

Discussion 14
Spring 2019

1. MMSE from Joint Density

Let the joint density of two random variables X and Y be

$$f_{X,Y}(x,y) = \frac{1}{4}(2x+y)I\{0 \leq x \leq 1\}I\{0 \leq y \leq 2\}.$$

Suppose you observe Y drawn from this joint density. Find $\text{MMSE}[X | Y = y]$.

2. Forrest Gump

Forrest Gump is running across the United States, and we would like to track his progress. Assume that on day $n \in \mathbb{N}$ he runs $X(n)$ miles, and the amount he runs each day is determined by the amount he ran on the previous day with some random noise in the following manner: $X(n) = \alpha X(n-1) + V(n)$. Unfortunately, the measurements of the distance he traveled on each day are also subject to some noise. Assume that $Y(n)$ gives the measured number of miles Forrest Gump traveled on day n and that $Y(n) = \beta X(n) + W(n)$. For this problem, assume that $X(0) \sim \mathcal{N}(0, \sigma_X^2)$, $W(n) \sim \mathcal{N}(0, \sigma_W^2)$, $V(n) \sim \mathcal{N}(0, \sigma_V^2)$ are independent.

- (a) Suppose that you observe $Y(0)$. Find the MMSE of $X(0)$ given this observation.
- (b) Express both $\mathbb{E}[Y(n) | Y(0), \dots, Y(n-1)]$ and $\mathbb{E}[X(n) | Y(0), \dots, Y(n-1)]$ in terms of $\hat{X}(n-1)$, where $\hat{X}(n-1)$ is the MMSE of $X(n-1)$ given the observations $Y(0), Y(1), \dots, Y(n-1)$.
- (c) Show that:

$$\hat{X}(n) = \alpha \hat{X}(n-1) + k_n[Y(n) - \alpha \beta \hat{X}(n-1)]$$

where

$$k_n = \frac{\text{cov}(X(n), \tilde{Y}(n))}{\text{var } \tilde{Y}(n)}$$

$$\text{and } \tilde{Y}(n) = Y(n) - L[Y(n) | Y(0), Y(1), \dots, Y(n-1)].$$

3. Hidden Markov Models

Figure 1 shows the life of Sinho. Some days he is Tired and some days he is Energetic. But he doesn't tell you whether he's Tired or not, and all you can observe is whether he Jumps, Eats, Runs, or Sleeps. We start on day 1 in the Energetic state and there is one transition per day.

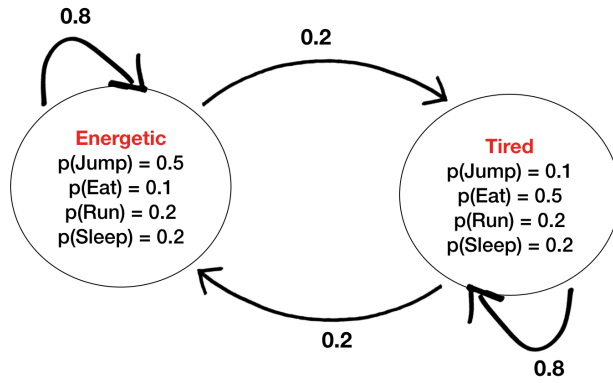


Figure 1: HMM model for Sinho.

For the questions below, we use the following notations:

- q_t : state on day t
- O_t : observation on day t

- What is $\mathbb{P}(q_2 = \text{Energetic} \mid O_2 = \text{Eat})$?
- What is $\mathbb{P}(O_3 = \text{Sleep} \mid O_2 = \text{Eat})$?