1. Inverses

In general, an *inverse* of a matrix "undoes" the operation that that matrix performs. Mathematically, we write this as

$$A^{-1}A = I \tag{1}$$

where A^{-1} is the inverse of A. Intuitively, this means that applying a matrix to a vector and then subsequently applying it's inverse is the same as leaving the vector untouched.

Properties of Inverses. For a matrix A, if its inverse exist, then:

$$A^{-1}A = AA^{-1} = I \tag{2}$$

$$(A^{-1})^{-1} = A \tag{3}$$

$$(kA)^{-1} = k^{-1}A^{-1} \qquad \text{for a nonzero scalar } k \tag{4}$$

$$(A^T)^{-1} = (A^{-1})^T \tag{5}$$

$$(AB)^{-1} = B^{-1}A^{-1} \qquad \text{assuming } A, B \text{ are both invertible} \tag{6}$$

(a) Prove that $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$

(b) Now consider the three matrices.

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \qquad C = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \qquad D = \begin{bmatrix} \frac{1}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{4}{5} \end{bmatrix}$$

i. What do each of these matrices do when you multiply them by a vector $\vec{x}$? Draw a picture.

ii. Intuitively, can these operations be undone? Why or why not? Make an intuitive argument.

iii. Are the matrices A, B, C, D invertible?

iv. Can you find anything in common about the rows (and columns) of A, B, C, D? (Bonus: How does this relate to the invertibility of A, B, C, D?)

v. Are all square matrices invertible?

vi. How can you find the inverse of a general $n \times n$ matrix?

Reference Definitions

Vector spaces: A vector space V is a set of elements that is closed under vector addition and scalar multiplication. For V to be a vector space, the following conditions must hold for every $\vec{u},\vec{v},\vec{z} \in V$ and for every $c,d \in \mathbb{R}$

No escape property (addition) $\vec{u}+\vec{v} \in V$,

No escape property (scalar multiplication) $c\vec{u} \in V$,

Commutativity $\vec{u}+\vec{v} = \vec{v}+\vec{u}$,

Associativity of vector addition $(\vec{u}+\vec{v})+\vec{z} = \vec{u}+(\vec{v}+\vec{z})$,

Additive identity There is $\vec{0} \in V$ such that for all $\vec{u}$, $\vec{0}+\vec{u} = \vec{u}+\vec{0} = \vec{u}$,

Existence of inverse For every $\vec{u}$, there is element $-\vec{u}$ such that $\vec{u}+(-\vec{u}) = 0$,

Associativity of scalar multiplication $c(d(\vec{u})) = (cd)\vec{u}$,

Distributivity of scalar sums $(c+d)\vec{u} = c\vec{u}+d\vec{u}$,

Distributivity of vector sums $c(\vec{u}+\vec{v}) = c\vec{u}+c\vec{v}$,

Scalar multiplication identity There is $1\vec{u} = \vec{u}$.

The most important of the above properties are the *no escape properties*. These demonstrate that the vector space is closed under addition and scalar multiplication. That is, if you add two vectors in V, your resulting vector will still be in V. If you multiply a vector in V by a scalar, your resulting vector will still be in V.

Subspaces: A subset W of a *vector space* V is a *subspace* of V if the following two conditions hold for any two vectors $\vec{u},\vec{v} \in W$, and any scalar $c \in \mathbb{R}$:

No escape property (addition) $\vec{u}+\vec{v} \in W$

No escape property (scalar multiplication) $c\vec{u} \in W$

Note that these are the only properties we need to establish to show that a subset of a vector space is a subspace! The other properties of the underlying vector space come for free, so to speak.

The vector spaces we will work with most commonly are $\mathbb{R}^n$ and $\mathbb{C}^n$, as well as their subspaces.

Basis: A *basis* for a vector space is a *set of linearly independent vectors* that *spans the vector space*.

So, if we want to check whether a set of vectors $\vec{v}_1,\vec{v}_2,...,\vec{v}_k$ forms a basis for a vector space V, we check for two important properties:

(a) $\vec{v}_1,\vec{v}_2,...,\vec{v}_k$ are linearly independent.

(b) $span(\vec{v}_1,\vec{v}_2,...,\vec{v}_k) = V$

As we move along, we'll learn how to identify and/or construct a basis, and we'll also learn some interesting properties of bases.

2. **Identifying a subspace: Proof exercise!**

Is the set

$$V = \left\{ \vec{v} : c \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + d \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \text{ where } c, d \in \mathbb{R} \right\}$$

a subspace of $\mathbb{R}^3$? Why/why not?

3. **Identifying a basis**

Does each of these sets describe a basis of some vector space?

$$V_1 = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\} \qquad V_2 = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\} \qquad V_3 = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

4. **Exploring dimensionality, linear independence and bases**

In this problem, we are going to talk about the connections between several concepts we have learned about in linear algebra – linear independence, dimensionality of a vector space/subspace, and basis.

Let's consider the vector space $\mathbb{R}^m$, and a set of n vectors $\vec{v}_1, \vec{v}_2, ... \vec{v}_n$ in $\mathbb{R}^m$.

(a) For the first part of the problem, let $m > n$. Can $\vec{v}_1, \vec{v}_2, ... \vec{v}_n$ form a basis of $\mathbb{R}^m$? Why/why not? What conditions would we need?

(b) Let $m = n$. Can $\vec{v}_1, \vec{v}_2, ... \vec{v}_n$ form a basis of $\mathbb{R}^m$? Why/why not? What conditions would we need?

(c) Now, let $m < n$. Can $\vec{v}_1, \vec{v}_2, ... \vec{v}_n$ form a basis of $\mathbb{R}^m$? What vector space could they form a basis for? (Hint: think about whether the vectors can now be linearly independent.)

5. **Constructing a basis**

Let's consider a subspace of $\mathbb{R}^3$, V, that has the following property: for every vector in V, the first entry is equal to two times the sum of the second and third entries. That is, if $\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \in V$, we have $a_1 = 2(a_2 + a_3)$.

Find a basis for V. What is the dimension of V?