UC Berkeley
Department of Electrical Engineering and Computer Sciences

EECS 126: Probability and Random Processes

Problem Set 8
Spring 2019

Issued: Friday, March 9, 2018 **Due:** Wednesday, March 14, 2018

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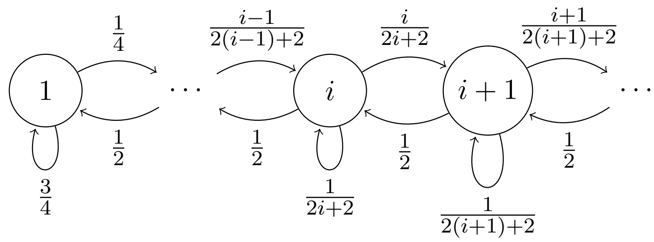
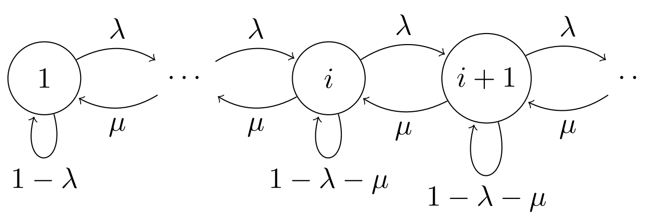
1. Markov Chains with Countably Infinite State Space

(a) Consider a Markov chain with state space $\mathbb{Z}_{>0}$ and transition probability graph:



Show that this Markov chain has no stationary distribution.

(b) Consider a Markov chain with state space $\mathbb{Z}_{>0}$ and transition probability graph:



Assume that $0 < \lambda < \mu$ and $0 < \lambda + \mu \leq 1$. Show that the probability distribution given by

$$\pi(i) = \left(\frac{\lambda}{\mu}\right)^{i-1}\left(1 - \frac{\lambda}{\mu}\right), \text{ for } i \in \mathbb{Z}_{>0},$$

is a stationary distribution of this Markov chain.

2. Choosing Two Good Movies

You have a database of a countably infinite number of movies. Each movie has a rating that is uniformly distributed in $\{0, 1, 2, 3, 4, 5\}$ and you want to find two movies such that the sum of their rating is greater than 7.5. Assume that you choose movies from the database one by one and keep the movie with the highest rating so far. You stop when you find that the sum of the ratings of the last movie you have chosen and the movie with the highest rating among all the previous movies is greater than 7.5.

(a) Define an appropriate Markov chain and use the first step equations in order to find the expected number of movies you will have to choose.

(b) Now assume that the ratings of the movies are uniformly distributed in the interval $[0, 5]$. Write the first step equations for the expected number of movies you will have to choose in this case.

3. **Expected return times and stationarity**

For this question, we have an irreducible, finite-state Markov chain $X_0, X_1, \cdots$, so it has a stationary distribution $\pi(x)$. In this question, we are going to show a remarkable property of expected self hitting times; namely, the stationary distribution is the reciprocal of the expected self-hitting time. To show this, we will first show that for any fixed state x, the stationary distribution is proportional to number of times you hit a particular state y before you come back to your original state.

(a) Count the number of times we hit y before coming back to x. Let $\pi_x^*(y) = \mathbb{E}[\sum_{i=0}^{T_x-1} \mathbb{1}_{X_i=y} \mid X_0 = x]$ where T_x is the first time we hit x again. Show that $\pi_x P = \pi_x$ (Hint: Do it differently for $\pi_x(y)$ for $y = x$ and $y \neq x$)

(b) Argue that $\pi(x) = \frac{1}{\mathbb{E}_x[T_x]}$ (Hint: Think of the normalizing constant for the vector π_x.)

4. **Poisson Branching**

Consider a branching process such that at generation n, each individual in the population survives until generation $n+1$ with probability $0 < p < 1$, independently, and a Poisson number (with parameter λ) of immigrants enters the population. Let X_n denote the number of people in the population at generation n.

(a) Suppose that at generation 0, the number of people in the population is a Poisson random variable with parameter λ_0. What is the distribution at generation 1? What is the distribution at generation n?

(b) What is the distribution of X_n as $n \to \infty$? What if at generation 0, the number of individuals is an arbitrary probability distribution over the non-negative integers; does the distribution still converge? (Justify the convergence rigorously.)

5. **Customers in a Store**

Consider two independent Poisson processes with rates λ_1 and λ_2. Those processes measure the number of customers arriving in store 1 and 2.

(a) What is the probability that a customer arrives in store 1 before any arrives in store 2?

(b) What is the probability that in the first hour exactly 6 customers arrive, in total, at the two stores?

(c) Given that exactly 6 have arrived, in total, at the two stores, what is the probability that exactly 4 went to store 1?

6. **Arrival Times of a Poisson Process**

Consider a Poisson process $(N_t, t \geq 0)$ with rate $\lambda = 1$. For $i \in \mathbb{Z}_{>0}$, let S_i be a random variable which is equal to the time of the i-th arrival.

(a) Find $\mathbb{E}[S_3 \mid N_1 = 2]$.

(b) Given $S_3 = s$, where $s > 0$, find the joint distribution of S_1 and S_2.

(c) Find $\mathbb{E}[S_2 \mid S_3 = s]$.

7. **[Bonus] Choosing Two Good Movies (cont.)**

The bonus question is just for fun. You are not required to submit the bonus question, but do give it a try and write down your progress.

Solve the first step equations that you derived in Part (b), in order to find the expected number of movies that you will have to choose.