

**Discussion 5**  
Spring 2019

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**1. Confidence Intervals: Chebyshev vs. Chernoff vs. CLT**

Let  $X_1, \dots, X_n$  be i.i.d. Bernoulli( $q$ ) random variables, with common mean  $\mu = \mathbb{E}[X_1] = q$  and variance  $\sigma^2 = \text{var}(X_1) = q(1 - q)$ . We want to estimate the mean  $\mu$ , and towards this goal we use the sample mean estimator

$$\bar{X}_n \triangleq \frac{X_1 + \dots + X_n}{n}.$$

Given some confidence level  $a \in (0, 1)$  we want to construct a confidence interval around  $\bar{X}_n$  such that  $\mu$  lies in this interval with probability at least  $1 - a$ .

- (a) Use Chebyshev's inequality in order to show that  $\mu$  lies in the interval

$$\left( \bar{X}_n - \frac{\sigma}{\sqrt{n}} \frac{1}{\sqrt{a}}, \bar{X}_n + \frac{\sigma}{\sqrt{n}} \frac{1}{\sqrt{a}} \right)$$

with probability at least  $1 - a$ .

- (b) Recall that in Homework 4 you showed that

$$\mathbb{P}(|\bar{X}_n - q| \geq \epsilon) \leq 2e^{-2n\epsilon^2}, \quad \text{for any } \epsilon > 0.$$

Use this inequality in order to show that  $\mu$  lies in the interval

$$\left( \bar{X}_n - \frac{1}{\sqrt{n}} \sqrt{2 \ln \frac{2}{a}}, \bar{X}_n + \frac{1}{\sqrt{n}} \sqrt{2 \ln \frac{2}{a}} \right)$$

with probability at least  $1 - a$ .

- (c) Show that if  $Z \sim \mathcal{N}(0, 1)$ , then

$$\mathbb{P}(|Z| \geq \epsilon) \leq 2e^{-\frac{\epsilon^2}{2}}, \quad \text{for any } \epsilon > 0.$$

- (d) Use the Central Limit Theorem, and Part (c) in order to heuristically argue that  $\mu$  lies in the interval

$$\left( \bar{X}_n - \frac{\sigma}{\sqrt{n}} \sqrt{2 \ln \frac{2}{a}}, \bar{X}_n + \frac{\sigma}{\sqrt{n}} \sqrt{2 \ln \frac{2}{a}} \right)$$

with probability at least  $1 - a$ .

- (e) Compare the three confidence intervals.

## 2. Gambling Game

Lets play a game. You stake a positive initial amount of money  $w_0$ . You toss a fair coin. If it comes up heads you earn an amount equal to three times your stake, so you quadruple your wealth. If it comes up tails you lose everything. There is one requirement though, you are not allowed to quit and have to keep playing, by staking all your available wealth, over and over again.

Let  $W_n$  be a random variable which is equal to your wealth after  $n$  plays.

- (a) Find  $\mathbb{E}[W_n]$  and show that  $\lim_{n \rightarrow \infty} \mathbb{E}[W_n] = \infty$ .
- (b) Since  $\lim_{n \rightarrow \infty} \mathbb{E}[W_n] = \infty$ , this game sounds like a good deal! But wait a moment!! Where does the sequence of random variables  $\{W_n\}_{n \geq 0}$  converges almost surely to?

## 3. Breaking a Stick

I break a stick  $n$  times, where  $n$  is a positive integer, in the following manner: the  $i$ th time I break the stick, I keep a fraction  $X_i$  of the remaining stick where  $X_i$  is uniform on the interval  $[0, 1]$  and  $X_1, X_2, \dots, X_n$  are i.i.d. Let  $P_n = \prod_{i=1}^n X_i$  be the fraction of the original stick that I end up with.

- (a) Show that  $P_n^{1/n}$  converges almost surely to some constant function.
- (b) Compute  $\mathbb{E}[P_n]^{1/n}$ .