Solutions: Provided by John Noonan.

1. Anti-Aliasing Filters

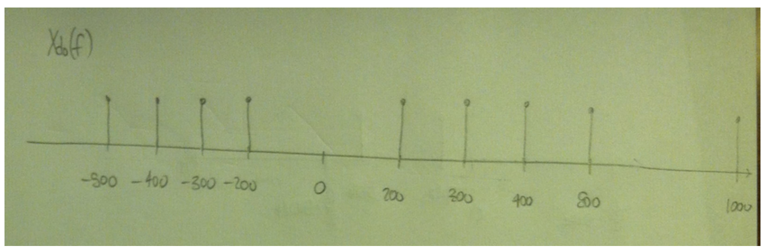
We want to digitize a signal $x_c(t) = \cos(2\pi 200t) + \cos(2\pi 300t)t + \cos(2\pi 400t) + \cos(2\pi 500t)$.

(a) What is the maximum sampling period T_s that will allow us to perfectly reconstruct the signal?

Solutions: The highest frequency is 500 Hz., so we need a 1 kHz. sampling frequency. Thus, the maximum sampling period $T_s = 10^{-3}s$.

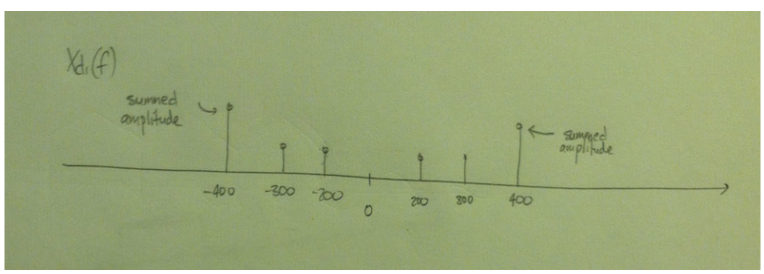
(b) If $T_s = 1/1200$ seconds, draw the spectrum $X_{d0}(f)$ of the sampled signal and label the key frequencies. Does aliasing occur? Why or why not?

Solutions: No, aliasing does not occur, so we can perfectly reconstruct the signal.



(c) If $T_s = 1/900$ seconds, draw the spectrum $X_{d1}(f)$ of the sampled signal and label the key frequencies. Does aliasing occur? Why or why not?

Solutions: Yes, aliasing does occur. Our sampling frequency is less than 2 * Nyquist frequency.



(d) Using the T_s from part (c), what filter $H_{a1}(f)$ could be applied **after** sampling to only preserve the frequencies of X_{d1} that are not "corrupted" by aliasing? (Any frequency information that does not match the original signal is considered corrupted.) Draw or write an equation to describe the filter. Out of the eight non-zero frequencies present in the original signal, how many are preserved?

Solutions: We can use the following filter to be applied after sampling.

$$H(f) = \begin{cases} 1 & -350 \leq f \leq 350 \\ 0 & \text{otherwise} \end{cases}$$

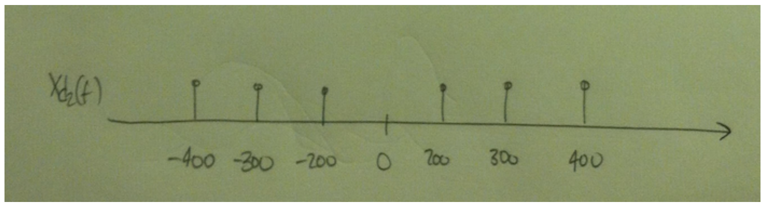
4 out of the 8 frequencies will be preserved: -300, -200, 200, 300.

(e) Using the T_s from part (c), explain how you could apply a filter $H_{a0}(f)$ **before** sampling to recover more of the original signal. Draw or write an equation to describe the filter. Then draw the spectrum $X_{d2}(f)$ of the sampled signal and label the key frequencies. Out of the eight non-zero frequencies present in the original signal, how many are preserved?

Solutions: We can use the following filter to be applied before sampling.

$$H(f) = \begin{cases} 1 & \text{-450} \leq \text{f} \leq 450 \\ 0 & \text{otherwise} \end{cases}$$

The cutoff is at $\frac{\omega_s}{2} = 450$.



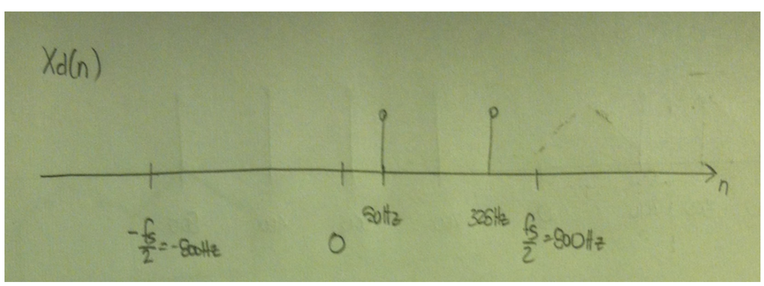
6 out of the 8 frequencies will be preserved.

2. Rocket ~~Science~~ Engineering

A vibration sensor mounted on a rocket during a test firing produces an output $x_c(t)$ that is proportional to the acceleration. The sensor output $x(t)$ is sampled at a rate of $f_s = 1000$Hz and a total of 10,000 samples are recorded. The sequence $x_d(n) = x_c(nT_s)$ is then processed using the Numpy `fft.fft` algorithm to compute $X[k]$, where $k = 1, 2, ..., 10000$.

(a) If $X[500]$, $X[3250]$, $X[6750]$, and $X[9500]$ are non-zero, and the remaining $X[k]$ are zero, sketch the spectrum of $x_d(n)$. Assuming no aliasing occurred, what frequencies were present in $x_c(t)$? Write a possible expression for $x_c(t)$.

Solutions:



There is 1000Hz of frequencies in 10000 sections, so that means that each 1 Hz of frequency is 10 samples, or there is 0.1Hz / sample. Thus, assuming no aliasing occurred, frequencies are present at 0.1 * 500 = 50 Hz and 0.1 * 3250 = 325Hz. The other two frequencies are out of range. One possible expression for $x_c(t)$ would be:

$$x_c(t) = K_1 cos(2\pi * 50t) + K_2 cos(2\pi * 325t)$$

(b) An engineer famliar with the dynamics of the rocket says that the highest frequency component determined in part (a) is to be exected but the lowest frequency component does not make sense. Further, a frequency component somewhere in the range of $500 < f < 1000$ Hz should be present. Given this information, what frequencies were actually present in $x_c(t)$? Write a more accurate expression for $x_c(t)$.

Solutions: When aliasing occurs, 0.1 Hz * 9500 = 950 Hz aliases down to 950 - 1000 = -50 Hz., and this is paired to 50 Hz. Similarly, 6750 * 0.1 Hz = 675 Hz aliases down to 675 - 1000 = -325 Hz., and this is paired to 325 Hz.

We know that 325 Hz. is expected. We also know that 50 Hz. is paired with -50 Hz., which is aliased to 950 Hz. Thus, the frequencies that are present are 325 Hz. and 950 Hz.

A more accurate expression for $x_c(t)$ would be:

$$x_c(t) = K_1 cos(2\pi * 325t) + K_2 cos(2\pi * 950t)$$