Linear Phase Filtering

$e^{j\omega n} \rightarrow$ [LTI $h(n)$] $\rightarrow e^{j\omega n} H(\omega)$

$H(\omega)$: DTFT of $h(n)$; Freq. Response filter at ω

$x(n) \rightarrow$ [LTI $h(n)$] $\rightarrow y(n)$

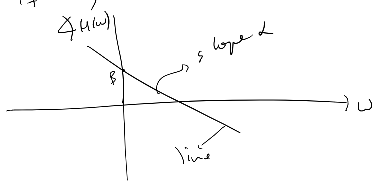
$H(\omega) = |H(\omega)| e^{j\measuredangle H(\omega)}$

$$Y(\omega) = X(\omega) H(\omega)$$

$$|Y(\omega)| = |H(\omega)|\,|X(\omega)|$$

$$\measuredangle Y(\omega) = \measuredangle H(\omega) + \measuredangle X(\omega)$$

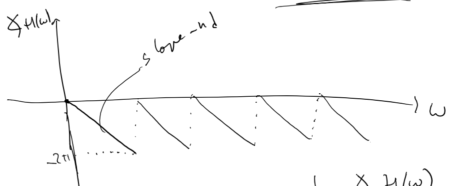
Def Linear phase LTI system: if $\measuredangle H(\omega)$ is linear in ω



Consider a pure delay LTI system

$h(n) = \delta(n - n_d)$

$|H(\omega)| = 1 \qquad \measuredangle H(\omega) = -\omega n_d$



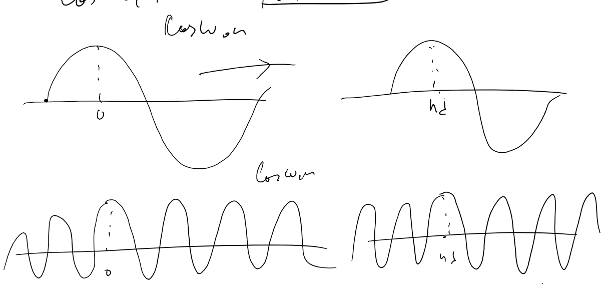
Def: Group delay $\triangleq -\frac{d}{d\omega} \measuredangle H(\omega)$

$\measuredangle H(\omega) = -\omega n_d \Rightarrow G.D. = -\frac{d}{d\omega}(-\omega n_d) = n_d$

intuitively n_d is the amount of delay for ALL frequencies; is independent of ω

$\Rightarrow$ All frequencies are delayed by the same amount as they pass through the system.

$\cos \omega_0 n + \cos \omega_1 n \rightarrow$ [$h(n) = \delta(n - n_d)$, G.D. n_d]



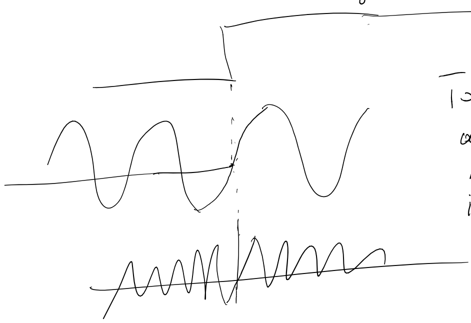
Observation: If Linear phase $\Rightarrow$ group delay is constant.

Linear phase of sinusoidal

⇒ relative phase of sinusoidal components of a signal remain intact as they go through the system.

→ no distortion added

Image Processing



To preserve edges in image processing it is good to have linear phase

Consider a narrow band signal.

$\omega_0 = 900\,MHz$

$\Delta\omega = 5\,kHz$

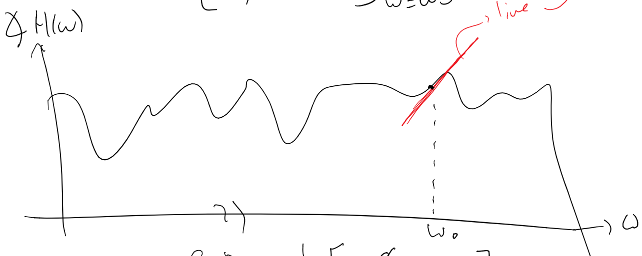
$\Delta\omega \ll \omega_0$

$\boxed{x(n) = s(n)\cos\omega_0 n}$ ←

Pass $x(n)$ through LTI filter $H(\omega)$ s.th. $|H(\omega)| = 1$. Can approximate $\angle H(\omega)$ at $\omega = \omega_0$ with a linear term

$$\left[\angle H(\omega)\right]_{\omega=\omega_0} = -\phi - \omega n_d$$

line



$$G.D = -\frac{d}{d\omega}\left[-\phi - \omega n_d\right] = n_d$$

Then: $x(n) \to \boxed{H(\omega)} \to y(n)$

Can show $y(n) = s(n - n_d)\cos(\omega_0 n - \omega_0 n_d - \phi)$

Conclusion: For a narrowband signal content around ω_0 delay is proportional to

$$GD = \left[-\frac{d}{d\omega}\angle H(\omega)\right]_{\omega=\omega_0}$$

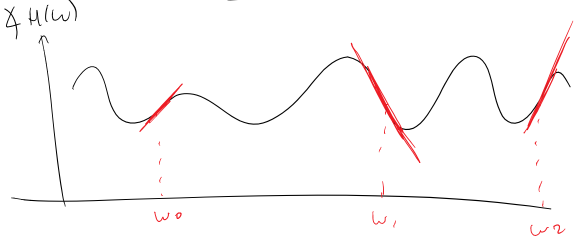
ω_0 = center frequency of narrowband signal

$x(n) = a\cos\omega_0 n + b\cos\omega_1 n + c\cos\omega_2 n$

$x(n) \to \boxed{h(n)} \to y(n)$

$x(n) = a\cos\omega_0 n + b\cos\omega_1 n + \ldots$
$c\cos\omega_2 n$

$x(n) \rightarrow \boxed{\begin{matrix} h(n) \\ H(\omega) \end{matrix}} \rightarrow y(n)$

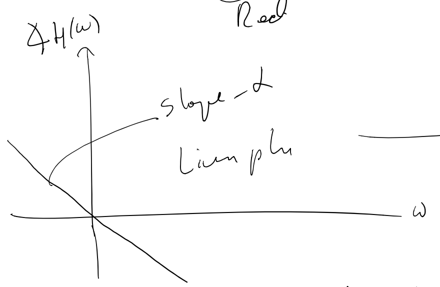


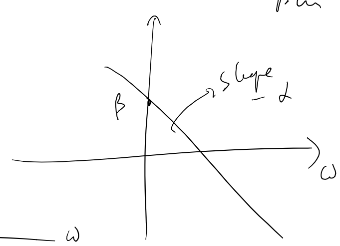
$a\cos\omega_0 n \longrightarrow a\cos(\omega_0 n - \omega_0 n_{d_0} - \phi_0)$
$b\cos\omega_1 n \longrightarrow b\cos(\omega_1 n - \omega_1 n_{d_1} - \phi_1)$
$c\cos\omega_2 n \longrightarrow c\cos(\omega_2 n - \omega_2 n_{d_2} - \phi_2)$

Def $H(\omega) = \underbrace{H_m(\omega)}_{\text{Real}} e^{-j\alpha\omega}$ linear phase

Generalized Linear Phase:

$H(\omega) = \underbrace{H_m(\omega)}_{\text{Real}} e^{j(\beta - \alpha\omega)}$ Generalized Linear phase

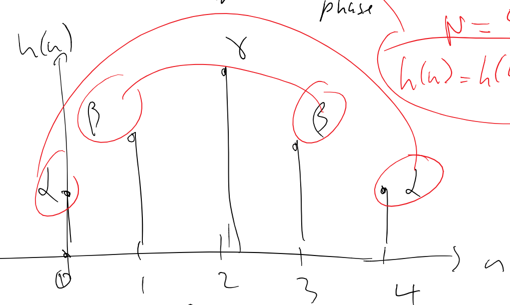




If imposes symmetry in FIR filter
Then $\rightarrow$ linear phase.

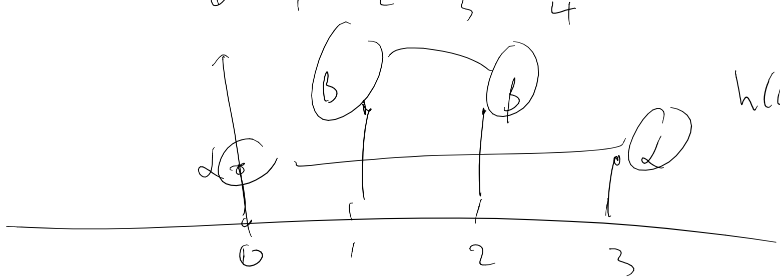
Show: If $h(n) = h(N-1-n)$ for FIR N Tap filter
filter with real coeff $\Longrightarrow$ linear phase

$N = 5$
$h(n) = h(4-n)$



$N = 4$

$h(n) = h(3-n)$



Proof: $H(\omega) = \sum_{n=0}^{N-1} h(n)\, e^{-j\omega n}$

$$H(\omega) = \sum_{n=0}^{N/2-1} h(n)\, e^{-j\omega n} + \underbrace{\sum_{n=N/2}^{N-1} h(n)\, e^{-j\omega n}}_{\text{change of variable}}$$

$m = N-1-n \leftarrow$

$$= \sum_{n=0}^{N/2-1} h(n)\, e^{-j\omega n} + \sum_{m=0}^{N/2-1} h(N-1-m)\, e^{-j\omega(N-1-m)}$$

Impose Assumption $h(n) = h(N-1-n)$

$$= \sum_{n=0}^{N/2-1} h(n) \left[e^{-j\omega n} + e^{-j\omega(N-1-n)} \right]$$

$$= e^{-j\omega\left(\frac{N-1}{2}\right)} \sum_{n=0}^{N/2-1} h(n) \left[e^{-j\omega n}\, e^{j\omega\frac{(N-1)}{2}} + e^{j\omega n}\, e^{-j\omega\frac{(N-1)}{2}} \right]$$

$$H(\omega) = e^{-j\omega\left(\frac{N-1}{2}\right)} \sum_{n=0}^{N/2-1} h(n) \left[2\cos\left(\omega n - \frac{\omega}{2}(N-1)\right) \right]$$

$\alpha = \frac{N-1}{2}$

Real $H_m(\omega)$

$$H(\omega) = H_m(\omega)\, e^{-j\alpha\omega}$$

$\longrightarrow$ linear phase.

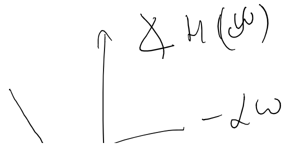
Q: what is $\measuredangle H(\omega)$

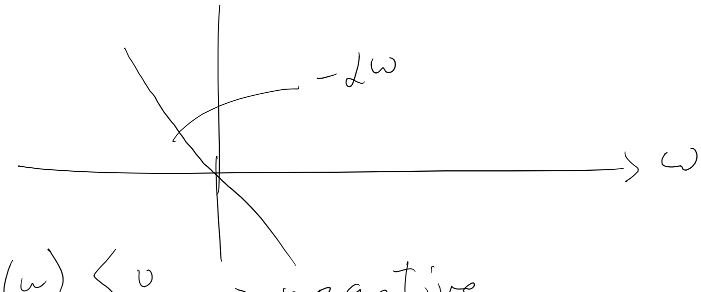
$$H(\omega) = |H(\omega)|\, e^{j\measuredangle H(\omega)} = H_m(\omega)\, e^{-j\alpha\omega}$$

Consider 2 Cases:

① $H_m(\omega) > 0$ positive $\Longrightarrow \measuredangle H(\omega) = e^{-j\alpha\omega}$

$\Longrightarrow H_m(\omega) = |H(\omega)|$





(2) $H_m(\omega) < 0 \longrightarrow$ negative

$$H(\omega) = |H_m(\omega)|\,(-1)\,e^{-j\alpha\omega}$$

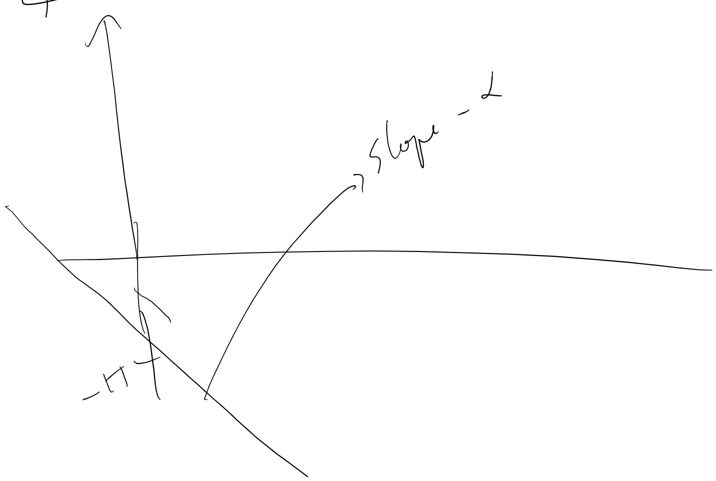
$$= |H_m(\omega)|\,e^{-j\pi}\,e^{-j\alpha\omega}$$

$$= |H_m(\omega)|\,e^{-j(\alpha\omega+\pi)}$$

$$= |H(\omega)|\,e^{j\measuredangle H(\omega)}$$

$\Longrightarrow \measuredangle H(\omega) = -\alpha\omega - \pi$

$\measuredangle H(\omega)$



Symetry in FIR filter can also decrease Multipl. count.

$$x(n) \longrightarrow \boxed{\begin{matrix} h(n) \\ h(n)=h(N-1-n) \end{matrix}} \longrightarrow y(n)$$

$$y(n) = \sum_{k=0}^{N-1} h(k)\,x(n-k)$$

$$= \sum_{k=0}^{N/2-1} h(k)\,x(n-k) + \sum_{k=N/2}^{N-1} h(k)\,x(n-k)$$

change variable

$k = N-1-m$

$\sum^{N/2-1}$ ... $\sum^{N/2-1}$ ...

$K = N - 1 - m$

$$y(n) = \sum_{k=0}^{N/2-1} h(k)\,x(n-k) + \sum_{m=0}^{N/2-1} h(N-1-m)\,x(n-N+1+m)$$

$$y(n) = \sum_{k=0}^{N/2-1} h(k)\,\underbrace{\left[\,x(n-k) + x(n-N+1+k)\,\right]}_{w(k)}$$

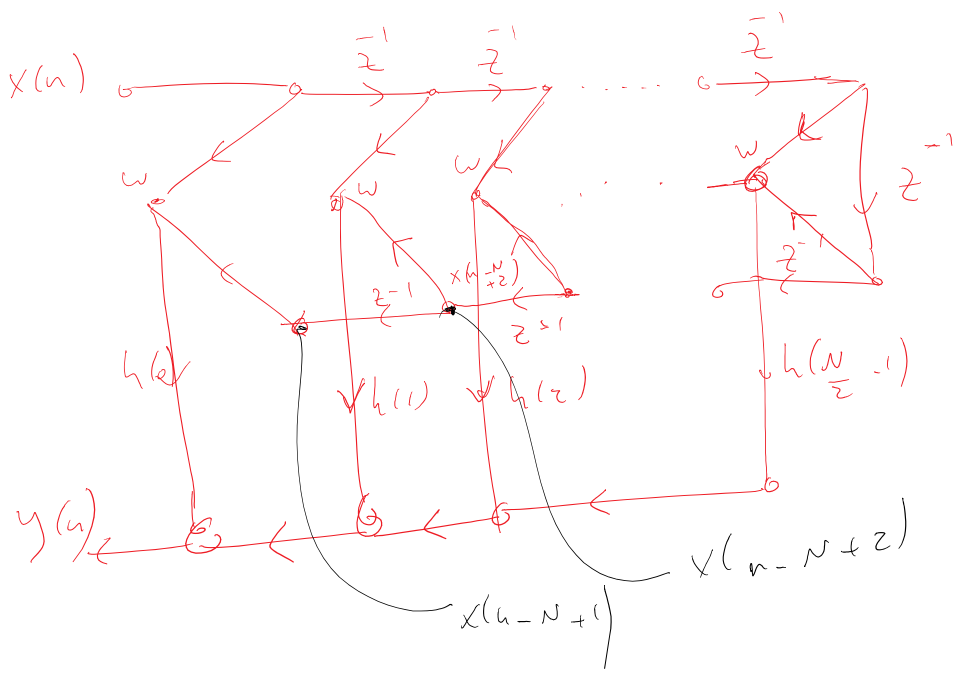
For each $y(n)$: $N/2$ mults
$N/2$ adds + $N/2$ adds to form $w(k)$

Regular:

$$\sum h(k)\,x(n-k)$$

N mults
N adds

$1/2$ as many mults



April 11th

Condition for Achieving Linear phase

$$H(\omega) = H_m(\omega)\, e^{j(\beta - \alpha\omega)}$$

α = group delay or

$$H(\omega) = \underbrace{H_m(\omega)}_{\substack{\text{Real} \\ + \\ \text{or} \\ -}} e^{j(\beta - \alpha\omega)}$$

α = group delay

$\measuredangle H(\omega) = \beta - \alpha\omega$

$-\frac{d}{d\omega}(\measuredangle H(\omega)) = \alpha$

$$H(\omega) = H_m(\omega)\cos(\beta - \alpha\omega) + j\, H_m(\omega)\sin(\beta - \alpha\omega)$$

$$\tan \measuredangle H(\omega) = \frac{\sin(\beta - \alpha\omega)}{\cos(\beta - \alpha\omega)} = \tan(\beta - \alpha\omega)$$

equ 1

Derive $\measuredangle H(\omega)$ in Terms of $h(n)$.

$$H(\omega) = \sum h(n) e^{-j\omega n}$$

$$H(\omega) = \sum_n h(n)\cos\omega n - j\sum_n h(n)\sin(\omega n)$$

$$\tan \measuredangle H(\omega) = \frac{-\sum_n h(n)\sin(\omega n)}{\sum_n h(n)\cos(\omega n)}$$

Equ 2

Combine 1 & 2 $\rightarrow$

$$\boxed{\frac{\sin(\beta - \alpha\omega)}{\cos(\beta - \alpha\omega)} = \frac{-\sum_n h(n)\sin\omega n}{\sum_n h(n)\cos\omega n}}$$

Necessary Condition for $h(n)$ to be GLP.

$$\Rightarrow \sin(\beta - \alpha\omega)\sum_n h(n)\cos\omega n + \cos(\beta - \alpha\omega)\sum_n h(n)\sin\omega n = 0$$

$$\boxed{\sum_{n=0}^{N-1} h(n)\sin[\omega(n - \alpha) + \beta] = 0}$$

$\rightarrow$ Necessary Condition for $h(n)$ To be G.L.P.

Case ①: $\beta = 0$ or π

$$\boxed{\sum_{n=0}^{N-1} h(n)\sin(\omega(n - \alpha)) = 0}$$

Can Show: If $N = 2\alpha + 1$

$h(n) = h(N - 1 - n)$

Can show: If $N = 2\alpha + 1$

$h(n) = h(N-1-n)$

Then we satisfy the condit.

Case ②: $\beta = \pi/2$ or $3\pi/2$

$$\sum_{n=0}^{N-1} h(n) \cos[(n-\alpha)\omega] = 0$$

Can show if $N = 2\alpha + 1$

$h(N-1-n) = -h(n)$

This is satisfied

Ex $N = 4$ # of Taps. even

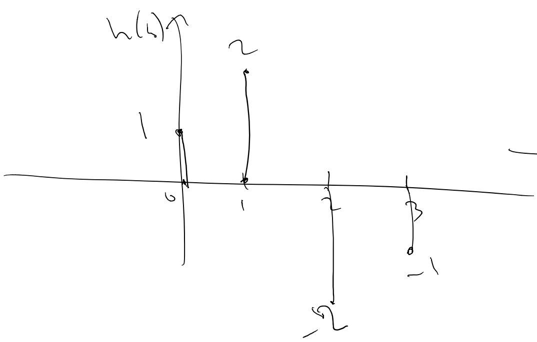
$h(3-n) = -h(n)$

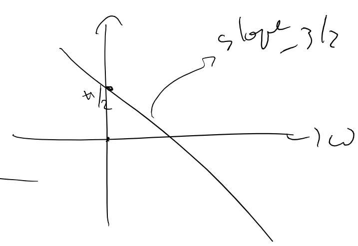
$\beta = \pi/2$

$N = 2\alpha + 1 \Rightarrow \alpha = 3/2$

$$H(\omega) = H_m(\omega)\, e^{j\left(\frac{\pi}{2} - \frac{3}{2}\omega\right)}$$

$\measuredangle H(\omega)$



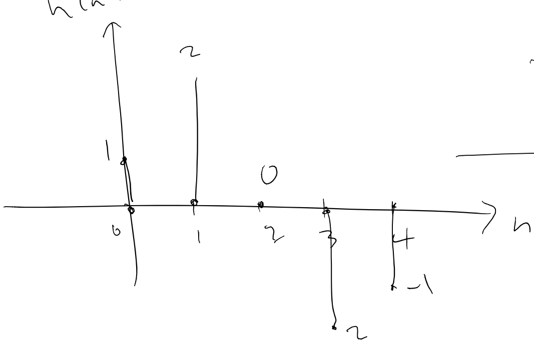


Ex $N = 5$

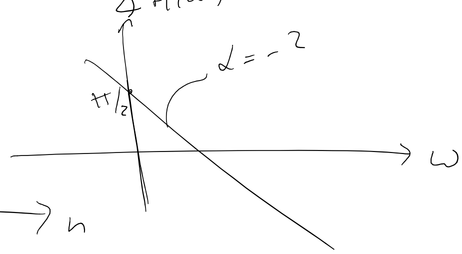
$N = 2\alpha + 1 = 5 \Rightarrow \alpha = 2$

$\beta = \pi/2$

$h(N-1-n) = -h(n) \Rightarrow h(4-n) = -h(n)$



$\measuredangle H(\omega)$



C... (ii) $\beta = 0$ ... $n < N$

Case ① $\beta = 0$

$$h(n) = \begin{cases} h(N-1-n) & 0 \le n < N \\ 0 & \text{otherwise} \end{cases}$$

$$H(\omega) = \underbrace{H_m(\omega)}_{\text{Real}} \, e^{j\left(\frac{N-1}{2}\right)\omega}$$

$\beta = 0$

$\alpha = \frac{N-1}{2}$

Case ② $\beta = \pi/2$

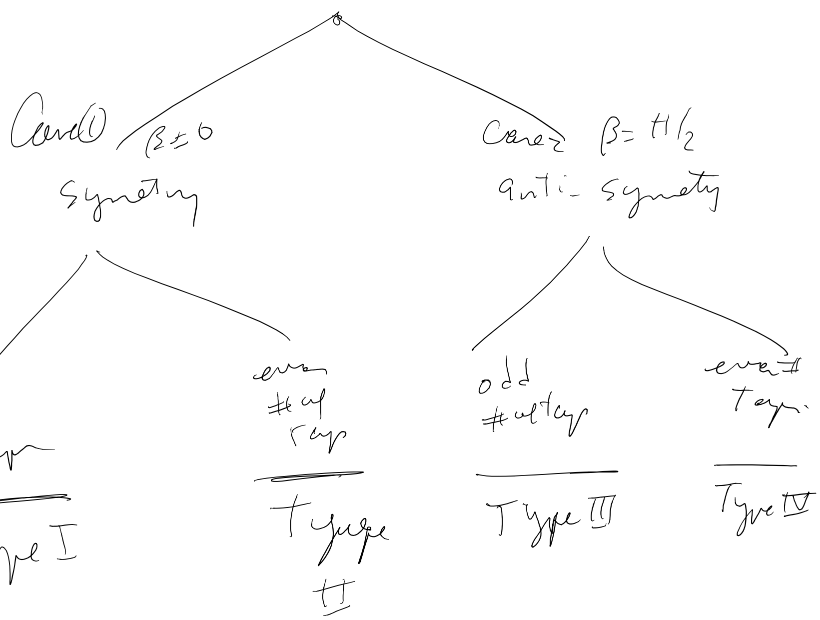
$$h(n) = \begin{cases} -h(N-1-n) & 0 \le n < N \\ 0 & \text{otherwise} \end{cases}$$

$$H(\omega) = H_m(\omega)\, e^{j(\beta - \alpha\omega)}$$

$\beta = \pi/2$

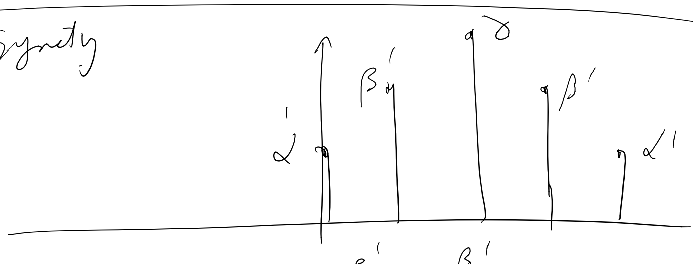
$\alpha = \frac{N-1}{2}$

$$H(\omega) = H_m(\omega)\, e^{j\pi/2}\, e^{-j\omega\left(\frac{N-1}{2}\right)}$$



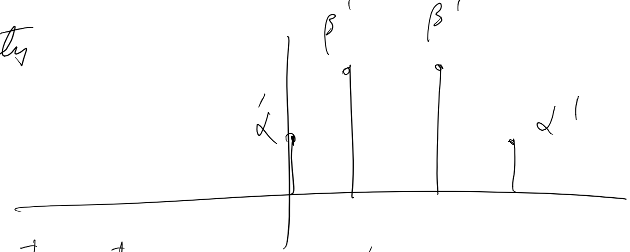
Type I Symmetry

$N = 5$



Type II Symmetry

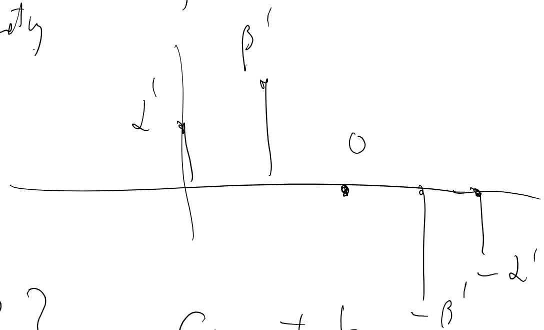
$N = 4$



Type III: Antisymmetry

N odd

$N = 5$



$$\left.\begin{array}{l} H(0) = 0 \\ H(\pi) = 0 \end{array}\right\} \rightarrow \text{Cannot be LPF, HPF}$$

Can be Bandpass.

Type IV

antisymmetry

and N even

$\beta = \pi/2$

Cannot be LPF

$H(\omega) = H_m(\omega)\, e^{j(\beta - \alpha\omega)}$

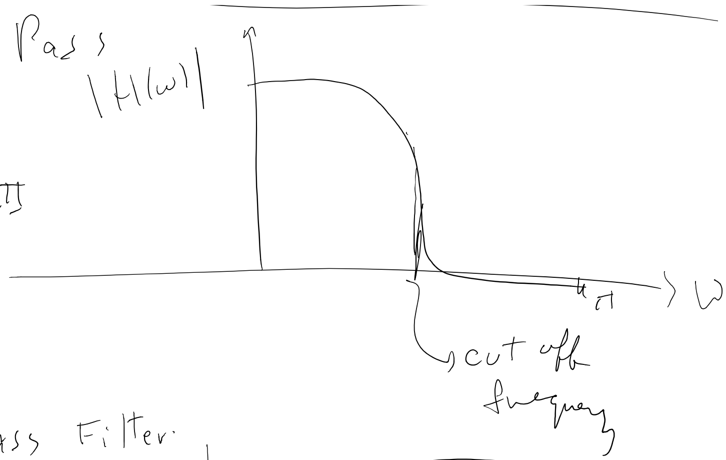
	Symmetry or antisymmetry	N even or odd	α	β	$H_m(\omega)$	Constraint
Type I	$h(n) = h(N-1-n)$	odd	$\frac{N-1}{2}$	0	$\sum_{0}^{\frac{N-1}{2}} a(n) \cos \omega n$	Real.
II	$h(n) = h(N-1-n)$	even	$\frac{N-1}{2}$	0	$\sum_{n=1}^{N/2} b(n) \cos \omega(n - \frac{1}{2})$	Real. $H(\pi) = 0$
III	$h(n) = -h(N-1-n)$	odd	$\frac{N-1}{2}$	$\pi/2$	$\sum_{n=1}^{\frac{N-1}{2}} c(n) \sin \omega n$	purely imaginary $H(0) = 0$ $H(\pi) = 0$
IV	$h(n) = -h(N-1-n)$	even	$\frac{N-1}{2}$	$\pi/2$	$\sum_{n=1}^{N/2} d(n) \sin \omega(n - \frac{1}{2})$	purely imaginary $H(0) = 0$

Low Pass

$|H(\omega)|$ 

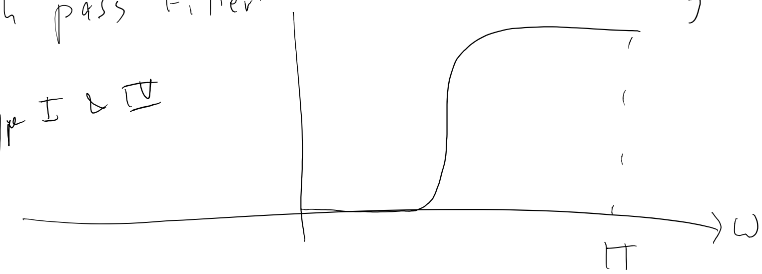
Low Pass

Type I & II



High pass Filter.

Type I & IV

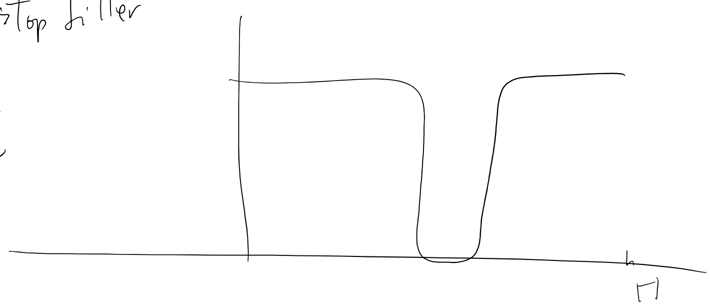


Band Pass

I, II, III, IV

Band stop filter

I



	LP	HP	BP	BS
I	✓	✓	✓	✓
II	✓	X	✓	X
III	X	X	✓	X
IV	X	✓	✓	X

IV | × | ✓ | ✓ | ×

Filter Design
LTI

FIR. The Determine coef. of h(n)

IIR.

Rational Transfer fn
$H(z) = \frac{P(z)}{Q(z)}$
determine coeff of P(z) and Q(z)

Diff. eqn.

not a rational xfer fn

Build Filter

1. Specification → Application dependent

2. Design ← Covered

3. Implementation

C prog on P.C

matlab

prog DSP chip
TI
A.D.

FPGA /

prog ASIC

ASIC

FIR

FIR

Windows   Optimum.

FIR Filter Design Using Windows

(1) Start with desired freq. Respon. $H_d(\omega)$

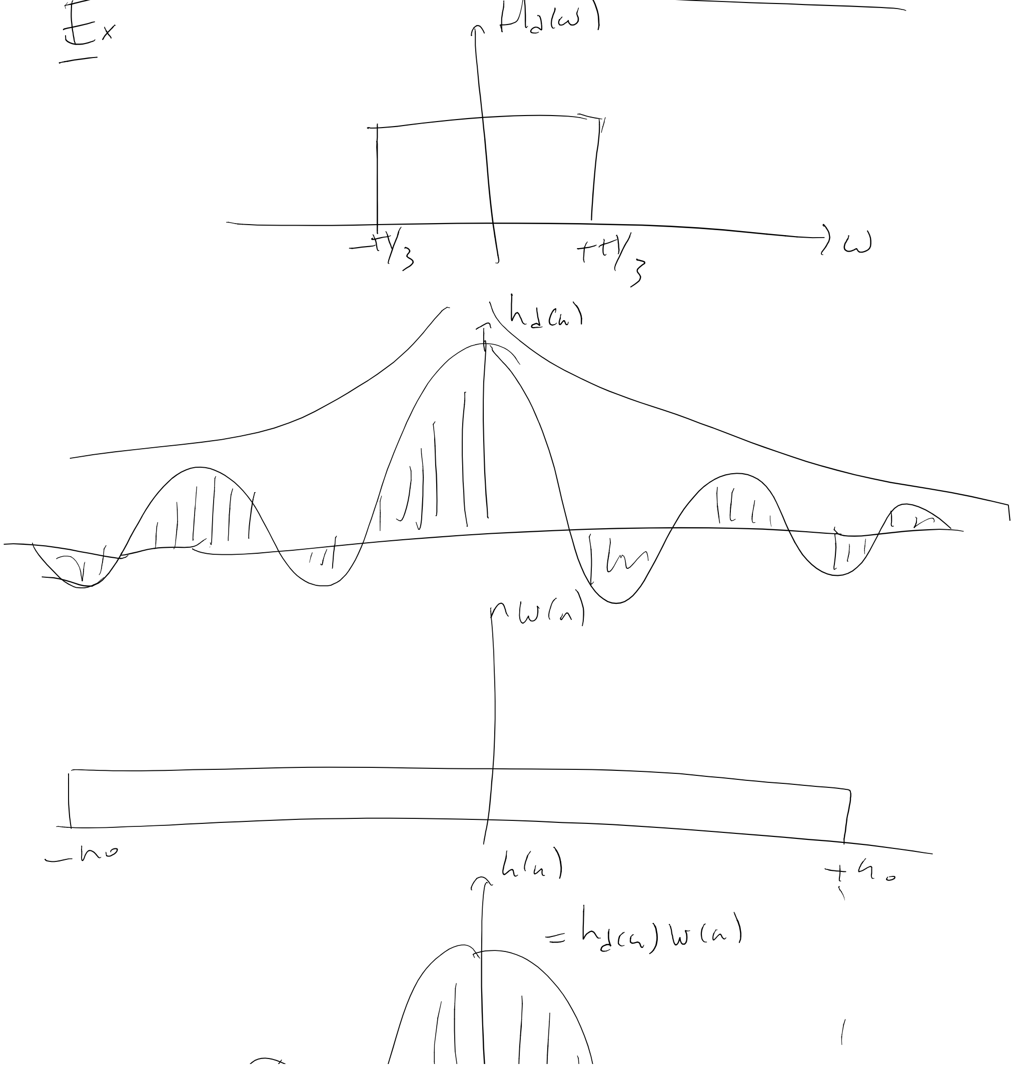
(2) Compute IDTFT$\{H_d(\omega)\} = h_d(n) =$

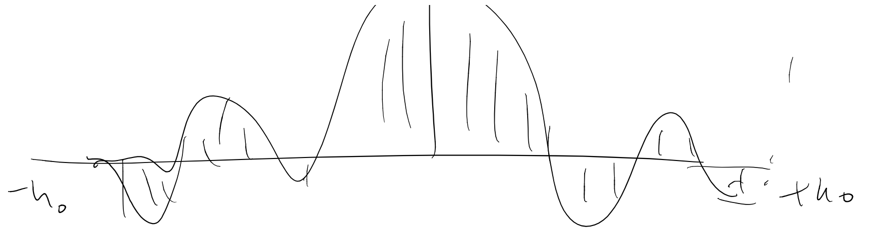
= desired Impulse respon.

(3) $h(n) = h_d(n)\, w(n)$

finite length window Funct.

Ex





Ex High Pass Filter

(2) $IDTFT\{H_d(\omega)\} =$

Assume Linear phase for both desired & final

Assume Type I $\quad \beta = 0, \quad \alpha = \frac{N-1}{2}$

$$\Longrightarrow H_d(\omega) = \underbrace{H_m(\omega)}_{\text{Real}} e^{-j\alpha\omega}$$

$$|H_d(\omega)| = \begin{cases} 1 & \pi - \omega_c \le \omega < \pi + \omega_c \\ 0 & \text{otherwise} \end{cases}$$

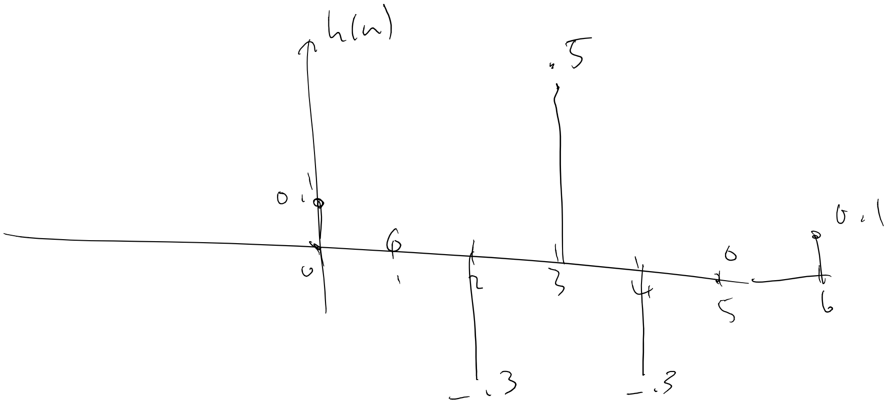
$$H_d(\omega) = \begin{cases} e^{-j\alpha\omega} & \pi - \omega_c < \omega < \pi + \omega_c \\ 0 & \text{otherwise} \end{cases}$$

$$IDTFT\{H_d(\omega)\} = \frac{1}{2\pi} \int_{\pi-\omega_c}^{\pi+\omega_c} e^{-j\alpha\omega} e^{j\omega n} \, d\omega$$

$$h_d(n) = \frac{(-1)^{n-\alpha}}{\pi(n-\alpha)} \sin\left[\omega_c(n-\alpha)\right]$$

(3) $h(n) = h_d(n) \, w(n)$

$w(n) \to 7$ Taps.

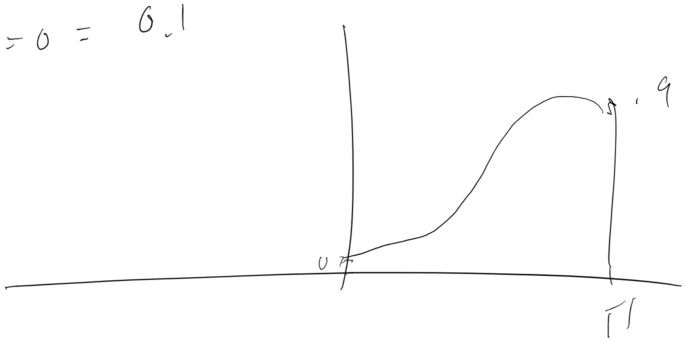


$$\left[H(\omega)\right]_{\omega=\pi} = \sum_n h(n)\, e^{-j\pi n}$$

$$= 0.1 - 0.3 - 0.5 - 0.3$$

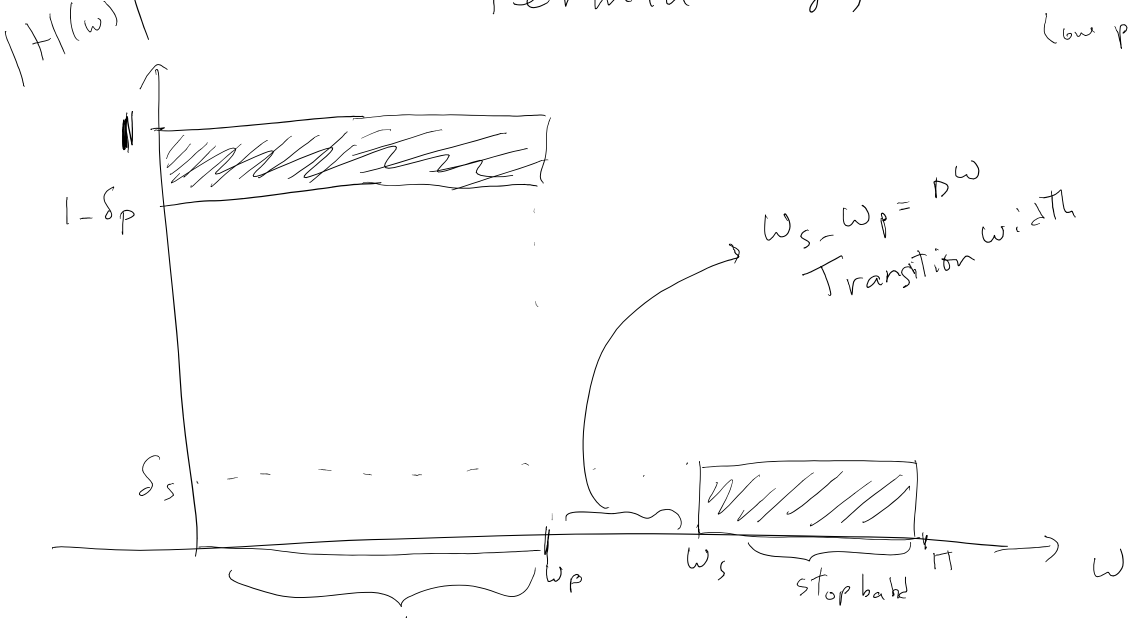
$$= -0.9$$

$$\left[H(\omega)\right]_{\omega=0} = 0.1$$



Terminology

Filter Spec
Low pass



Passban

$$0 \leq \omega \leq \omega_p \qquad \text{passband}$$

$$\omega_s \leq \omega \leq \pi \qquad \text{stopband}$$

δ_p = passband ripple $\qquad$ δ_s = stopband ripple.

For Window: (1) Transition width depends on width of the mainlobe of F.T. window fn

(2) Ripple (δ_p, δ_s) depend on sidelobe of F.T. of window

One way to reduce $\Delta\omega$ is to increase window size (length in time domain).

Another way: Different window shape.

$\Rightarrow$ Same length: Rectangular $\rightarrow$ small mainlobe width

Blackman $\rightarrow$ large mainlobe width

Ripple δ_p δ_s:

- shape of $w(n)$ affects sidelobe
- size of $w(n)$ does <u>not</u> significantly affect the side lobe behavi

w(n) shape → sidelobe and mainlobe.

size length $\xrightarrow{\text{only}}$ mainlobe

Strategy:

1. use shape → sidelobe ripple
2. use size length → mainlobe behavior

Showed Fig chp 7 of OAS

Kaiser Window:

$$w(n) = \begin{cases} I_0\left[\beta\left(1-\left(\frac{n-\alpha}{\alpha}\right)^2\right)^{1/2}\right] & 0 \le n \le M \\ 0 & \text{otherwise} \end{cases}$$

I_0 = zeroth order modified Bess

x^3 $\quad x^4$

el fn.

x^6

$+ \overline{} + \dots$

I_0 = zeroth

$$I_0(x) = 1 + \frac{x^2}{2^2 (1!)^2} + \frac{x^4}{2^4 (\ldots}$$

Solution differential eqn

$$\frac{x^2 d^2y}{dx^2} + x\frac{dy}{dx} - (x \ldots$$

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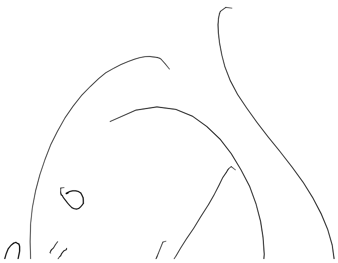
① $\Delta W = W_s - W_p =$

② ripple $\delta \longrightarrow A = -2 \ldots$

$$\alpha = \frac{M}{2} = \frac{N-1}{2}$$



$$M = 2\alpha = \frac{A - 8}{2.285 \ldots}$$

$$\frac{}{(2!)^2} + \frac{x^6}{2^6 (3!)^2} + \dots$$

$$\left(\quad + n^2 \right) y = 0$$

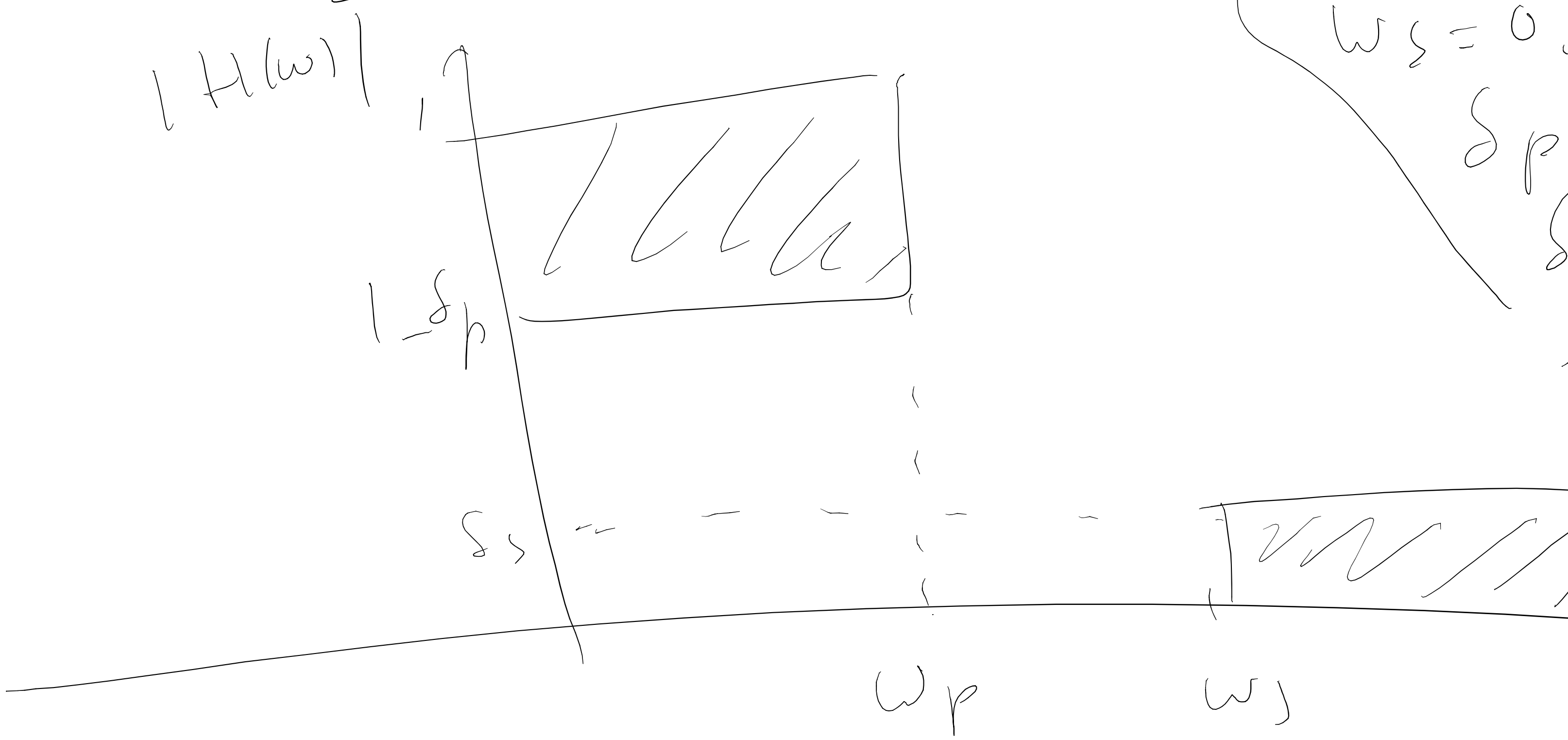
nth order
modified
Bessel fn

$\log_{10}$ δ

$$M = 2\alpha = \frac{\quad}{2.285\,\Delta\omega}$$

$$\beta = \begin{cases} 0.1102(A - 8.7) \\ .5842(A-21)^{0.4} \\ \\ 0 \end{cases}$$

Ex LPF



$\omega_p = 0.$
$\omega_s = 0.$
δ_p
δ

$A > 50$

$21 \le A < 50$

otherwise

4π $\rightarrow \Delta\omega = 0.2\pi$

$.6\pi$

$= 0.01$ $\rightarrow \delta = 0.001$

$\delta = 0.001$ $\delta = -60\,dB$

$A = 20 \log_{10}$

$\Rightarrow M = \underline{37} \rightarrow N = \underline{38}$

$\beta' = 6$

π

$\eta_n = \eta_d$