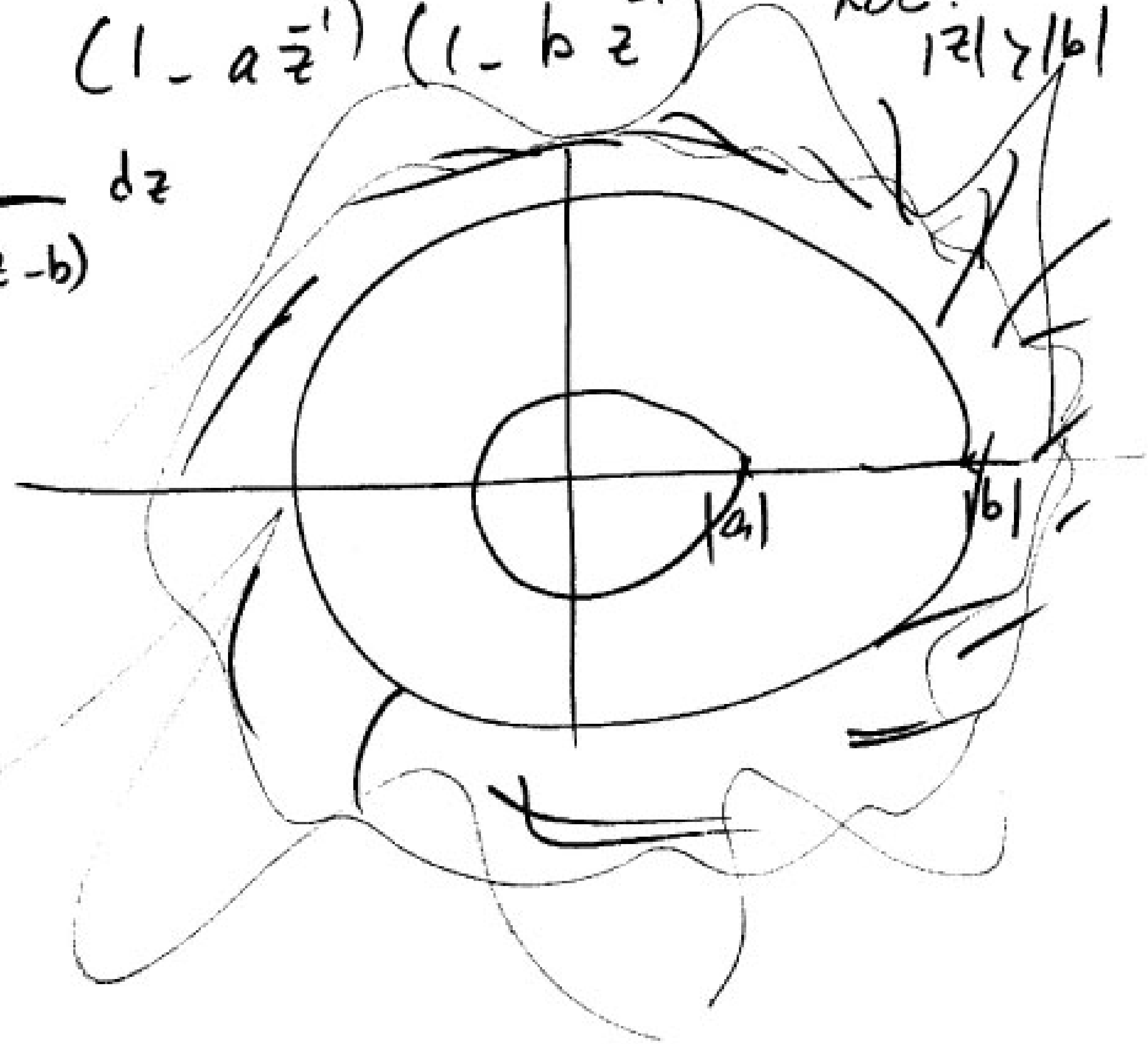
C.R.T. To Compute Inverse Z.T.

$|b| > |a|$

Ex $\quad X(z) = \dfrac{1}{(1 - a z^{-1})(1 - b z^{-1})}$

ROC: $|z| > |b|$

$$x(n) = \frac{1}{2\pi j} \oint_C \frac{z^{n+1}}{(z-a)(z-b)} \, dz$$



Two cases: Pole structure is different for different values of n

$n < -1 \Rightarrow$ poles at origin.

Consider

1. $n \geq -1$
2. $n < -1$

1. $\Rightarrow$ 2 poles inside contour a, b.

$$X(n) = \underbrace{\operatorname{Res}\left[F(z)\right]_{z=a}}_{z_0 = a \quad s=1} + \underbrace{\operatorname{Res}\left[F(z)\right]_{z=b}}_{z_0=b,\ s=1 \quad \phi(z) = \frac{z^{n+1}}{(z-a)}}$$

$$X(n) \quad \left[\phi(z) = \frac{z^{n+1}}{z-b}\right]_{z=a} + \left[\frac{z^{n+1}}{(z-a)}\right]_{z=b}$$

$$\boxed{X(n) = \frac{a^{n+1}}{a-b} + \frac{b^{n+1}}{b-a} \qquad n \geq -1}$$

(2) $n < -1$

$n = -2 \Longrightarrow$ $z = 0, \quad z = a, \quad z = b.$

$n = -3 \Longrightarrow$ double pole at $z = 0$, $z = a$, $z = b$

$n = -4 \Longrightarrow$ triple pole at $z = 0$, $z = a$, $z = b$.

Consider $n = -2$ $\quad F(z) = \dfrac{1}{z(z-a)(z-b)}$

$x(n) =$ Res at $z=0$ + Res $z=a$ + Res for $z=b$.

$$= \left[\frac{1}{(z-a)(z-b)}\right]_{z=0} + \left[\frac{1}{z(z-b)}\right]_{z=a} + \left[\frac{1}{z(z-a)}\right]_{z=b}$$

$$= \frac{1}{ab} + \frac{1}{a(a-b)} + \frac{1}{b(b-a)} = \emptyset$$

$n = -3, -4, \ldots \longrightarrow x(n) = 0$

Combining Case 1 + 2:

$$x(n) = u(n+1)\left(\frac{a^{n+1}}{a-b} + \frac{b^{n+1}}{b-a}\right)$$

$$\boxed{x(n) = u(n)\left(\frac{a^{n+1}}{a-b} + \frac{b^{n+1}}{b-a}\right)}$$

~~RH~~ Right handed Seq.

Partial Fraction Expansion

$$X(z) = \frac{1}{(1-az^{-1})(1-bz^{-1})} = \frac{P(z)}{Q(z)}$$

$P(z)$, $Q(z)$ → polynomials in z.

$$X(z) = \frac{A}{(1-az^{-1})} + \frac{B}{(1-bz^{-1})}$$

common denominator

$$X(z) = \frac{A(1-bz^{-1}) + B(1-az^{-1})}{(1-az^{-1})(1-bz^{-1})} =$$

$$A(1-bz^{-1}) + B(1-az^{-1}) = 1$$

True for all powers of z

Equate coeff of z^{-1} on Right & left:

$$\cancel{A(1-bz^{-1}) + B(1-az^{-1}) = 1}$$

$$-Ab - Ba = 0$$

Equate coef of z^{0}

$$A + B = 1$$

$$\Longrightarrow$$

$$A = \frac{a}{a-b} \qquad B = \frac{b}{b-a}$$

$$X(z) = \frac{a/(a-b)}{1-az^{-1}} + \frac{b/(b-a)}{1-bz^{-1}} \qquad |z| > |b|$$

$$\downarrow \qquad\qquad \downarrow$$

$$\ldots u(n) + \ldots u(n)$$

Same answer as before.

Initial Value Theorem

If $x(n)=0$ for $n<0$ then

$$x(0) = \underbrace{\lim_{z\to\infty} X(z)}_{RHS.}$$

$$RHS = \lim_{z\to\infty}\left[\sum_{n=-\infty}^{+\infty} x(n)\, z^{-n}\right]$$

$$= \lim_{z\to\infty}\left[\sum_{n=0}^{\infty} x(n)\, z^{-n}\right]$$

$$= \lim_{z\to\infty}\left[x(0) + x(1)\,z^{-1} + x(2)\,z^{-2} + \cdots\right]$$

$$= x[0]$$

Bag of Tricks for Inverse Z.T.

1. Cauchy Residue Th.
2. Partial Fract. Exp.
3. Direct way: inspection.
4. table look up. $\longrightarrow$ Look at table 3.1 on p. 104
5. Use properties
6. Series expansion

$$X(z) = 5 + 20\, z^{-6}$$

$$X(z) = \sum_n x(n) = x(0) + x(1) z^{-1} + x(2) z^{-2} + \dots + x(-1) z + x(-2) z^2 + \dots$$

$$x(n) = \begin{cases} 5 & n=0 \\ 20 & n=6 \end{cases}$$

Series Exp

Ex $\log(1 + az^{-1}) = X(z)$ ROC $|az^{-1}| < 1$

Apply Series Exp: $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$

$|x| < 1$

$$\log(1 + az^{-1}) = az^{-1} - \frac{(az^{-1})^2}{2} + \frac{(az^{-1})^3}{3} - \frac{(az^{-1})^4}{4} + \cdots$$

$$X(z) = \sum_n x(n) z^{-n} = \cdots + x(-2)z^2 + x(-1)z + x(0) + x(1)z^{-1} + x(+1)z^{-2} + \cdots$$

$$x(n) = \begin{cases} 0 & n \leq 0 \\ a & n = 1 \\ -\dfrac{a^2}{2} & n = 2 \\ \dfrac{a^3}{3} & n = 3 \end{cases}$$

$$\boxed{x(n) = u(n-1)\,\frac{a^n}{n}\,(-1)^{n+1}} \longleftrightarrow \log(1 + a z^{-1})$$