UC Berkeley
Department of Electrical Engineering and Computer Sciences

EECS 126: Probability and Random Processes

Discussion 10
Spring 2019

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1. **Generating Erdös-Renyi Random Graphs**

   True/False: Let G_1 and G_2 be independent Erdös-Renyi random graphs on n vertices with probabilities p_1 and p_2, respectively. Let $G = G_1 \cup G_2$, that is, G is generated by combining the edges from G_1 and G_2. Then, G is an Erdös-Renyi random graph on n vertices with probability $p_1 + p_2$.

2. **Isolated Vertices**

   Consider a Erdös-Renyi random graph $\mathcal{G}(n, p(n))$, where n is the number of vertices and $p(n)$ is the probability that a specific edge appears in the graph. Let X_n be the number of isolated vertices in $\mathcal{G}(n, p(n))$. Show that

$$\mathbb{E}[X_n] \xrightarrow{n\to\infty} \begin{cases} \infty, & p(n) \ll \dfrac{\ln n}{n}, \\ \exp(-c), & p(n) = \dfrac{\ln n + c}{n}, \\ 0, & p(n) \gg \dfrac{\ln n}{n}, \end{cases}$$

   where the notation $p(n) \ll f(n)$ means that $p(n)/f(n) \to 0$ as $n \to \infty$, and $p(n) \gg f(n)$ means $p(n)/f(n) \to \infty$ as $n \to \infty$. Show also that in the third case, $p(n) \gg (\ln n)/n$, we have $X_n \to 0$ in probability as well.

3. **Sub-Critical Forest**

   Consider a random graph $\mathcal{G}(n, p(n))$ where $p(n) \ll 1/n$ (this is called the **sub-critical phase**). Show that the probability that $\mathcal{G}(n, p(n))$ is a forest, i.e. contains no cycles, tends to 1 as $n \to \infty$. [If X_n is the number of cycles, compute $\mathbb{E}[X_n]$ and show that $\mathbb{E}[X_n] \to 0$. Then, apply the First Moment Method.]