1. Finding the bright cave

Nara the one-handed druid and Kody the one-handed ranger find themselves in dire straits. Before them is a cliff with four cave entrances arranged in a square: two upper caves and two lower caves. Each entrance emits a certain amount of light, and the two wish to find exactly the amount of light coming from each cave. Here's the catch: after contracting a particularly potent strain of ghoul fever, our intrepid heroes are only able to see the total intensity of light before them (so their eyes operate like a single-pixel camera). Kody and Nara are capable adventurers. But they don't know any linear algebra - and they need your help.

Kody proposes an imaging strategy where he uses his hand to completely block the light from two caves at a time. He is able to take measurements using the following four masks (black means the light is blocked from that cave):

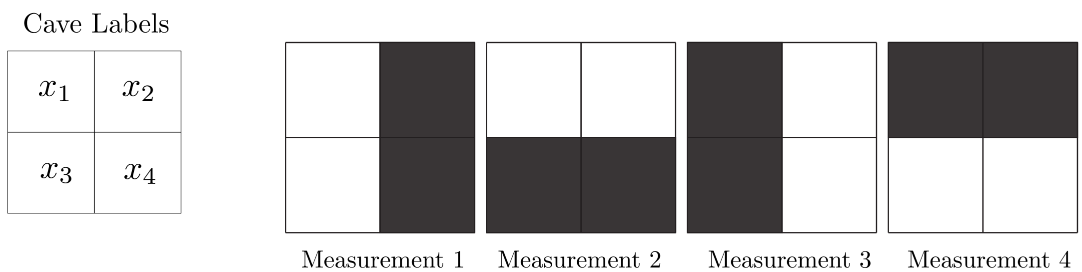


Figure 1: Four image masks.

(a) Let $\vec{x}$ be the four-element vector that represents the magnitude of light emanating from the four cave entrances. **Write a matrix K that performs the masking process in Figure 1 on the vector $\vec{x}$, such that $K\vec{x}$ is the result of the four measurements.**

(b) Does Kody's set of masks give us a unique solution for all four caves' light intensities? Why or why not?

(c) Nara, in her infinite wisdom, places her one hand diagonally across the entrances, covering two of the cave entrances. However her hand is not wide enough, letting in 50% of the light from the caves covered and 100% of the light from the caves not covered. The following diagram shows the percentage of light let through from each cave:

50%	100%
100%	50%

Does this additional measurement give them enough information to solve the problem? Why or why not?

2. Derivations & Dependence

(a) Suppose for some non-zero vector $\vec{x}$, $\mathbf{A}\vec{x} = \vec{0}$. Prove that the columns of $\mathbf{A}$ are linearly dependent.

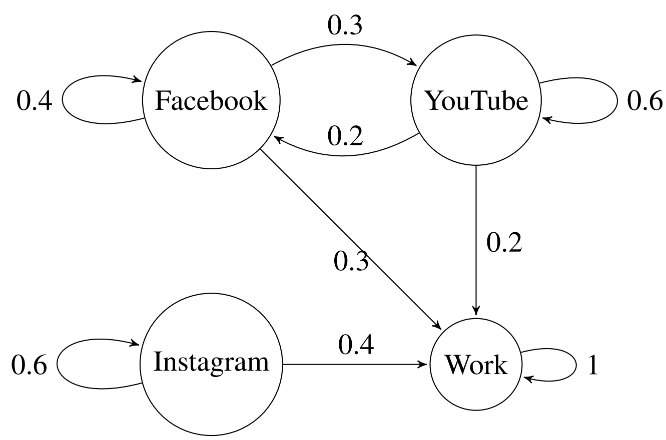
(b) Now suppose there exist two unique vectors $\vec{x}_1$ and $\vec{x}_2$ that both satisfy $\mathbf{A}\vec{x} = \vec{b}$, that is $\mathbf{A}\vec{x}_1 = \vec{b}$ and $\mathbf{A}\vec{x}_2 = \vec{b}$. Prove that the columns of $\mathbf{A}$ are linearly dependent.

(c) Prove that if a matrix's columns are linearly dependent, there will be either infinite or no solutions to $A\vec{x} = \vec{b}$. What is the physical interpretation of this statement?

(d) Suppose we have an experiment where we have n measurements of linear combinations of n unknowns. Show that if at least one of the experiment's measurements could be predicted from the other measurements, there would be either infinite or no solutions. In other words, prove that if a $n \times n$ matrix A has rows that are linearly dependent, there will be either infinite or no solutions to $A\vec{x} = \vec{b}$.

3. Social Media

As a tech-savvy Berkeley student, the distractions of social media are always calling you away from productive stuff like homework for your classes. You're curious–are you the only one who spends hours switching between Facebook or YouTube? How do other students manage to get stuff done and balance pursuing Insta-fame? You conduct an experiment, collect some data, and notice Berkeley students tend to follow a pattern of behavior similar to the below figure. So, for example, if 100 students are on Facebook, in the next timestep 30 of them will click a link and move to YouTube.



(a) What is the corresponding transition matrix?

(b) There are 750 of you in the class. Suppose on a given Monday night (the day before HW is due), there are 350 EE16A students on Facebook, 225 on YouTube, 100 on Instagram and 75 actually doing work. In the next timestep, how many people will be doing each activity? In other words, after you apply the matrix one timestep, what is the state vector?

(c) If the entries in each of the column vectors of your state transition matrix summed to 1, what would this mean with respect to the students on social media? (What is the physical interpretation?)

(d) I want to predict how many students will be on each website n timesteps in the future. How would I formulate that mathematically? Without working it out, can you predict roughly how many students will be in each state 1000 timesteps/days in the future?

(e) **Extra for Experts:** Suppose instead of having 'Work' as an explicit state, we assume that any student not on Facebook/Youtube/Instagram is working. Work is like the "void", and if a student is "leaked" from any of the other states we assume s/he has gone to work and will never come back. How would you reformulate this problem? Redraw the figure and rewrite the appropriate transition matrix. What are the major differences between this problem and the previous one?