EECS 16A Designing Information Devices and Systems I
Spring 2017 Babak Ayazifar, Vladimir Stojanovic Homework 4

This homework is due February 22, 2017, at 23:59.
Self-grades are due February 27, 2017, at 23:59.

Submission Format

Your homework submission should consist of **two** files.

- `hw4.pdf`: A single pdf file that contains all your answers (any handwritten answers should be scanned) as well as your IPython notebook saved as a pdf.

  If you do not attach a pdf of your IPython notebook, you will not receive credit for problems that involve coding. Make sure your results and plots are showing.

- `hw4.ipynb`: A single IPython notebook with all your code in it.

  In order to receive credit for your IPython notebook, you must submit both a "printout" and the code itself.

Submit each file to its respective assignment in Gradescope.

1. Vector Matching

Consider a vector $\vec{y} \in \mathbb{R}^N$. Suppose we had two vectors, $\vec{v}_1$ and $\vec{v}_2$, where $||\vec{v}_1|| = ||\vec{v}_2|| = 1$. We wish to determine which vector is a better match to $\vec{y}$. See Figure 1 for a visualization of this in $\mathbb{R}^2$. These figures are meant to help you visualize at a lower-dimension, and these results are applicable for higher dimensions.

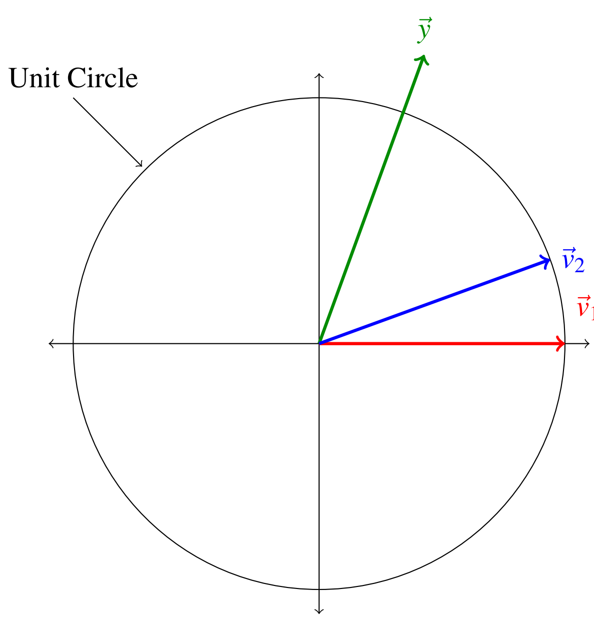
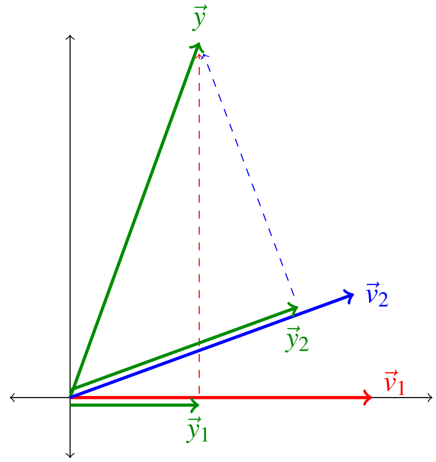


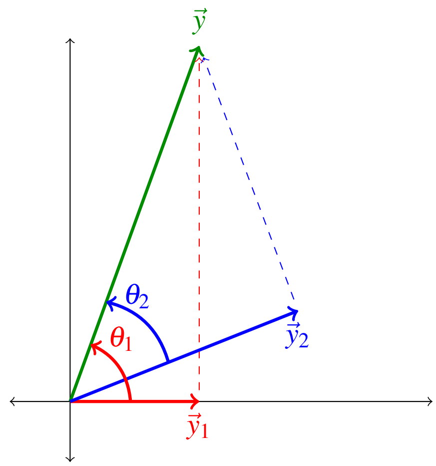
Figure 1

(a) Express $\vec{y}_1 = \text{proj}_{\vec{v}_1}\vec{y}$ and $\vec{y}_2 = \text{proj}_{\vec{v}_2}\vec{y}$ in terms dot products. $\text{proj}_{\vec{v}_i}\vec{y}$ is the projection of $\vec{y}$ onto $\vec{v}_i$.



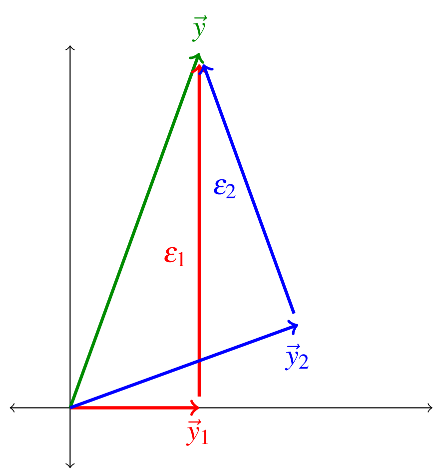
(a)

(b) Express $||\vec{y}_i||$ in terms of $||\vec{y}||$ and θ_i, the angle between $\vec{y}$ and $\vec{v}_i$.



(b) Note that $\vec{y}_1$ and $\vec{y}_2$ have changed colors.

(c) Let $\vec{\varepsilon}_1$ and $\vec{\varepsilon}_2$ be the error vectors, where $\vec{\varepsilon}_i = \vec{y} - \vec{y}_i$. Express $||\vec{\varepsilon}_i||^2$ in terms of $\vec{y}$ and $\vec{v}_i$.



(c)

(d) We claim that if $||\vec{\varepsilon}_1||^2 < ||\vec{\varepsilon}_2||^2$, then $\vec{v}_1$ is a better match to $\vec{y}$ than $\vec{v}_k$ is. Write this inequality in terms of $\vec{y}$ and $\vec{v}_i$.

2. Four Fundamental Subspaces

Consider a matrix $\mathbf{A}$:

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & -3 & 4 \\ 3 & -3 & -5 & 8 \\ 1 & -1 & -1 & 2 \end{bmatrix}$$

(a) Find the column space of $\mathbf{A}$. What is the dimension of this space?

(b) Find the nullspace of $\mathbf{A}$. What is the dimension of this space?

(c) Find the row space of $\mathbf{A}$. What is the dimension of this space?

(d) Find the left nullspace of $\mathbf{A}$. What is the dimension of this space?

3. Image Compression

In this question, we explore how eigenvalues and eigenvectors can be used for image compression. We have seen that a grayscale image can be represented as a data grid. Say a symmetric, square image is represented by a symmetric matrix $\mathbf{A}$, such that $\mathbf{A}^T = \mathbf{A}$. We've been transforming the images to vectors in the past to make it easier to process them as data, but here we will understand them as 2D data. Let $\lambda_1 \cdots \lambda_n$ be the eigenvalues of $\mathbf{A}$ with corresponding eigenvectors $\vec{v}_1 \cdots \vec{v}_n$. Then, the matrix can be represented as

$$\mathbf{A} = \lambda_1 \vec{v}_1 \vec{v}_1^T + \lambda_2 \vec{v}_2 \vec{v}_2^T + \cdots + \lambda_n \vec{v}_n \vec{v}_n^T$$

However, the matrix $\mathbf{A}$ can also be *approximated* with the k largest eigenvalues and corresponding eigenvectors. That is,

$$\mathbf{A} \approx \lambda_1 \vec{v}_1 \vec{v}_1^T + \lambda_2 \vec{v}_2 \vec{v}_2^T + \cdots + \lambda_k \vec{v}_k \vec{v}_k^T$$

(a) Construct appropriate matrices $\mathbf{V}$, $\mathbf{W}$ (using $\vec{v}_i$'s as rows and columns) and a matrix Λ with the eigenvalues λ_i as components such that

$$\mathbf{A} = \mathbf{V} \Lambda \mathbf{W}$$

(b) Use the IPython notebook `prob4.ipynb` and the image file `pattern.npy`. Use the `numpy.linalg` command `eig` to find the $\mathbf{V}$ and Λ matrices for the image. Mathematically, how many eigenvectors are required to fully capture the information within the image?

(c) In the IPython notebook, find an approximation for the image using the 100 largest eigenvalues and eigenvectors.

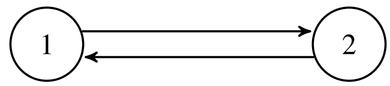
(d) Repeat part (c) with $k = 50$. By further experimenting with the code, what seems to be the lowest value of k that retains most of the salient features of the given image?

4. Counting the paths of a Random Surfer

In class, we discussed the behavior of a random web-surfer who jumps from webpage to webpage. We would like to know how many possible paths there are for a random surfer to get from a page to another page. To do this, we represent the webpages as a graph. If page 1 has a link to page 2, we have a directed edge from page 1 to page 2. This graph can further be represented by what is known as an "adjacency matrix", $\mathbf{A}$, with elements a_{ij}. We define $a_{ji} = 1$ if there is link from page i to page j. Matrix operations on the adjacency matrix make it very easy to compute the number of paths to get from a particular webpage i to webpage j.

This path counting aspect actually is an implicit part of the how the "importance scores" for each webpage are described. Recall that the "importance score" of a website is the steady-state frequency of the fraction of people on that website.

Consider the following graphs.

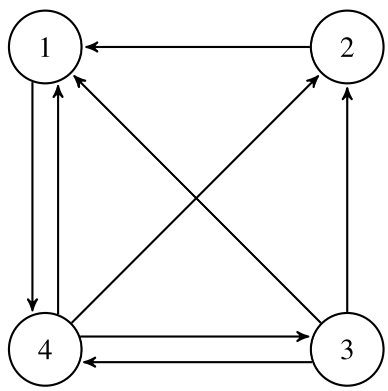


Graph A

(a) Write out the adjacency matrix for graph A.

(b) For graph A: How many one-hop paths are there from webpage 1 to webpage 2? How many two-hop paths are there from webpage 1 to webpage 2? How about three-hop paths?

(c) For graph A: What are the importance scores of the two webpages?

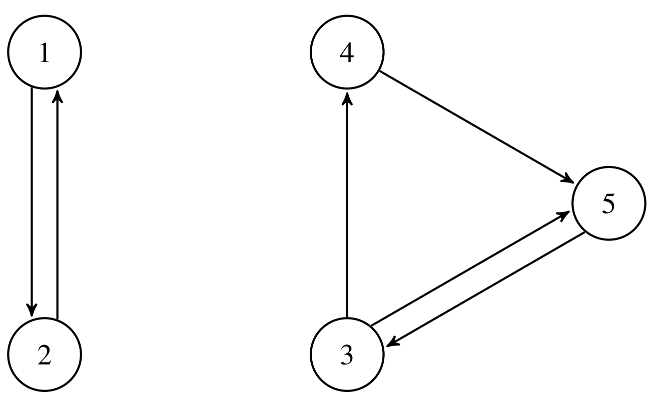


Graph B

(d) Write out the adjacency matrix for graph B.

(e) For graph B: How many two-hop paths are there from webpage 1 to webpage 3? How many three-hop paths are there from webpage 1 to webpage 2?

(f) For graph B: What are the importance scores of the webpages? You may use your IPython notebook for this.



Graph C

(g) Write out the adjacency matrix for graph C.

(h) For graph C: How many paths are there from webpage 1 to webpage 3?

(i) For graph C: What are the importance scores of the webpages? How is graph (c) different from graph (b), and how does this relate to the importance scores and eigenvalues and eigenvectors you found?

5. Homework process and study group

Who else did you work with on this homework? List names and student ID's. (In case of hw party, you can also just describe the group.) How did you work on this homework?
Working in groups of 3-5 will earn credit for your participation grade.

6. (PRACTICE) Mechanical Problem

Compute the eigenvalues and eigenvectors of the following matrices.

(a) $\begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix}$

(b) $\begin{bmatrix} 22 & 6 \\ 6 & 13 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

(d) $\begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$ (What special matrix is this?)