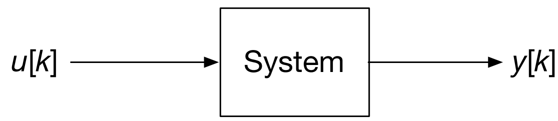
EE 16B Designing Information Devices and Systems II
Fall 2015 Section 10B

Solutions: Courtesy of Quincy Huynh.

1. Open-Loop System



Consider the open-loop system shown above, with $A = \begin{bmatrix} 0.9 & 0.8 \\ 0.5 & 0.6 \end{bmatrix}$, $B = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, and $C = \begin{bmatrix} 0 & 1 \end{bmatrix}$.

(a) What is the size of the state vector $x(k)$? The input vector $u(k)$? The output vector $y(k)$?

Solutions: A is 2x2, B is 2x1 and C is 1x2. From the equations:

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) \\ y(k) &= Cx(k) \end{aligned}$$

$\therefore$ $x(k)$ is a 2x1, $u(k)$ is a 1x1 and $y(k)$ is a 1x1.

(b) Assuming $x(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $u(k) = 0$ for all k, find the state $x(k)$ of the system for $k = 0$ to 3.

Solutions:

$$\begin{aligned} x(k) &= A^k x(0) \\ x(0) &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ x(1) &= \begin{bmatrix} 0.9 & 0.8 \\ 0.5 & 0.6 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ x(1) &= \begin{bmatrix} 1.7 \\ 1.1 \end{bmatrix} \\ x(2) &= \begin{bmatrix} 0.9 & 0.8 \\ 0.5 & 0.6 \end{bmatrix}^2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ x(2) &= \begin{bmatrix} 2.41 \\ 1.51 \end{bmatrix} \\ x(3) &= \begin{bmatrix} 0.9 & 0.8 \\ 0.5 & 0.6 \end{bmatrix}^3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ x(3) &= \begin{bmatrix} 3.377 \\ 2.111 \end{bmatrix} \end{aligned}$$

(c) Calculate the eigenvalues of matrix A.

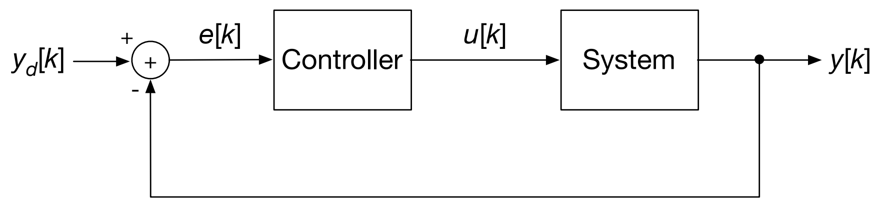
Solutions:

$$det(A-\lambda I) = \begin{vmatrix} 0.9-\lambda & 0.8 \\ 0.5 & 0.6-\lambda \end{vmatrix}$$
$$0 = (0.9-\lambda)(0.6-\lambda) - 0.4$$
$$0 = \lambda^2 - 1.5\lambda + 0.14$$
$$0 = (\lambda - 1.4)(\lambda - 0.1)$$
$$\lambda = 1.4, 0.1$$

(d) Would you consider this a "stable" system? Explain your answer.

Solutions: Since $\exists\ \lambda$ such that $\lambda > 1$, the system is not stable. This means that as $k \to \infty$, $x(k) \to \infty$.

2. Closed-Loop System



Consider the open-loop system shown above, with the same A, B, and C as in problem 1. The controller is implemented with parameter $K = 0.6$.

(a) Find the dimensions of the all of the vectors and matrices in the system.
Vectors: $x(k), y_d(k), e(k), u(k), y(k)$ Matrices: A, B, C, K, A_{CL}, and B_{CL}.

Solutions: A is 2x2, B is 2x1 and C is 1x2. From the equations:

$$x(k+1) = A_{CL}x(k) + B_{CL}y_d(k)$$
$$x(k+1) = (A - BKC)x(k) + (BK)y_d(k)$$
$$y(k) = Cx(k)$$

$\therefore$ $x(k)$ is a 2x1, $y_d(k)$ is a 1x1, $e(k)$ is a 1x1, $u(k)$ is a 1x1, $y(k)$ is a 1x1.
A_{CL} is a 2x2 and B_{CL} is a 2x1.

(b) Find A_{CL} and B_{CL}, the new state matrices that define the closed-loop system.

Solutions:

$$A_{CL} = A - BKC$$
$$A_{CL} = \begin{bmatrix} 0.9 & 0.8 \\ 0.5 & 0.6 \end{bmatrix} - 0.6 \begin{bmatrix} 2 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix}$$
$$\mathbf{A_{CL}} = \begin{bmatrix} 0.9 & -0.4 \\ 0.5 & 0 \end{bmatrix}$$
$$B_{CL} = BK$$
$$\mathbf{B_{CL}} = \begin{bmatrix} 1.2 \\ 0.6 \end{bmatrix}$$

(c) Assuming $x(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $y_d(k) = 0$ for all k, find the state $x(k)$ of the system for $k = 0$ to 3.

Solutions: Since $y_d(k) = 0 \; \forall \; k$, then $B_{CL} y_d(k) = 0 \; \forall \; k$.

$$x(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$x(1) = \begin{bmatrix} 0.9 & -0.4 \\ 0.5 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$x(1) = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$$

$$x(2) = \begin{bmatrix} 0.9 & -0.4 \\ 0.5 & 0 \end{bmatrix}^2 \text{âĂć} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$x(2) = \begin{bmatrix} 0.25 \\ 0.25 \end{bmatrix}$$

$$x(3) = \begin{bmatrix} 0.9 & -0.4 \\ 0.5 & 0 \end{bmatrix}^3 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$x(3) = \begin{bmatrix} 0.125 \\ 0.125 \end{bmatrix}$$

(d) Calculate the eigenvalues of matrix A_{CL}.

Solutions:

$$det(A - \lambda I) = \begin{vmatrix} 0.9 - \lambda & -0.4 \\ 0.5 & -\lambda \end{vmatrix}$$

$$0 = (0.9 - \lambda)(-\lambda) + 0.2$$

$$0 = \lambda^2 - 0.9\lambda + 0.2$$

$$0 = (\lambda - 0.5)(\lambda - 0.4)$$

$$\lambda = 0.5, 0.4$$

(e) Would you consider this a “stable” system? Explain your answer.

Solutions: Since both $\lambda < 1$, the system is stable. This means that as $k \to \infty$, $x(k) \to 0$. Notice that the initial condition $x(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is the eigenvector corresponding to $\lambda = 0.5$, so $x(k)$ became a smaller scaled version of $x(0)$ as k increased.