UC Berkeley
Department of Electrical Engineering and Computer Sciences

EECS 126: Probability and Random Processes

Problem Set 9
Spring 2019

Issued: Thursday, March 21, 2019 **Due:** Wednesday, April 3, 2019

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1. **Bus Arrivals at Cory Hall**

   Starting at time 0, the 52 line makes stops at Cory Hall according to a Poisson process of rate λ. Students arrive at the stop according to an independent Poisson process of rate μ. Every time the bus arrives, all students waiting get on.

   (a) Given that the interarrival time between bus $i-1$ and bus i is x, find the distribution for the number of students entering the ith bus. (Here, x is a given number, not a random quantity.)

   (b) Given that a bus arrived at 9:30 AM, find the distribution for the number of students that will get on the next bus.

   (c) Find the distribution of the number of students getting on the next bus to arrive after 9:30 AM, assuming that time 0 was infinitely far in the past.

2. **Basketball II**

   Team A and Team B are playing an untimed basketball game in which the two teams score points according to independent Poisson processes. Team A scores points according to a Poisson process with rate λ_A and Team B scores points according to a Poisson process with rate λ_B. The game is over when one of the teams has scored k more points than the other team. Find the probability that Team A wins.

3. **Frogs**

   Three frogs are playing near a pond. When they are in the sun they get too hot and jump in the lake at rate 1. When they are in the lake they get too cold and jump onto the land at rate 2. The rates here refer to the rate in exponential distribution. Let X_t be the number of frogs in the sun at time $t \geq 0$.

   (a) Find the stationary distribution for $(X_t)_{t\geq 0}$.

   (b) Check the answer to (a) by noting that the three frogs are independent two-state Markov chains.

4. **Taxi Queue**

   Empty taxis pass by a street corner at a Poisson rate of two per minute and pick up a passenger if one is waiting there. Passengers arrive at the street corner at a Poisson rate of one per minute and wait for a taxi only if there are less than four persons waiting; otherwise they leave and never return. John arrives at the street corner at a given time. Find his expected waiting time, given that he joins the queue. Assume that the process is in steady state.

5. **Two-Server System**

A company has two servers (the second server is a backup in case the first server fails, so only one server is ever used at a time). When a server is running, the time until it breaks down is exponentially distributed with rate μ. When a server is broken, it is taken to the repair shop. The repair shop can only fix one server at a time, and its repair time is exponentially distributed with rate λ. Find the long-run probability that no servers are operational.

6. **Poisson Queues**

A continuous-time queue has Poisson arrivals with rate λ, and it is equipped with infinitely many servers. The servers can work in parallel on multiple customers, but they are non-cooperative in the sense that a single customer can only be served by one server. Thus, when there are k customers in the queue ($k \in \mathbb{N}$), k servers are active. Suppose that the service time of each customer is exponentially distributed with rate μ and they are i.i.d.

(a) Argue that the queue-length is a Markov chain. Draw the transition diagram of the Markov chain.

(b) Prove that for all finite values of λ and μ the Markov chain is positive-recurrent and find the invariant distribution.

7. **[Bonus] Jukes-Cantor Model**

The bonus question is just for fun. You are not required to submit the bonus question, but do give it a try and write down your progress.

In this question we consider a CTMC model for the evolution of DNA over time. Consider a CTMC $(X_t)_{t\geq 0}$ on the states $\mathcal{X} := \{A, C, G, T\}$ with transition rate matrix

$$Q = \begin{bmatrix} -3\lambda & \lambda & \lambda & \lambda \\ \lambda & -3\lambda & \lambda & \lambda \\ \lambda & \lambda & -3\lambda & \lambda \\ \lambda & \lambda & \lambda & -3\lambda \end{bmatrix}, \qquad \text{for } \lambda > 0.$$

For $x, y \in \mathcal{X}$, what is $P_t(x, y) := \mathbb{P}(X_t = y \mid X_0 = x)$? What happens as $t \to \infty$?

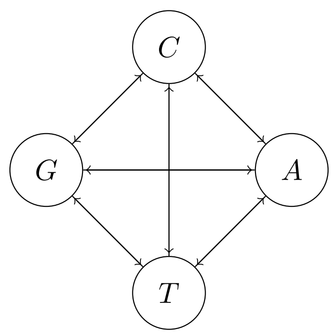

Figure 1: All edges in the rate diagram have rate λ.