

Acknowledgements

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Introduction to Linear Algebra — the EECS Way

In this note, we will teach the basics of linear algebra and relate it to the work we will see in labs and EECS in general. We will define important terms and show how to do certain calculations.

What is Linear Algebra and why is it important?

- Linear algebra is the study of vectors and their transformations
- A lot of objects in EECS can be treated as vectors and studied with linear algebra.
- Linearity is a good first-order approximation to the complicated real world

¹Proper credit here also acknowledges the strong cultural influence of the faculty team that made EECS 70 and shaped it into a required course: Alistair Sinclair, Satish Rao, David Tse, Umesh Vazirani, Christos Papadimitriou, David Wagner, and Anant Sahai, along with pioneering TA's like Thomas Vidick, all helped make and polish the notes for 70. The 16AB+70 sequence all follows the aesthetic that mathematical ideas should have practical punchlines. 16AB emphasizes the rooting of mathematical intuition in physical examples as well as virtual ones.

- There exist good fast algorithms to do many of these manipulations in computers.

As you will see in the homeworks and labs, these concepts can be used to do many interesting things in real-world-relevant application scenarios.

Tomography: Linear Algebra in Imaging

Let's start with an example of linear algebra that relates to this module and uses key concepts from this note: tomography. Tomography is taking an image section by section by using a penetrating wave, such as X-rays. CT Scans in medical imaging are perhaps the most famous such example.

Here is a specific toy example.

A grocery store employee just had a truck load of bottles given to him. Each stack contains these bottles: Milk (M), Juice (J) or Empty (O). One such stack could look like this:

$$\begin{bmatrix} M & J & O \\ M & J & O \\ M & O & J \end{bmatrix} \quad (1)$$

He cannot see directly into the stack but he needs to sort them somehow (without opening every single one). However, suppose he has a device that can tell him how much light an object absorbs. Now, he can shine a light through different angles to figure out how much total light the bottles in those columns absorb.

Suppose milk absorbs 3 units of light, juice absorbs 2 units of light and an empty bottle absorbs 1 unit of light.

Let's assign variables to the amount of light absorbed at each position:

$$\begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix} \quad (2)$$

If we shine light in a straight line, we can determine the amount of light absorbed as the sum of the light absorbed by each bottle. For example if the rows absorb 5, 6, and 4 units of light, we can write down the following equations:

$$x_{11} + x_{12} + x_{13} = 5 \quad (3)$$

$$x_{21} + x_{22} + x_{23} = 6 \quad (4)$$

$$x_{31} + x_{32} + x_{33} = 4 \quad (5)$$

Likewise, if the 3 columns absorb 6, 3, and 6 units respectively, we have the additional equations:

$$x_{11} + x_{21} + x_{31} = 6 \quad (6)$$

$$x_{12} + x_{22} + x_{32} = 3 \quad (7)$$

$$x_{13} + x_{23} + x_{33} = 6 \quad (8)$$

One such configuration that satisfies those equations is:

$$\begin{bmatrix} 3 & 1 & 1 \\ 2 & 1 & 3 \\ 1 & 1 & 2 \end{bmatrix} \quad (9)$$

If this is true, we can deduce that the original stack looked like:

$$\begin{bmatrix} M & O & O \\ J & O & M \\ O & O & J \end{bmatrix} \quad (10)$$

However, the following configuration also satisfies the above equations:

$$\begin{bmatrix} 1 & 1 & 3 \\ 3 & 1 & 2 \\ 2 & 1 & 1 \end{bmatrix} \quad (11)$$

In order to distinguish between these two possible configurations, we need to shine light at different angles, perhaps diagonally. Are there other possible configurations? How many different directions do we need to shine light through before we are certain of the configuration? By the end of this module, you will have the tools to answer these questions.

In lecture itself, we will work out specific examples to make this more clear. The important thing is that just taking enough measurements is not enough — the experiments that you design (namely which combinations of things you wish to measure) are important in that collectively, they have to allow you to extract the information that you want to extract. This doesn't always happen. Sometimes, the experiments that you might think of end up being redundant in subtle ways.

Vector and Matrix

Let us first introduce two central terms in linear algebra — vectors and matrices.

Definition 1.1 (Vector): A vector is a collection of numbers. Suppose we have a collection of n real numbers, x_1, x_2, \dots, x_n . We can represent this collection as a single point in an n -dimensional space, denoted:

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}. \quad (12)$$

We call \vec{x} a **vector**. Each x_i (for i between 1 and n) is called a **component**, or **element**, of the vector. The **size** of a vector is the number of components it contains.

Example 1.1 (Vector of size 2):

$$\vec{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

In the above example, \vec{x} is a vector with two components.

Definition 1.2 (Matrix): A **matrix** is a rectangular array of numbers, written as:

$$A = \begin{bmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{m1} & \cdots & A_{mn} \end{bmatrix} \quad (13)$$

Each A_{ij} (where i is the row index and j is the column index) is a **component**, or **element** of the matrix A .

Example 1.2 (4×3 Matrix):

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 5 & 7 \\ 4 & 8 & 12 \end{bmatrix}$$

In the example above, A has $m = 4$ rows and $n = 3$ columns (a 4×3 matrix). We will give you a bit more intuitions and examples on the role of vectors and matrices in science and engineering applications in the next few lectures.

System of Linear Equations

We can represent a system of linear equations in matrix or vector form. Suppose we have a system of linear equations below

$$\begin{aligned} x_1 + x_2 &= 1 \\ 2x_1 + x_2 &= 3 \end{aligned} \quad (14)$$

We can pull out the numbers on the left and right separately, and represent the system as

$$\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$

In general, we can represent any system of m linear equations with n variables in the form

$$A\vec{x} = \vec{b}$$

where A is a $m \times n$ matrix, \vec{x} is a vector containing the n variables, and \vec{b} is a vector of size n containing the constants. Why do we care about writing a system of linear equations in the above representation? We will see later in the course that decoupling the coefficients and the variables makes it easier to perform analysis on the system, especially if the system is large.

Example 1.3 (System of linear equations in matrix form): Recall the toy tomography example where we shine light in a straight line. If the rows absorb 5, 6, and 4 units of light, we have the system of equations,

$$x_{11} + x_{12} + x_{13} = 5 \quad (15)$$

$$x_{21} + x_{22} + x_{23} = 6 \quad (16)$$

$$x_{31} + x_{32} + x_{33} = 4. \quad (17)$$

We can rewrite the system as

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{12} \\ x_{13} \\ x_{21} \\ x_{22} \\ x_{23} \\ x_{31} \\ x_{32} \\ x_{33} \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \\ 4 \end{bmatrix}.$$

Gaussian Elimination

Gaussian Elimination is a strategy to solve systems of linear equations. When we model experiments as measuring some (possibly weighted) combination of things that we are interested in, each measurement taken results in a linear equation. On the left hand side is the experiment — the particular weighed combination of things that are being examined and this manifests as a weighted sum of variables — and on the right hand side is the result of the experiment, the measurement itself.

The goal of Gaussian Elimination is to solve a system of linear equations such as

$$\begin{aligned} x - 2y &= 1 \\ 2x + y &= 7 \end{aligned} \tag{18}$$

by adding or subtracting multiples of one equation from another.

Example 1.4 (System of 2 equations): To solve the system of equations above, we subtract 2 times the first equation from the second one, to get:

$$\begin{aligned} x - 2y &= 1 \\ 5y &= 5 \end{aligned} \tag{19}$$

Then we divide the second equation by 5 to solve for y and substitute its value into the first equation to get the solution:

$$\begin{aligned} x &= 3 \\ y &= 1 \end{aligned} \tag{20}$$

Soon we will present a general method to solve systems of linear equations that can be extended to any number of equations. Right now we will use an example with 3 equations:

Example 1.5 (System of 3 equations): Suppose we would like to solve the following system of 3 equations:

$$\begin{aligned} 2y + z &= 1 \\ 2x + 6y + 4z &= 10 \\ x - 3y + 3z &= 14 \end{aligned} \tag{21}$$

The first step is to eliminate x from all but one equation. Since x is the first variable to be eliminated, we want the equation containing it to be at the top. However, the first equation does not contain x . To solve this problem, we swap the first two equations:

$$\begin{array}{rcl} 2x + 6y + 4z & = & 10 \\ & 2y + z & = 1 \\ x - 3y + 3z & = & 14 \end{array} \quad (22)$$

Now we can divide the first equation by 2 and subtract it from the third equation to get:

$$\begin{array}{rcl} x + 3y + 2z & = & 5 \\ & 2y + z & = 1 \\ - & 6y + z & = 9 \end{array} \quad (23)$$

Now there is only one equation containing x , of the remaining 2 equations, we want only 1 of them to contain y , so we add 3 times the second equation to the third:

$$\begin{array}{rcl} x + 3y + 2z & = & 5 \\ & 2y + z & = 1 \\ & & 4z = 12 \end{array} \quad (24)$$

Now we can solve for $z = 3$, and then eliminate z from the rest of the equations:

$$\begin{array}{rcl} x + 3y & = & -1 \\ & 2y & = -2 \\ & & z = 3 \end{array} \quad (25)$$

Realizing that we can solve for y now, we get $y = -1$ and eliminate y from the remaining equation to solve for x .

$$\begin{array}{rcl} x & = & 2 \\ & y & = -1 \\ & & z = 3 \end{array} \quad (26)$$

How about solving systems of more equations? To do so, we need to look at the operations that we have performed in the previous example and see if we can come up with an *algorithm* for solving a system of any number of linear equations. Remember that we can perform an operation to an equation as long as it doesn't change the solutions to the system.

First, we would like to abstract out the variables by representing the equation as an *augmented matrix*. For example, the following system of two equations can be represented as a matrix:

$$\begin{bmatrix} 5x + 3y & = & 5 \\ -4x + y & = & 2 \end{bmatrix} \quad \begin{bmatrix} 5 & 3 & 5 \\ -4 & 1 & 2 \end{bmatrix}$$

There are 3 basic operations that we will perform to the rows of a matrix, corresponding to the list of equations:

1. Multiplying a row by a scalar. For example we multiply the first row by 2:

$$\begin{bmatrix} 10x + 6y & = & 10 \\ -4x + y & = & 2 \end{bmatrix} \quad \begin{bmatrix} 10 & 6 & 10 \\ -4 & 1 & 2 \end{bmatrix}$$

2. Switching rows. We swap the 2 rows:

$$\begin{cases} -4x + y = 2 \\ 5x + 3y = 5 \end{cases} \quad \begin{bmatrix} -4 & 1 & 2 \\ 5 & 3 & 5 \end{bmatrix}$$

3. Adding scalar multiple of a row to another row. We add the 2 times the first row and the second:

$$\begin{cases} 5x + 3y = 5 \\ 6x + 7y = 12 \end{cases} \quad \begin{bmatrix} 5 & 3 & 5 \\ 6 & 7 & 12 \end{bmatrix}$$

Here we present some examples of solving systems of equations using Gaussian Elimination.

Example 1.6 (Equations with exactly one solution):

$$\begin{cases} 2x + 4y + 2z = 8 \\ x + y + z = 6 \\ x - y - z = 4 \end{cases} \quad \begin{bmatrix} 2 & 4 & 2 & 8 \\ 1 & 1 & 1 & 6 \\ 1 & -1 & -1 & 4 \end{bmatrix}$$

Divide row 1 by 2.

$$\begin{cases} x + 2y + z = 4 \\ x + y + z = 6 \\ x - y - z = 4 \end{cases} \quad \begin{bmatrix} 1 & 2 & 1 & 4 \\ 1 & 1 & 1 & 6 \\ 1 & -1 & -1 & 4 \end{bmatrix}$$

Subtract row 1 from row 2 and 3.

$$\begin{cases} x + 2y + z = 4 \\ -y = 2 \\ -3y - 2z = 0 \end{cases} \quad \begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & -1 & 0 & 2 \\ 0 & -3 & -2 & 0 \end{bmatrix}$$

Subtract 3 times of row 2 from row 3 and then multiply row 2 by -1.

$$\begin{cases} x + 2y + z = 4 \\ y = -2 \\ -2z = -6 \end{cases} \quad \begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & -2 & -6 \end{bmatrix}$$

Divide row 3 by -2 then subtract it from row 1.

$$\begin{cases} x + 2y = 1 \\ y = -2 \\ z = 3 \end{cases} \quad \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Subtract 2 times row 2 from row 1.

$$\begin{cases} x = 5 \\ y = -2 \\ z = 3 \end{cases} \quad \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

This system of equations has a unique solution.

Example 1.7 (Equations with an infinite number of solutions):

$$\begin{cases} x+y+2z = 2 \\ y+z = 0 \\ 2x+y+3z = 4 \end{cases} \quad \begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & 1 & 1 & 0 \\ 2 & 1 & 3 & 4 \end{bmatrix}$$

Subtract 2 times row 1 from row 3.

$$\begin{cases} x+y+2z = 2 \\ y+z = 0 \\ -y-z = 0 \end{cases} \quad \begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & -1 & 0 \end{bmatrix}$$

Add row 2 to row 3.

$$\begin{cases} x+y+2z = 2 \\ y+z = 0 \\ 0 = 0 \end{cases} \quad \begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Subtract row 2 from row 1.

$$\begin{cases} x+z = 2 \\ y+z = 0 \\ 0 = 0 \end{cases} \quad \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This is the best we can do. Notice that the third equation is redundant, thus this system of equations has an infinite number of solutions.

Example 1.8 (Equations with no solution):

$$\begin{cases} x+4y+2z = 2 \\ x+2y+8z = 0 \\ x+3y+5z = 3 \end{cases} \quad \begin{bmatrix} 1 & 4 & 2 & 2 \\ 1 & 2 & 8 & 0 \\ 1 & 3 & 5 & 3 \end{bmatrix}$$

Subtract row 1 from the row 2 and row 3.

$$\begin{cases} x+4y+2z = 2 \\ -2y+6z = -2 \\ -y+3z = 1 \end{cases} \quad \begin{bmatrix} 1 & 4 & 2 & 2 \\ 0 & -2 & 6 & -2 \\ 0 & -1 & 3 & 1 \end{bmatrix}$$

Divide row 2 by -2.

$$\begin{cases} x+4y+2z = 2 \\ y-3z = 1 \\ -y+3z = 1 \end{cases} \quad \begin{bmatrix} 1 & 4 & 2 & 2 \\ 0 & 1 & -3 & 1 \\ 0 & -1 & 3 & 1 \end{bmatrix}$$

Add row 2 to row 3.

$$\begin{cases} x+4y+2z = 2 \\ y-3z = 1 \\ 0 = 2 \end{cases} \quad \begin{bmatrix} 1 & 4 & 2 & 2 \\ 0 & 1 & -3 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

Now we get a contradiction $0 = 2$, which means that there is no solution to this system of equations.

Now we can generalize this strategy to an arbitrary number of equations. First we eliminate the first unknown from all but one equation, then among the remaining equations eliminate the second unknown from all but one equation. Repeat this until you reach one of three situations:

1. We have no all-zero rows and an equation with one unknown. This means that the system of equations have a unique solution. We can solve for that unknown and eliminate it from every equation. Repeat this until every equation has one unknown left and the system of equations is solved.
2. We have a all-zero row in the matrix corresponding to the equation $0 = 0$. Then there are fewer equations than unknowns and the system of linear equations is underdetermined. There are an infinite number of solutions.
3. We have a all-zero row in the matrix corresponding to the equation $0 = a$ where $a \neq 0$. This means that the system of linear equations is inconsistent and there are no solutions.

In discussion section, you will see this explained. We will also release supplemental notes and iPython code that illustrates how Gaussian Elimination can proceed.