1. Unit Spheres

The unit sphere of a given norm $\|\cdot\|$ is the set of vectors x for which $\|x\| = 1$. For the following norms (in $\mathbb{R}^2$), plot their unit sphere:

(a) $\|\cdot\|_1$

(b) $\|\cdot\|_2$

(c) $\|\cdot\|_\infty$

2. Matrix Inner Products

First, a definition: the trace of a square matrix $\mathbf{A}$, denoted $\mathrm{tr}(\mathbf{A})$, is the sum of its diagonal entries.

An inner product for matrices $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{m \times n}$ is:

$$\langle \mathbf{A}, \mathbf{B} \rangle = \mathrm{tr}(\mathbf{A}^T \mathbf{B})$$

(a) Let a_i be the i^{th} column of $\mathbf{A}$ and b_i be the columns of $\mathbf{B}$. Show that:

$$\mathrm{tr}(\mathbf{A}^T \mathbf{B}) = \sum_{i=1}^{n} \langle a_i, b_i \rangle$$

(b) Confirm that $\langle \mathbf{A}, \mathbf{B} \rangle$ is in fact an inner product by showing the following:

i. $\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$

ii. $\langle \alpha u, v \rangle = \alpha \langle u, v \rangle$

iii. $\langle u, v \rangle = \langle v, u \rangle$

iv. $\langle u, u \rangle \geq 0$ with equality if and only if $u = 0$.

3. Row Space

Consider:

$$\mathbf{V} = \begin{bmatrix} 2 & 4 & 6 \\ 4 & 0 & 4 \\ 6 & 4 & 10 \\ -2 & 4 & 2 \end{bmatrix} \tag{1}$$

Row reducing this matrix yields:

$$\mathbf{U} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{2}$$

(a) Show that the row space of U and V are the same. Argue that in general, Gaussian elimination preserves the row space.

(b) Show that the null space of U and V are the same. Argue generally that Gaussian elimination preserves the null space.