UC Berkeley
Department of Electrical Engineering and Computer Sciences

EECS 126: Probability and Random Processes

Problem Set 3
Spring 2019

Issued: February 7, 2019 **Due:** 11:59 PM, Wedenesday, February 13, 2019

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1. **Graphical Density**

   Figure 1 shows the joint density $f_{X,Y}$ of the random variables X and Y.

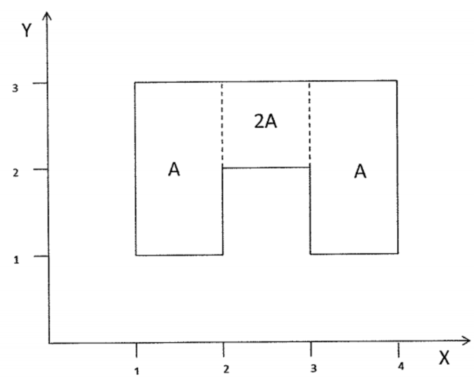

Figure 1: Joint density of X and Y.

   (a) Find A and sketch f_X, f_Y, and $f_{X|X+Y \le 3}$.
   (b) Find $\mathbb{E}[X \mid Y = y]$ for $1 \le y \le 3$ and $\mathbb{E}[Y \mid X = x]$ for $1 \le x \le 4$.
   (c) Find $\text{cov}(X, Y)$.

2. **Conditional Distribution of a Poisson Random Variable with Exponentially Distributed Parameter**

   Let X have a Poisson distribution with parameter $\lambda > 0$. Suppose λ itself is random, having an exponential density with parameter $\theta > 0$.

   (a) Show that
   $$\mathbb{E}(\lambda^k) = \frac{k!}{\theta^k}, \qquad k \in \mathbb{N}$$
   (b) What is the distribution of X?
   (c) Determine the conditional density of λ given $X = k$, where $k \in \mathbb{N}$.

3. **Gaussian Densities**

   (a) Let $X_1 \sim \mathcal{N}(0, \sigma_1^2)$, $X_2 \sim \mathcal{N}(0, \sigma_2^2)$, where X_1 and X_2 are independent. Convolve the densities of X_1 and X_2 to show that $X_1 + X_2 \sim \mathcal{N}(0, \sigma_1^2 + \sigma_2^2)$.

(b) Show that all linear combinations of i.i.d. finitely many Gaussians are Gaussian.

(c) Let $X \sim \mathcal{N}(0, \sigma^2)$; find $\mathbb{E}[X^n]$ for $n \in \mathbb{N}$.

(d) Let $Z \sim \mathcal{N}(0,1)$. For a random vector $(X_1, \ldots, X_n)$ where n is a positive integer and $X_1, \ldots, X_n$ are real-valued random variables, the expectation of $(X_1, \ldots, X_n)$ is the vector of elementwise expectations of each random variable and the **covariance matrix** of $(X_1, \ldots, X_n)$ is the $n \times n$ matrix whose (i,j) entry is $\mathrm{cov}(X_i, X_j)$ for all $i, j \in \{1, \ldots, n\}$. Find the mean and covariance matrix of $(Z, \mathbb{1}\{Z > c\})$ in terms of ϕ and Φ, the standard Gaussian PDF and CDF respectively.

4. **Joint Density for Exponential Distribution**

(a) If $X \sim Exp(\lambda)$ and $Y \sim Exp(\mu)$, X and Y independent, compute $\mathbb{P}(X < Y)$.

(b) If X_k, $1 \leq k \leq n$ are exponentially distributed with parameters $\lambda_1, \ldots, \lambda_n$, show that,

$$\mathbb{P}(X_i = \min_{1 \leq k \leq n} X_k) = \frac{\lambda_i}{\sum_{j=1}^{n} \lambda_j}$$

5. **Matrix Sketching**

Matrix sketching is an important technique in randomized linear algebra to do large computations efficiently. For example, to compute the multiplication $\mathbf{A}^T \times \mathbf{B}$ of two large matrices $\mathbf{A}$ and $\mathbf{B}$, we can use a random sketch matrix $\mathbf{S}$ to compute a "sketch" $\mathbf{SA}$ of $\mathbf{A}$ and a "sketch" $\mathbf{SB}$ of $\mathbf{B}$. Such a sketching matrix has the property that $\mathbf{S}^T\mathbf{S} \approx \mathbf{I}$ so that the approximate multiplication $\mathbf{A}^T\mathbf{S}^T\mathbf{SB}$ is close to $\mathbf{A}^T\mathbf{B}$.

In this problem, we will discuss two popular sketching schemes and understand how they help in approximate computation. Let $\hat{\mathbf{I}} = \mathbf{S}^T\mathbf{S}$ and the dimension of sketch matrix $\mathbf{S}$ be $d \times n$ (typically $d \ll n$).

(a) (**Gaussian-sketch**) Define

$$\mathbf{S} = \frac{1}{\sqrt{d}} \begin{bmatrix} S_{11} & \ldots & \ldots & S_{1n} \\ \vdots & \ddots & & \vdots \\ S_{d1} & \ldots & \ldots & S_{dn} \end{bmatrix}$$

such that S_{ij}'s are chosen i.i.d. from $\mathcal{N}(0,1)$ for all $i \in [1,d]$ and $j \in [1,n]$. Find the element-wise mean and variance of the matrix $\hat{\mathbf{I}} = \mathbf{S}^T\mathbf{S}$, that is, find $\mathbb{E}[\hat{I}_{ij}]$ and $\mathrm{Var}[\hat{I}_{ij}]$ for all $i \in [1,d]$ and $j \in [1,n]$.

(b) (**Count-sketch**) For each column $j \in [1,n]$ of $\mathbf{S}$, choose a row i uniformly randomly from $[1,d]$ such that

$$S_{ij} = \begin{cases} 1, & \text{with probability } 0.5 \\ -1, & \text{with probability } 0.5 \end{cases}$$

and assign $S_{kj} = 0$ for all $k \neq i$. An example of a 3×8 count-sketch is

$$\mathbf{S} = \begin{bmatrix} 0 & -1 & 1 & 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \end{bmatrix}$$

Again, find the element-wise mean and variance of the matrix $\hat{\mathbf{I}} = \mathbf{S}^T\mathbf{S}$.

Note that for sufficiently large d, the matrix $\hat{\mathbf{I}}$ is close to the identity matrix for both cases. We will use this fact in the lab to do an approximate matrix multiplication.

6. **Records** Let n be a positive integer and $X_1, X_2, \ldots, X_n$ be a sequence of i.i.d. continuous random variable with common probability density f_X. For any integer $2 \leq k \leq n$, define X_k as a record-to-date of the sequence if $X_k > X_i$ for all $i = 1, \ldots, k-1$. (X_1 is automatically a record-to-date.)

   (a) Find the probability that X_2 is a record-to-date.
   Hint: You should be able to do it without rigorous computation.

   (b) Find the probability that X_n is a record-to-date.

   (c) Find the expected number of records-to-date that occur over the first n trials (Hint: Use indicator functions.) Compute this when $n \to \infty$.