

**Problem Set 6**

Spring 2019

**Issued:** February 27, 2018

**Due:** 11:59 PM, Wednesday, March 6, 2019

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**1. The Weak Law of Large Numbers**

Let  $X_1, X_2, \dots$  be a sequence of independent identically distributed random variables with common mean  $\mu$  and associated transform  $M_X$ . We assume that  $M_X(s)$  is finite when  $s \in (-d, d)$ , where  $d$  is some positive number. Let

$$\bar{X}_n = \frac{X_1 + \dots + X_n}{n}.$$

- (a) Show that the transform associated with  $\bar{X}_n$  satisfies

$$M_{\bar{X}_n}(s) = M_X(s/n)^n.$$

- (b) Suppose that the transform  $M_X(s)$  has a first order Taylor series expansion around  $s = 0$ , of the form

$$M_X(s) = a + bs + o(s),$$

where  $o(s)$  is a function that satisfies  $\lim_{s \rightarrow 0} o(s)/s = 0$ . Find  $a$  and  $b$  in terms of  $\mu$ .

- (c) Show that

$$\lim_{n \rightarrow \infty} M_{\bar{X}_n}(s) = e^{\mu s}, \quad \text{for all } s \in (-d, d).$$

*Hint:* If  $\{a_n\}_{n \in \mathbb{N}}$  is a real sequence with  $\lim_{n \rightarrow \infty} a_n = a \in \mathbb{R}$ , then  $\lim_{n \rightarrow \infty} \left(1 + \frac{a_n}{n}\right)^n = e^a$ .

- (d) Deduce that  $\bar{X}_n \xrightarrow{d} \mu$ .

*Note:* From this together with Problem 6 we can conclude that  $\bar{X}_n \xrightarrow{\mathbb{P}} \mu$ .

**2. Huffman Questions**

Consider a set of  $n$  objects. Let  $X_i = 1$  or  $0$  accordingly as the  $i$ -th object is good or defective. Let  $X_1, X_2, \dots, X_n$  be independent with  $\mathbb{P}(X_i = 1) = p_i$ ; and  $p_1 > p_2 > \dots > p_n > 1/2$ . We are asked to determine the set of all defective objects. Any yes-no question you can think of is admissible.

- (a) Propose an algorithm based on Huffman coding in order to identify all defective objects.
- (b) If the longest sequence of questions is required by nature's answers to our questions which are based on Huffman coding, then what (in words) is the last question we should ask? And what two sets are we distinguishing with this question?

*Note:* This problem is related to the ‘Entropy and Information Content’ section of Lab 4.

### 3. Number of Parameters

- (a) Let  $Y_0, Y_1, \dots, Y_n$  be binary random variables. How many parameters are required to parametrize the joint distribution  $\mathbb{P}(Y_0 = y_0, Y_1 = y_1, \dots, Y_n = y_n)$ ?
- (b) Let  $Z_0, Z_1, \dots, Z_n$  be binary, independent random variables. How many parameters are required to parametrize the joint distribution  $\mathbb{P}(Z_0 = z_0, Z_1 = z_1, \dots, Z_n = z_n)$ ?
- (c) Let  $X_0, X_1, \dots, X_n, \dots$  be a Markov chain with state space  $S = \{0, 1\}$ , initial distribution  $\pi_0$  and transition probability matrix  $P$ . How many parameters are required to parametrize the joint distribution  $\mathbb{P}(X_0 = x_0, X_1 = x_1, \dots, X_n = x_n)$ , where  $X_0, X_1, \dots, X_n$  are the first  $n + 1$  random variables of the Markov chain given above?
- (d) Say we want to construct a countably infinite sequence  $Z_0, Z_1, \dots, Z_n, \dots$  of independent random variables, which is not a time-invariant Markov chain. Which defining property must this sequence violate? Give a concrete example of such sequence of independent random variables.

### 4. Backwards Markov Property

Let  $(X_n)_{n \in \mathbb{N}}$  be a Markov chain with state space  $\mathcal{S}$ . Show that for every  $m, k \in \mathbb{N}$ , with  $m \geq 1$ , we have

$$\mathbb{P}(X_k = i_0 \mid X_{k+1} = i_1, \dots, X_{k+m} = i_m) = \mathbb{P}(X_k = i_0 \mid X_{k+1} = i_1),$$

for all states  $i_0, i_1, \dots, i_m \in \mathcal{S}$ .

### 5. [Bonus] The CLT Implies the WLLN

*The bonus question is just for fun. You are not required to submit the bonus question, but do give it a try and write down your progress.*

- (a) Let  $\{X_n\}_{n \in \mathbb{N}}$  be a sequence of random variables. Show that if  $X_n \xrightarrow{d} c$ , where  $c$  is a constant, then  $X_n \xrightarrow{\mathbb{P}} c$ .
- (b) Let  $\{X_n\}_{n \in \mathbb{N}}$  be a sequence of i.i.d. random variables, with mean  $\mu$  and finite variance  $\sigma^2$ . Show that the CLT implies the WLLN.