UC Berkeley
Department of Electrical Engineering and Computer Sciences

EECS 126: Probability and Random Processes

Discussion 12
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1. **Hypothesis Testing for Gaussian Distribution**

Assume that X has prior probabilities $\mathbb{P}(X = 0) = \mathbb{P}(X = 1) = 1/2$. Further

- If $X = 0$, then $Y \sim \mathcal{N}(\mu_0, \sigma_0^2)$.
- If $X = 1$, then $Y \sim \mathcal{N}(\mu_1, \sigma_1^2)$.

Assume $\mu_0 < \mu_1$ and $\sigma_0 < \sigma_1$.

Using the Bayesian formulation of hypothesis testing, find the optimal *decision rule* $r : \mathbb{R} \to \{0, 1\}$ with respect to the minimum expected cost criterion

$$\min_{r:\mathbb{R}\to\{0,1\}} \mathbb{E}[I\{r(Y) \neq X\}].$$

2. **Hypothesis Testing for Uniform Distribution**

Assume that

- If $X = 0$, then $Y \sim \text{Uniform}[-1, 1]$.
- If $X = 1$, then $Y \sim \text{Uniform}[0, 2]$.

Using the Neyman-Pearson formulation of hypothesis testing, find the optimal randomized *decision rule* $r : [-1, 2] \to \{0, 1\}$ with respect to the criterion

$$\begin{aligned} \min_{\text{randomized } r:[-1,2]\to\{0,1\}} \ & \mathbb{P}\big(r(Y) = 0 \mid X = 1\big) \\ \text{s.t. } & \mathbb{P}\big(r(Y) = 1 \mid X = 0\big) \leq \beta, \end{aligned}$$

where $\beta \in [0, 1]$ is a given upper bound on the false positive probability.

3. **Cauchy-Schwarz Inequality**

In this problem, we will introduce an important inequality called the **Cauchy-Schwarz Inequality**. Let X, Y be random variables with finite non-zero variance. Then, the inequality states that $|\,\mathbb{E}[XY]| \leq \sqrt{\mathbb{E}[X^2]\,\mathbb{E}[Y^2]}$.

(a) Starting with the simple facts $\mathbb{E}[(X - Y)^2] \geq 0$ and $\mathbb{E}[(X + Y)^2] \geq 0$, show that

$$|\,\mathbb{E}[XY]| \leq \frac{\mathbb{E}[X^2] + \mathbb{E}[Y^2]}{2}. \tag{1}$$

(b) Actually, $(\mathbb{E}[X^2] + \mathbb{E}[Y^2])/2 \geq \sqrt{\mathbb{E}[X^2]\,\mathbb{E}[Y^2]}$ by a famous inequality known as the Arithmetic Mean-Geometric Mean (AM-GM) Inequality, so our bound is too loose. We can sharpen it by observing that the LHS is unchanged if we replace X by λX and Y by Y/λ (where $\lambda > 0$). Applying the above bound to λX and Y/λ instead, optimize over $\lambda > 0$ to deduce the Cauchy-Schwarz Inequality.