FIRST Name Inner LAST Name ~~Product~~ Peace

Lab Day & Time: 24/7 SID (All Digits): 123456789

- **(10 Points)** On *every* page, print legibly your name, ALL digits of your SID, and your lab day and time. These points divide equally for each page.
- This quiz should take up to 90 minutes to complete. However, you may use up to a maximum of 120 minutes *in one sitting*, to work on the quiz.
- **This quiz is closed book.** Collaboration is not permitted. You may not use or access, or cause to be used or accessed, any reference in print or electronic form at any time during the quiz, except one double-sided $8.5" \times 11"$ sheet of handwritten, original notes having no appendage. Computing, communication, and other electronic devices (except dedicated timekeepers) must be turned off.
- The primary purpose of this quiz is to enable you to assess your fluency with the subject matter of the course as Drop Date nears. Please give yourself a sufficient buffer between now and Drop Date, so you can take, self-grade, and decide whether you wish to stay with the course or opt to take it in a future semester.
- **The quiz printout consists of pages numbered 1 through 10.** Verify that your copy of the quiz is free of printing anomalies and contains all of the ten numbered pages.
- Please write neatly and legibly, because *if you can't read it, you can't grade it.*
- For each problem, limit your work to the space provided specifically for that problem. *No other work should be considered in grading your quiz. No exceptions.*
- Unless explicitly waived by the specific wording of a problem, you must explain your responses (and reasoning) succinctly, but clearly and convincingly.
- We hope you do a *fantastic* job on this quiz.

FIRST Name Inner LAST Name ~~Product~~ Peace

Lab Day & Time: 24/7 SID (All Digits): 123456789

Q1.1 (20 Points) [Coded Messages]

Arif and his roommate Rajiv are excited about what they're learning in their EECS courses at UC Berkeley. In one of their many nerdy activities, they've developed a special coded-message system to get food for unexpected guests who drop by their apartment. They have many "friends" who appear unannounced, often several at a time! However, Arif and Rajiv are hospitable souls; they never turn away friends who knock on their door to visit.

One late afternoon, when Arif is at home, a group of unexpected guests appear. Immediately, Arif assesses the situation and sends Rajiv a text message that contains a coded sequence of four real numbers. You may think of the numbers in the sequence as the entries in a vector $\boldsymbol{b}$. The vector $\boldsymbol{b}$ is designed to convey to Rajiv how much of each of *four* agreed-upon menu items (dishes) to bring home from their favorite eatery *House of Curries*, located on College Ave.

Rajiv knows—by prior agreement with Arif—that the vector $\boldsymbol{b}$ is a linear combination of four *mutually-orthogonal* vectors $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_4$, each of which represents the known codeword for a corresponding dish.

Generically, the vector $\boldsymbol{b}$ that Arif sends to Rajiv can be written as

$$\boldsymbol{b} = x_1\,\boldsymbol{a}_1 + x_2\,\boldsymbol{a}_2 + x_3\,\boldsymbol{a}_3 + x_4\,\boldsymbol{a}_4 \tag{1}$$

$$= \underbrace{\begin{bmatrix} \boldsymbol{a}_1 & \boldsymbol{a}_2 & \boldsymbol{a}_3 & \boldsymbol{a}_4 \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}}_{\boldsymbol{x}} = \mathbf{A}\boldsymbol{x}, \tag{2}$$

where $\boldsymbol{a}_k \perp \boldsymbol{a}_\ell$ for all $k \neq \ell$, and the coefficients $x_1, \ldots, x_4$ denote the *nonnegative integer* quantities of the dishes that Arif has requested. Rajiv's task is to process the vector $\boldsymbol{b}$ to determine the coefficients in the linear combination of Equation (1) or, equivalently, the unknown vector $\boldsymbol{x}$ in Equation (2). Each coefficient x_n, where $n = 1, \ldots, 4$, indicates to Rajiv how much of the corresponding Dish n to bring home from House of Curries.

The four mutually-orthogonal vectors (codewords), as well as the particular dish that each represents, are given below:

$$\boldsymbol{a}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \text{ Vegetable Samosa} \qquad \boldsymbol{a}_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} \text{ Bhaingan Bharta}$$

$$\boldsymbol{a}_3 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} \text{ Chicken Tikka Masala} \qquad \boldsymbol{a}_4 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} \text{ Lamb Chops Tandoori}$$

FIRST Name Inner LAST Name ~~Product~~ Peace

Lab Day & Time: 24/7 SID (All Digits): 123456789

Rajiv receives the following vector from Arif:

$$\boldsymbol{b} = \begin{bmatrix} 12 \\ 2 \\ 0 \\ 2 \end{bmatrix}.$$

Determine how much of each dish *Vegetable Samosa*, *Bhaingan Bharta*, *Chicken Tikka Masala*, and *Lamb Chops Tandoori* Arif wants Rajiv to bring home.

$\langle \underline{a}_n, \underline{a}_n \rangle = 4 \quad n = 1,2,3,4$

$\underline{a}_k \perp \underline{a}_\ell \quad k \neq \ell \Rightarrow \langle \underline{a}_k, \underline{a}_\ell \rangle = 0 \quad k \neq \ell$

Method I:

$$\underline{b} = x_1 \underline{a}_1 + x_2 \underline{a}_2 + x_3 \underline{a}_3 + x_4 \underline{a}_4 = \sum_{n=1}^{4} x_n \underline{a}_n$$

$$\langle \underline{b}, \underline{a}_1 \rangle = \left\langle \sum_{n=1}^{4} x_n \underline{a}_n, \underline{a}_1 \right\rangle = \sum_{n=1}^{4} x_n \langle \underline{a}_n, \underline{a}_1 \rangle$$

Due to the mutual orthoganality of $\underline{a}_1, \ldots, \underline{a}_4$, only one inner product on the right-hand side is nonzero:

$$\langle \underline{a}_n, \underline{a}_1 \rangle = \begin{cases} 0 & n \neq 1 \\ 4 & n = 1 \end{cases}$$

So $\langle \underline{b}, \underline{a}_1 \rangle = x_1 \langle \underline{a}_1, \underline{a}_1 \rangle \Rightarrow x_1 = \dfrac{\langle \underline{b}, \underline{a}_1 \rangle}{\langle \underline{a}_1, \underline{a}_1 \rangle}$

In general, we have $x_n = \dfrac{\langle \underline{b}, \underline{a}_n \rangle}{\langle \underline{a}_n, \underline{a}_n \rangle} = \dfrac{1}{4} \langle \underline{b}, \underline{a}_n \rangle \quad n = 1, \ldots, 4$

$\langle \underline{b}, \underline{a}_1 \rangle = [12 \; 2 \; 0 \; 2] \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = 16 \Rightarrow x_1 = \dfrac{16}{4} = 4$ Samosas

$\langle \underline{b}, \underline{a}_2 \rangle = [12 \; 2 \; 0 \; 2] \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} = 12 - 2 - 2 = 8 \Rightarrow x_2 = \dfrac{8}{4} = 2$ Bhaingans

$\langle \underline{b}, \underline{a}_3 \rangle = [12 \; 2 \; 0 \; 2] \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} = 12 - 2 + 2 = 12 \Rightarrow x_3 = \dfrac{12}{4} = 3$ Chicken Tikkas

$\langle \underline{b}, \underline{a}_4 \rangle = [12 \; 2 \; 0 \; 2] \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} = 12 - 2 + 2 = 12 \Rightarrow x_4 = \dfrac{12}{4} = 3$ Lamb Tandooris

Method II:

$$\underline{b} = x_1\underline{a}_1 + x_2\underline{a}_2 + x_3\underline{a}_3 + x_4\underline{a}_4$$

$$= \underbrace{[\underline{a}_1 \ \underline{a}_2 \ \underline{a}_3 \ \underline{a}_4]}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}}_{\underline{x}} = A\underline{x}$$

Matrix A has mutually-orthogonal columns,

so $A^T A = \begin{bmatrix} \underline{a}_1^T \\ \vdots \\ \underline{a}_4^T \end{bmatrix} [\underline{a}_1 \cdots \underline{a}_4] = \begin{bmatrix} \langle \underline{a}_1, \underline{a}_1 \rangle & \cdots & \langle \underline{a}_1, \underline{a}_4 \rangle \\ \vdots & \ddots & \vdots \\ \langle \underline{a}_4, \underline{a}_1 \rangle & \cdots & \langle \underline{a}_4, \underline{a}_4 \rangle \end{bmatrix}$

$= \begin{bmatrix} 4 & & 0 \\ & \ddots & \\ 0 & & 4 \end{bmatrix} = 4I$ All off-diagonal entries of A^TA are zero due to orthogonality of the columns.

$$A\underline{x} = \underline{b} \Rightarrow A^T A \underline{x} = A^T\underline{b} \Rightarrow 4I\underline{x} = A^T\underline{b} \Rightarrow$$

$$\underline{x} = \frac{1}{4}A^T\underline{b} = \frac{1}{4}\begin{bmatrix} \underline{a}_1^T \\ \vdots \\ \underline{a}_4^T \end{bmatrix}\underline{b} = \frac{1}{4}\begin{bmatrix} \langle \underline{a}_1, \underline{b} \rangle \\ \langle \underline{a}_2, \underline{b} \rangle \\ \langle \underline{a}_3, \underline{b} \rangle \\ \langle \underline{a}_4, \underline{b} \rangle \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 3 \\ 3 \end{bmatrix}$$

Note: $A^{-1} = \frac{1}{4}A^T$ in this case.

Orthogonality is your friend!

FIRST Name Inner LAST Name ~~Product~~ Peace

Lab Day & Time: 24/7 SID (All Digits): 123456789

Q1.2 (35 Points) [A Three-Node Network of Web Sites]

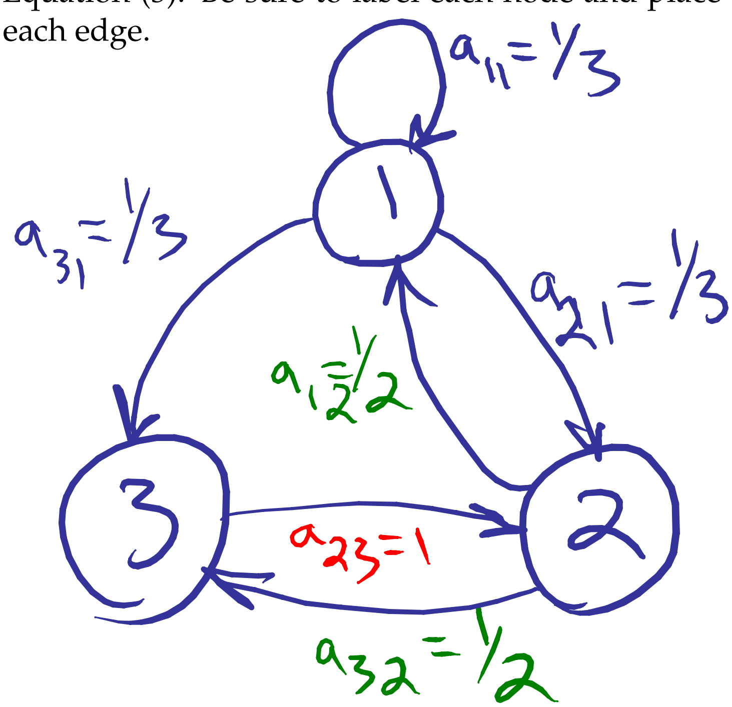
Consider a network of three web pages visited by a total population of web surfers. The surfer population size remains unchanged throughout our analysis time window; that is, no person joins the population of surfers and no one leaves it. The state-evolution equation for the way the surfers visit the three nodes on the network is given by

$$\forall n \in \{0, 1, 2, \ldots\}, \quad \boldsymbol{s}[n+1] = \mathbf{A}\, \boldsymbol{s}[n], \quad \text{where} \quad \mathbf{A} = \begin{bmatrix} \frac{1}{3} & \frac{1}{2} & 0 \\ \frac{1}{3} & 0 & 1 \\ \frac{1}{3} & \frac{1}{2} & 0 \end{bmatrix}. \tag{3}$$

In Equation (3), the state vector $\boldsymbol{s}[n] = \begin{bmatrix} s_1[n] \\ s_2[n] \\ s_3[n] \end{bmatrix}$ is defined so that $s_\ell[n]$ denotes the *fraction* of the surfers at Node ℓ at time n; for example, $s_3[n]$ denotes the fraction of the surfers at Node 3 at time n. Accordingly, note that $\boldsymbol{s}[n]$ is a nonnegative vector and its entries sum to unity:

$$\mathbf{1}^\mathsf{T} \boldsymbol{s}[n] = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} s_1[n] \\ s_2[n] \\ s_3[n] \end{bmatrix} = s_1[n] + s_2[n] + s_3[n] = 1, \qquad \forall n \in \{0, 1, 2, \ldots\}.$$

(a) (10 Points) Provide a well-labeled directed graph that captures the dynamic flow of surfers among the three nodes in a manner consistent with the State-Evolution Equation (3). Be sure to label each node and place the appropriate weight on each edge.



FIRST Name Inner LAST Name ~~Product~~ Peace

Lab Day & Time: 24/7 SID (All Digits): 123456789

(b) (10 Points) Is it possible to determine the state $s[n]$ from the subsequent state $s[n+1]$? If you answer in the affirmative, express $s[n]$ in terms of $s[n+1]$ as follows

$$s[n] = \mathbf{Q}\, s[n+1],$$

and determine the matrix $\mathbf{Q}$ explicitly, finding numerical values for each of its entries.

If you claim that it is *not* possible to draw any reverse-time inference on the state, provide a concise, yet clear and convincing explanation to justify your claim.

No, it's not possible. For such a matrix Q to exist, it must be $Q = A^{-1}$. But A has no inverse, because its first and third rows are identical—and, so, linearly dependent. The matrix A is singular.

(c) (15 Points) You're tasked with devising ranking scores akin to Google's PageRank for this network of three nodes. Determine a score vector $s^* = \begin{bmatrix} s_1^* \\ s_2^* \\ s_3^* \end{bmatrix}$ whose entries are the ranking scores for the corresponding nodes. Ensure that your score vector is nonnegative and sums to unity—that is, $\mathbf{1}^\top s^* = 1$.

The vector $\underline{s}^*$ must satisfy the equilibrium equation $A\underline{s}^* = \underline{s}^*$, which is equivalent to $(I-A)\underline{s}^* = \underline{0}$.

$$(I-A)\underline{s}^* = \begin{bmatrix} 2/3 & -1/2 & 0 \\ -1/3 & 1 & -1 \\ -1/3 & -1/2 & 1 \end{bmatrix} \underline{s}^* = \underline{0} \xRightarrow[\text{Elim}]{\text{Gaussian}} \begin{bmatrix} 2/3 & -1/2 & 0 \\ 0 & 3/2 & -2 \\ 0 & -3/2 & 2 \end{bmatrix} \underline{s}^* = \underline{0} \Rightarrow$$

$$\begin{bmatrix} 2/3 & -1/2 & 0 \\ 0 & 3/2 & -2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} s_1^* \\ s_2^* \\ s_3^* \end{bmatrix} = \underline{0} \Rightarrow$$

s_3^*: free variable; $\frac{3}{2}s_2^* - 2s_3^* = 0 \Rightarrow s_2^* = \frac{4}{3}s_3^*$; $\frac{2}{3}s_1^* - \frac{1}{2}s_2^* = 0 \Rightarrow s_1^* = s_3^*$

$$\underline{s}^* = \begin{bmatrix} 1 \\ 4/3 \\ 1 \end{bmatrix} s_3^* \xrightarrow[\text{so } \mathbb{1}^\top \underline{s}^* = \mathbb{1}^\top]{\text{Let } s_3^* = 3/10} \underline{s}^* = \begin{bmatrix} 3/10 \\ 4/10 \\ 3/10 \end{bmatrix} \text{ Score vector}$$

FIRST Name Inner LAST Name ~~Product~~ Peace

Lab Day & Time: 24/7 SID (All Digits): 123456789

Q1.3 (45 Points) [The Incredible Circulants]

Consider the 3×3 *canonical circulant matrix*

$$\mathbf{C} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}. \tag{4}$$

The matrix $\mathbf{C}$ is called a *circulant matrix*, because each row (column) is a circularly-shifted version of its prior adjacent row (column). A circular shift is a shift with wrap-around; for example, if we shift the first row of $\mathbf{C}$ rightward, the right entry 1 wraps around to the left, resulting in the row vector $[1\ 0\ 0]$—which is the second row of $\mathbf{C}$. A rightward circular shift of the last row produces the first row. A similar feature holds for the columns, where the circular shift is downward, with wraparound from the bottom to the top. And a downward circular shift of the last column produces the first column.

Circulant matrices have important applications in many areas of engineering, particularly in signal processing—such as in the Discrete Fourier Transform (DFT), which you'll study in EE 16B, and in cyclic codes for error correction, which you'll learn about in an upper-division course on digital communications or in an advanced course on information and coding theory. Later in the third module of EE 16A, we'll study one application—positioning (an example of which is GPS).

In this problem, we use basic linear-algebraic concepts to explore a few salient properties of circulant matrices.

(a) (5 Points) Consider a vector

$$\boldsymbol{s} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \in \mathbb{R}^3.$$

Determine the column vector $\boldsymbol{y} = \mathbf{C}\boldsymbol{s}$ and the row vector $\boldsymbol{z}^\mathsf{T} = \boldsymbol{s}^\mathsf{T}\mathbf{C}$. Also, explain, in plain words, what each of these multiplications by $\mathbf{C}$ does to the entries of $\boldsymbol{s}$ and $\boldsymbol{s}^\mathsf{T}$.

$$\underline{y} = C\underline{s} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} c \\ a \\ b \end{bmatrix}$$

← Colored entries in $\underline{s}$ moved by corresponding like colored entries 1 in C.

Downward circular shift of $\underline{s}$.

$$\underline{z}^T = \underline{s}^T C = [a\ b\ c] \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = [b\ c\ a]$$

Leftward circular shift of $\underline{s}^T$.

FIRST Name Inner LAST Name ~~Product~~ Peace

Lab Day & Time: 24/7 SID (All Digits): 123456789

(b) (15 Points) In this part, we explore the integer powers $\mathbf{C}^k$ of the canonical circulant matrix $\mathbf{C}$ (note that k is allowed to be a negative integer). As is the case for nonzero scalars, the zeroth power of any nonzero square matrix is the identity matrix; accordingly, $\mathbf{C}^0 = \mathbf{I}$, where $\mathbf{I}$ is the 3×3 identity matrix.

(i) (5 Points) Show that

$$\mathbf{C}^2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

Base on Part (a) we know that pre-multiplication by C performs a downward circular shift of the rows

Also determine $\mathbf{C}^3$ and $\mathbf{C}^4$.

$$C^2 = CC = C\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

Downward circular shift of C

$$C^3 = CC^2 = C\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

downward circ shift of C^2

$$C^4 = C^3 C = IC = C = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

(ii) (6 Points) Determine $\mathbf{C}^{999}$, $\mathbf{C}^{1000}$, $\mathbf{C}^{1001}$. Also, determine $\mathbf{C}^k$, where k is an arbitrary nonnegative integer.

$$C^{999} = (C^3)^{333} = I^{333} = I; \quad C^{1000} = C^{999} C = IC = C$$

$$C^{1001} = C^{999} C^2 = IC^2 = C^2$$

$$C^k = \begin{cases} I & \text{if } k \bmod 3 = 0 \\ C & \text{if } k \bmod 3 = 1 \\ C^2 & \text{if } k \bmod 3 = 2 \end{cases}$$

← multiples of 3

← division by 3 has remainder 1

← division by 3 has remainder 2

FIRST Name Inner LAST Name ~~Product~~ Peace

Lab Day & Time: 24/7 SID (All Digits): 123456789

(iii) (4 Points) Determine $\mathbf{C}^{-1}$ and $\mathbf{C}^{\mathsf{T}}$. Explain whether your expressions for $\mathbf{C}^k$ in the previous part hold if k is a negative integer.

You can find C^{-1} through the usual method of elimination, or you can simply notice that

$$C^T = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = C^2 \Rightarrow C^T C = C^2 C = C^3 = I \Rightarrow C^T = C^{-1}$$

The formula for C^k in the prev part also holds at negative k.

(c) (15 Points) As you know, an arbitrary pair of square matrices $\mathbf{A}$ and $\mathbf{B}$ do *not* commute—that is, $\mathbf{AB} \neq \mathbf{BA}$, in general. However, certain matrices *do* commute. And when you come across commuting matrices, it might feel as though drums have rolled and fireworks have adorned the evening sky, for you have encountered celebrities!

A non-empty set of matrices is said to form a *commuting family* if every pair of matrices in that set commute with each other. In this part, we study the commuting family for which the canonical circulant shift matrix $\mathbf{C}$ of Equation (4) is the matriarch or patriarch, whichever you prefer. In particular, we consider the commuting family

$$\mathcal{F}_{\mathbf{C}} = \left\{ \mathbf{A} \in \mathbb{R}^{3\times 3} \middle| \, \mathbf{AC} = \mathbf{CA} \right\}.$$

(i) (4 Points) Is the following assertion true?

The commuting family $\mathcal{F}_{\mathbf{C}}$ forms a subspace of $\mathbb{R}^{3\times 3}$.

If you answer in the affirmative, prove that $\mathcal{F}_{\mathbf{C}}$ is a subspace. If you claim that $\mathcal{F}_{\mathbf{C}}$ is not a subspace, show that it fails to satisfy at least one necessary property of subspaces.

We know $\mathbb{R}^{3\times3}$ is a vector space, and $\mathcal{F}_C \subset \mathbb{R}^{3\times 3}$. To show $\mathcal{F}_C$ is a subspace, we need only prove that the closure axioms hold. Let $A, B \in \mathcal{F}_C \Rightarrow$

$$(A+B)C = AC + BC = CA + CB = C(A+B) \Rightarrow A+B \in \mathcal{F}_C$$

$$\alpha AC = \alpha CA = C(\alpha A) \Rightarrow \alpha A \in \mathcal{F}_C$$

QED.

FIRST Name ~~Inner~~ Inner LAST Name ~~Product~~ Peace

Lab Day & Time 24/7 SID (All Digits): 123456789

(ii) (8 Points) Show that a 3×3 matrix $\mathbf{A}$ is in $\mathcal{F}_{\mathbf{C}}$, if, and only if, it is expressible in the form

$$\mathbf{A} = \begin{bmatrix} a & b & c \\ c & a & b \\ b & c & a \end{bmatrix}. \qquad (5)$$

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \qquad AC = \begin{bmatrix} b & c & a \\ e & f & d \\ h & i & g \end{bmatrix}$$

Leftward circ. shift of the cols of A

$$CA = \begin{bmatrix} g & h & i \\ a & b & c \\ d & e & f \end{bmatrix}$$

Downward circ shift of the rows of A

Setting $CA = AC$, we have $b = g = f$, $c = h = d$, and $\underbrace{a = i = e}_{\text{diag elems}}$ $\Rightarrow A = \begin{bmatrix} a & b & c \\ c & a & b \\ b & c & a \end{bmatrix}$

(iii) (3 Points) Show that the circulant matrix $\mathbf{A}$ of Equation (5) can be written as a quadratic polynomial in the canonical circulant matrix $\mathbf{C}$ of Equation (4). In particular, show that

$$\mathbf{A} = \alpha_0 \underbrace{\mathbf{I}}_{\mathbf{C}^0} + \alpha_1 \mathbf{C} + \alpha_2 \mathbf{C}^2,$$

and determine the coefficients α_0, α_1, and α_2 in terms of the parameters a, b, and c in Equation (5).

$$A = \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix} + \begin{bmatrix} 0 & b & 0 \\ 0 & 0 & b \\ b & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & c \\ c & 0 & 0 \\ 0 & c & 0 \end{bmatrix}$$

$$= a \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{C^0 = I} + b \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}}_{C^2} + c \underbrace{\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}}_{C}$$

$$\Rightarrow A = aI + cC + bC^2$$

$\alpha_0 = a$

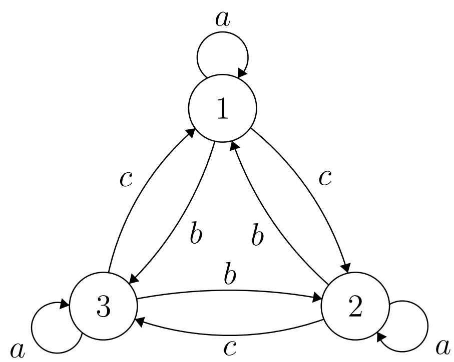
$\alpha_1 = c$

$\alpha_2 = b$

FIRST Name Inner LAST Name ~~Product~~ Peace

Lab Day & Time: 24/7 SID (All Digits): 123456789

(d) (10 Points) Consider a network of three water reservoirs. At the end of each day water transfers among the reservoirs according to the directed graph shown below.



The parameter a denotes the *fraction* of the water of each reservoir that stays in the same reservoir at the end of each day n. The parameters b and c denote the fraction of the water in each reservoir that transfers to an adjacent reservoir at the end of each day, according to the directed graph above. Assume that the reservoir system is conservative, meaning that no water enters or leaves the system. In that case, $a + b + c = 1$. Needless to say, a, b, and c are nonnegative parameters.

The state-evolution equation governing the water flow dynamics in the reservoir system is given by $\boldsymbol{s}[n+1] = \mathbf{A}\,\boldsymbol{s}[n]$, where $\boldsymbol{s}[n] \in \mathbb{R}^3$ is the nonnegative state vector that shows the water content of each of the three reservoirs at the end of Day n, and the 3×3 matrix $\mathbf{A}$ is the state-transition matrix. Let the initial state be such that $\mathbf{1}^\mathsf{T}\boldsymbol{s}[0] = s_1[0] + s_2[0] + s_3[0] = 9$ million liters.

Determine the state-transition matrix $\mathbf{A}$. Also, if you believe that the system has an equilibrium state $\boldsymbol{s}*$—that is, a state for which the following is true: $\boldsymbol{s}[n+1] = \boldsymbol{s}[n] = \boldsymbol{s}*$—then determine that equilibrium state. If you believe that the system has no equilibrium state, explain why it does not.

By symmetry, we expect $\underline{s}^* = \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix}$, because every node has the same set of backlinks and forward links. We know $a+b+c=1 \Rightarrow$

$A = \begin{bmatrix} a & b & c \\ c & a & b \\ b & c & a \end{bmatrix}$ circulant. $A\mathbb{1} = \mathbb{1} \Rightarrow \underline{s}^* = \alpha\mathbb{1}$. Choose $\alpha = 1/3$ to ensure $\mathbb{1}^T\underline{s}^* = \mathbb{1}^T$.

$\mathbb{1} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

A Few Takeaway Points from the Circulant Matrix Problem:

$$C = \left[\begin{array}{cc|c} 0 & 0 & 1 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right] = \left[\begin{array}{c|c} \underline{0}^T & 1 \\ \hline I & \underline{0} \end{array}\right]$$

In general, an $N \times N$ canonical circulant matrix is $C = \begin{bmatrix} \underline{0}_{N-1}^T & 1 \\ I_{N-1} & \underline{0}_{N-1} \end{bmatrix}$

where $\underline{0}_{N-1}$ is a zero vector of size $N-1$, and I_{N-1} is the $(N-1)\times(N-1)$ identity matrix.

Pre-Multiplying a Matrix R by C:

$$CR = C \begin{bmatrix} \underline{r}_1^T \\ \vdots \\ \underline{r}_N^T \end{bmatrix} = \begin{bmatrix} \underline{r}_N^T \\ \underline{r}_1^T \\ \vdots \\ \underline{r}_{N-1}^T \end{bmatrix}$$

Downward circular shift of the rows of R

Post Multiplying a Matrix A by C:

$$AC = [\underline{a}_1 \cdots \underline{a}_N]C = [\underline{a}_2 \cdots \underline{a}_N \ \underline{a}_1]$$

Leftward circular shift of the columns of A.

C^{-1}

C belongs to the set of permutation matrices, so like all permutation matrices $C^T = C^{-1}$!