This homework is optional.

This homework contains practice problems only. You are not required to turn in this assignment, but you are responsible for its content.

1. **Sports Rank**

Every year in college sports, specifically football and basketball, debate rages over team rankings. The rankings determine who will get to compete for the ultimate prize, the national championship. However, ranking teams is quite challenging in the setting of college sports for a few reasons: there is uneven paired competition (not every team plays each other), sparsity of matches (every team plays a small subset of all the teams available), and there is no well-ordering (team A beats team B who beats team C who beats A). In this problem we will come up with an algorithm to rank the teams, with real data drawn from the 2014 Associated Press (AP) poll of the top 25 college football teams.

Given N teams we want to determine the rating r_i for the ith team for $i = 1, 2, \ldots, N$, after which the teams can be ranked in order from highest to lowest rating. Given the wins and losses of each team we can assign each team a score s_i.

$$s_i = \sum_j^N q_{ij} r_j, \tag{1}$$

where q_{ij} represents the number of times team i has beaten team j divided by the number [1] of games played by team i. If we define the vectors $\vec{s} = \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_N \end{bmatrix}$, and $\vec{r} = \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_N \end{bmatrix}$ we can express their relationship as a system of equations

$$\vec{s} = \mathbf{Q}\vec{r}, \tag{2}$$

where $\mathbf{Q} = \begin{bmatrix} q_{11} & q_{12} & \cdots & q_{1N} \\ q_{21} & q_{22} & \cdots & q_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ q_{N1} & q_{N2} & \cdots & q_{NN} \end{bmatrix}$ is an $N \times N$ matrix.

(a) Consider a specific case where we have three teams, team A, team B, and team C. Team A beats team C twice and team B once. Team B beats team A twice and never beats team C. Team C beats team B three times. What is the matrix $\mathbf{Q}$?

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[1]We normalize by the number of games played to prevent teams from getting a high score by just repeatedly playing against weak opponents

(b) Returning to the general setting, if our scoring metric is good, then it should be the case that teams with better ratings have higher scores. Let's make the assumption that $s_i = \lambda r_i$ with $\lambda > 0$. Show that $\vec{r}$ is an eigenvector of $\mathbf{Q}$.

To find our rating vector we need to find an eigenvector of $\mathbf{Q}$ with all nonnegative entries (ratings can't be negative) and a positive eigenvalue. If the matrix $\mathbf{Q}$ satisfies certain conditions (beyond the scope of this course) the dominant eigenvalue λ_D, i.e. the largest eigenvalue in absolute value, is positive and real. In addition, the dominant eigenvector, i.e. the eigenvector associated with the dominant eigenvalue, is unique and has all positive entries. We will now develop a method for finding the dominant eigenvector for a matrix if it is unique.

(c) Given $\vec{v}$, an eigenvector of $\mathbf{Q}$ with eigenvalue λ, and a real nonzero number c, express $\mathbf{Q}^n c\vec{v}$ in terms of $\vec{v}$, c, n, and λ

(d) Now given multiple eigenvectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$ of $\mathbf{Q}$, their eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_m$, and real nonzero numbers $c_1, c_2, \ldots, c_m$, express $\mathbf{Q}^n(\sum_{i=1}^m c_i\vec{v}_i)$ in terms of $\vec{v}$'s, λ's, and c's.

(e) Assuming that $|\lambda_1| > |\lambda_i|$ for $i = 2, \ldots, m$, argue or prove

$$\lim_{n\to\infty} \frac{1}{\lambda_1^n} \mathbf{Q}^n \left(\sum_{i=1}^m c_i\vec{v}_i \right) = c_1\vec{v}_1 \tag{3}$$

Hints:

   i. For sequences of vectors $\{\vec{a}_n\}$ and $\{\vec{b}_n\}$, $\lim_{n\to\infty}(\vec{a}_n + \vec{b}_n) = \lim_{n\to\infty}\vec{a}_n + \lim_{n\to\infty}\vec{b}_n$.

   ii. For a scalar w with $|w| < 1$, $\lim_{n\to\infty} w^n = 0$.

(f) Now further assuming that λ_1 is positive show

$$\lim_{n\to\infty} \frac{\mathbf{Q}^n(\sum_{i=1}^m c_i\vec{v}_i)}{\|\mathbf{Q}^n(\sum_{i=1}^m c_i\vec{v}_i)\|} = \frac{c_1\vec{v}_1}{\|c_1\vec{v}_1\|} \tag{4}$$

Hints:

   i. Divide the numerator and denominator by λ_1^n and use the result from the previous part.

   ii. For the sequence of vectors $\{\vec{a}_n\}$, $\lim_{n\to\infty}\|\vec{a}_n\| = \|\lim_{n\to\infty}\vec{a}_n\|$.

   iii. For a sequence of vectors $\{\vec{a}_n\}$ and a sequence of scalars $\{\alpha_n\}$, if $\lim_{n\to\infty}\alpha_n$ is not equal to zero then the $\lim_{n\to\infty}\frac{\vec{a}_n}{\alpha_n} = \frac{\lim_{n\to\infty}\vec{a}_n}{\lim_{n\to\infty}\alpha_n}$.

Let's assume that any vector $\vec{b}$ in $\mathbb{R}^N$ can be expressed as a linear combination of the eigenvectors of any square matrix $\mathbf{A}$ in $\mathbb{R}^{N\times N}$, i.e. $\mathbf{A}$ has N rows and N columns.

Let's tie it all together. Given the eigenvectors of $\mathbf{Q}$, $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_N$, we arbitrarily choose the dominant eigenvector to be $\vec{v}_1 = \vec{v}_D$. If we can find a vector $\vec{b} = \sum_{i=1}^m c_i\vec{v}_i$, such that c_1 is nonzero, then [2]

$$\lim_{n\to\infty} \frac{\mathbf{Q}^n\vec{b}}{\|\mathbf{Q}^n\vec{b}\|} = \frac{c_1\vec{v}_D}{\|c_1\vec{v}_D\|}. \tag{5}$$

This is the idea behind the power iteration method, which is a method for finding the unique dominant eigenvector (up to scale) of a matrix whenever one exists. In the IPython notebook we will use this method to rank our teams. Note: For this application we know the rating vector (which will be the dominant eigenvector) has all positive entries, but c_1 might be negative resulting in our method returning a vector with all negative entries. If this happens, we simply multiply our result by -1.

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[2]If we select a vector at random, c_1 will be almost certainly non-zero.

(g) From the method you implemented in the IPython notebook, name the top five teams, the fourteenth team, and the seventeenth team.

2. The Dynamics of Romeo and Juliet's Love Affair

In this problem, we will study a discrete-time model of the dynamics of Romeo and Juliet's love affair— adapted from Steven H. Strogatz's original paper, *Love Affairs and Differential Equations*, Mathematics Magazine, 61(1), p.35, 1988, which describes a continuous-time model.

Let $R[n]$ denote Romeo's feelings about Juliet on day n, and let $J[n]$ quantify Juliet's feelings about Romeo on day n. If $R[n] > 0$, it means that Romeo loves Juliet and inclines toward her, whereas if $R[n] < 0$, it means that Romeo is resentful of her and inclines away from her. A similar interpretation holds for $J[n]$, which represents Juliet's feelings about Romeo.

A larger $|R[n]|$ represents a more intense feeling of love (if $R[n] > 0$) or resentment (if $R[n] < 0$). If $R[n] = 0$, it means that Romeo has neutral feelings toward Juliet on day n. Similar interpretations hold for larger $|J[n]|$ and the case of $J[n] = 0$.

We model the dynamics of Romeo and Juliet's relationship using the following coupled system of linear evolutionary equations:

$$R[n+1] = aR[n] + bJ[n], \quad n = 0, 1, 2, \ldots \tag{6}$$

and

$$J[n+1] = cR[n] + dJ[n], \quad n = 0, 1, 2, \ldots, \tag{7}$$

which we can rewrite as

$$\vec{s}[n+1] = \mathbf{A}\,\vec{s}[n], \tag{8}$$

where

$$\vec{s}[n] = \begin{bmatrix} R[n] \\ J[n] \end{bmatrix}$$

denotes the state vector and

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

the state transition matrix, for our dynamic system model.

The parameters a and d capture the linear fashion in which Romeo and Juliet respond to their own feelings, respectively, about the other person. It's reasonable to assume that $a, d > 0$, to avoid scenarios of fluctuating day-to-day mood swings. Within this positive range, if $0 < a < 1$, then the effect of Romeo's own feelings about Juliet tend to fizzle away with time (in the absence of influence from Juliet to the contrary), whereas if $a > 1$, Romeo's feelings about Juliet intensify with time (in the absence of influence from Juliet to the contrary). A similar interpretation holds when $0 < d < 1$ and $d > 1$.

The parameters b and c capture the linear fashion in which the other person's feelings influence $R[n]$ and $J[n]$, respectively. These parameters may or may not be positive. If $b > 0$, it means that the more Juliet shows affection for Romeo, the more he loves her and inclines toward her. If $b < 0$, it means that the more Juliet shows affection for Romeo, the more resentful he feels and the more he inclines away from her. A similar interpretation holds for the parameter c.

All in all, each of Romeo and Juliet has four romantic styles, which makes for a combined total of sixteen possible dynamic scenarios. The fate of their interactions depends on the romantic style each of them exhibits, the initial state, and the values of the entries in the state transition matrix $\mathbf{A}$. In this problem, we'll explore a subset of the possibilities.

(a) Consider the case where $a+b=c+d$ in the state-transition matrix

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

   i. Show that

   $$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

   is an eigenvector of $\mathbf{A}$, and determine its corresponding eigenvalue λ_1. Also determine the other eigenpair $(\lambda_2, \vec{v}_2)$. Your expressions for λ_1, λ_2, and $\vec{v}_2$ must be in terms of one or more of the parameters a, b, c, and d.

   ii. Consider the following state-transition matrix:

   $$\mathbf{A} = \begin{bmatrix} 0.75 & 0.25 \\ 0.25 & 0.75 \end{bmatrix}.$$

      i. Determine the eigenpairs for this system.
      ii. Determine all the *fixed points* of the system. That is, find the set of points such that if Romeo and Juliet start at, or enter, any of those points, they'll stay in place forever: $\{\vec{s}_* | \mathbf{A}\vec{s}_* = \vec{s}_*\}$. Show these points on a diagram where the x and y-axes are $R[n]$ and $J[n]$.
      iii. Determine representative points along the state trajectory $\vec{s}[n]$, $n = 0, 1, 2, \ldots$, if Romeo and Juliet start from the initial state

      $$\vec{s}[0] = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

      iv. Suppose the initial state is $\vec{s}[0] = \begin{bmatrix} 3 & 5 \end{bmatrix}^T$. Determine a reasonably simple expression for the state vector $\vec{s}[n]$. Find the limiting state vector

      $$\lim_{n \to \infty} \vec{s}[n].$$

(b) Consider the setup in which

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

In this scenario, if Juliet shows affection toward Romeo, Romeo's love for her increases, and he inclines toward her. The more intensely Romeo inclines toward her, the more Juliet distances herself. The more Juliet withdraws, the more Romeo is discouraged and retreats into his cave. But the more Romeo inclines away, the more Juliet finds him attractive and the more intensely she conveys her affection toward him. Juliet's increasing warmth increases Romeo's interest in her, which prompts him to incline toward her—again!

Predict the outcome of this scenario before you write down a single equation.

Then determine a complete solution $\vec{s}[n]$ in the simplest of terms, assuming an initial state given by $\vec{s}[0] = \begin{bmatrix} 1 & 0 \end{bmatrix}^T$. As part of this, you must determine the eigenvalues and eigenvectors of the $\mathbf{A}$.

Plot (by hand, or otherwise without the assistance of any scientific computing software package), on a two-dimensional plane (called a *phase plane*)—where the horizontal axis denotes $R[n]$ and the

vertical axis denotes $J[n]$—representative points along the trajectory of the state vector $\vec{s}[n]$, starting from the initial state given in this part. Describe, in plain words, what Romeo and Juliet are doing in this scenario. In other words, what does their state trajectory look like? Determine $\|\vec{s}[n]\|^2$ for all $n = 0, 1, 2, \ldots$ to corroborate your description of the state trajectory.

3. **Homework process and study group**

Who else did you work with on this homework? List names and student ID's. (In case of hw party, you can also just describe the group.) How did you work on this homework?
Working in groups of 3-5 will earn credit for your participation grade.