1. Polynomials as a Vector Space

Let $\mathbb{P}_2$ be the set of polynomials of degree at most two (that is, at^2+bt+c).

(a) Give a basis for $\mathbb{P}_2$.

(b) Consider the linear transformations

$$T_1(f(t)) = 2f(t)$$
$$T_2(f(t)) = f'(t)$$

For each, find the transformation matrix with respect to the basis from part (a).

(c) Suppose that $\{x_0, x_1, x_2\}$ form a basis for $\mathbb{P}_2$, and that the following polynomials have the corresponding coordinates in this basis.

$$(1,1,1) \rightarrow 2t^2+3t$$
$$(1,0,-1) \rightarrow t+1$$
$$(0,2,0) \rightarrow 4t+2$$

Find the basis vectors x_0, x_1, x_2.

2. Mechanical Projection

Reference Definitions

Inner Product Algebraic definition: $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix} \in \mathbb{R}^N, \vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{bmatrix} \in \mathbb{R}^N : \langle \vec{x}, \vec{y} \rangle = \sum_{i=1}^{N} x_i \cdot y_i$.

Euclidean Norm The *Euclidean Norm* of a vector $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix} \in \mathbb{R}^N$ is $\|\vec{x}\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_N^2}$

Vector Scaling Let $c \in \mathbb{R}$ and $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix} \in \mathbb{R}^N$. Recall that $c \cdot \vec{x} = \begin{bmatrix} c \cdot x_1 \\ c \cdot x_2 \\ \vdots \\ c \cdot x_N \end{bmatrix}$.

In $\mathbb{R}^n$, the projection of vector $\vec{b}$ onto vector $\vec{a}$ is:

$$proj_{\vec{a}}\vec{b} = \frac{\langle \vec{b}, \vec{a} \rangle}{||\vec{a}||^2}\vec{a} = \frac{\langle \vec{b}, \vec{a} \rangle}{||\vec{a}||}\frac{\vec{a}}{||\vec{a}||}$$

where $\hat{a}$ is the normalized $\vec{a}$, i.e. a unit vector with the same direction as $\vec{a}$.

(a) Project $\begin{bmatrix} 5 \\ 2 \end{bmatrix}$ onto $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ - that is, onto the x-axis. Graph these two vectors and the projection.

(b) Project $\begin{bmatrix} 5 \\ 2 \end{bmatrix}$ onto $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ - that is, onto the y-axis. Graph these two vectors and the projection.

(c) Project $\begin{bmatrix} 4 \\ -2 \end{bmatrix}$ onto $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$. Graph these two vectors and the projection.

(d) Project $\begin{bmatrix} 4 \\ -2 \end{bmatrix}$ onto $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Graph these two vectors and the projection.