UC Berkeley
Department of Electrical Engineering and Computer Sciences

EECS 126: PROBABILITY AND RANDOM PROCESSES

Problem Set 11
Spring 2019

Issued: Thursday April 18, 2019 **Due:** 11:59 PM, Wednesday April 24, 2019

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1. **Flipping Coins and Hypothesizing**

   You flip a coin until you see heads. Let $X = 0$ be the hypothesis that the bias of the coin (the probability of heads) is p, and $X = 1$ be the hypothesis that the bias of the coin is q, for $q > p$. Solve the hypothesis testing problem: maximize $\mathbb{P}[\hat{X} = 1 \mid X = 1]$ subject to $\mathbb{P}[\hat{X} = 1 \mid X = 0] \leq \beta$ for $\beta \in [0, 1]$.

2. **BSC Hypothesis Testing**

   Recall that we looked at the MLE and the MAP of the BSC in last week's homework. Now we will examine the BSC in the hypothesis testing framework.

   You are testing a digital link that corresponds to a BSC with some error probability $\epsilon \in [0.1, 0.5)$. You observe n inputs and outputs of the BSC, where n is a positive integer. You want to solve a hypothesis problem to detect that $\epsilon > 0.1$ with a probability of false alarm at most equal to 0.05. Assume that n is very large and use the CLT.

   Hint: The null hypothesis is $\epsilon = 0.1$. The alternate hypothesis is $\epsilon > 0.1$, which is a **composite hypothesis** (this means that under the alternate hypothesis, the probability distribution of the observation is not completely determined; compare this to a **simple hypothesis** such as $\epsilon = 0.3$, which *does* completely determine the probability distribution of the observation). The Neyman-Pearson Lemma we learned in class applies for the case of a simple null hypothesis and a simple alternate hypothesis, so it does not directly apply here.

   To fix this, fix some specific $\epsilon' > 0.1$ and use the Neyman-Pearson Lemma to find the optimal hypothesis test for the hypotheses $\epsilon = 0.1$ vs. $\epsilon = \epsilon'$. Then, argue that the optimal decision rule does not depend on the specific choice of ϵ'; thus, the decision rule you derive will be *simultaneously* optimal for testing $\epsilon = 0.1$ vs. $\epsilon = \epsilon'$ for all $\epsilon' > 0.1$.

3. **Projections**

   The following exercises are from the note on the Hilbert space of random variables. See the notes for some hints.

   (a) Let $\mathcal{H} := \{X : X \text{ is a real-valued random variable with } \mathbb{E}[X^2] < \infty\}$. Prove that $\langle X, Y \rangle := \mathbb{E}[XY]$ makes $\mathcal{H}$ into a real inner product space.

   (b) Let U be a subspace of a real inner product space V and let P be the projection map onto U. Prove that P is a linear transformation.

(c) Suppose that U is finite-dimensional, $n := \dim U$, with basis $\{v_i\}_{i=1}^n$. Suppose that the basis is orthonormal. Show that $Py = \sum_{i=1}^n \langle y, v_i \rangle v_i$. (Note: If we take $U = \mathbb{R}^n$ with the standard inner product, then P can be represented as a matrix in the form $P = \sum_{i=1}^n v_i v_i^\mathsf{T}$.)

4. **Exam Difficulties**

   The difficulty of an EE 126 exam, Θ, is uniformly distributed on $[0, 100]$, and Alice gets a score X that is uniformly distributed on $[0, \Theta]$. Alice gets her score back and wants to estimate the difficulty of the exam.

   (a) What is the LLSE for Θ?

   (b) What is the MAP of Θ?

5. **Photodetector LLSE**

   Consider a photodetector in an optical communications system that counts the number of photons arriving during a certain interval. A user conveys information by switching a photon transmitter on or off. Assume that the probability of the transmitter being on is p. If the transmitter is on, the number of photons transmitted over the interval of interest is a Poisson random variable Θ with mean λ, and if it is off, the number of photons transmitted is 0. Unfortunately, regardless of whether or not the transmitter is on or off, photons may be detected due to "shot noise". The number N of detected shot noise photons is a Poisson random variable N with mean μ, independent of the transmitted photons. Given the number of detected photons, find the LLSE of the number of transmitted photons.

6. **[Bonus] p-Value**

   The bonus question is just for fun. You are not required to submit the bonus question, but do give it a try and write down your progress.

   Let us define what the p-value of a hypothesis test is. Given an observation Y and a constraint of β on the PFA, the Neyman-Pearson rule will either declare that the alternate hypothesis is true or not. The constraint on the PFA controls the trade-off between declaring the alternate hypothesis to be true when it is not (false alarm), and declaring the alternate hypothesis to be true when it is (correct detection). Therefore, for very high values of β, the hypothesis test will declare that the alternate hypothesis is true, and for very low values of β, the hypothesis test will declare that the null hypothesis is true.

   (Intuitively, the smaller the value of β, the more conservative the resulting hypothesis test is, i.e., it will be more reluctant to declare that the alternate hypothesis is true.)

   The **p-value** of the observation is the smallest value of β such that the alternate hypothesis is declared true.

   Think about this carefully, and explain why the p-value is *not* the probability that the alternate hypothesis is true.