A Quick Introduction on Compressive Sensing

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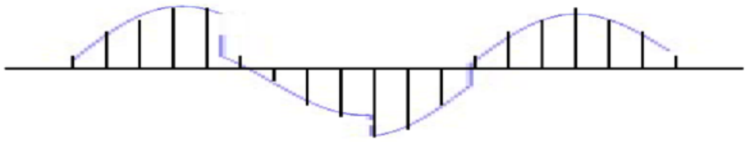
Pressure is on DSP

- Shannon/Nyquist sampling theorem
  – For general band-limited signal, no information loss if we sample at 2x signal bandwidth
- DSP revolution:
  – sample first and ask questions later

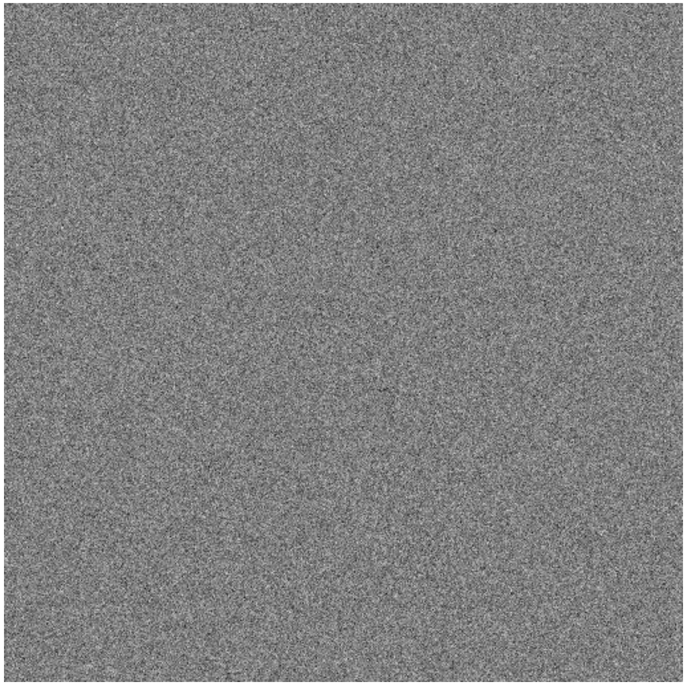


- Increasing pressure on DSP hardware, algorithms
  – ever faster sampling and processing rates
  – ever larger dynamic range
  – ever larger, higher-dimensional data
  – ever lower energy consumption

  …

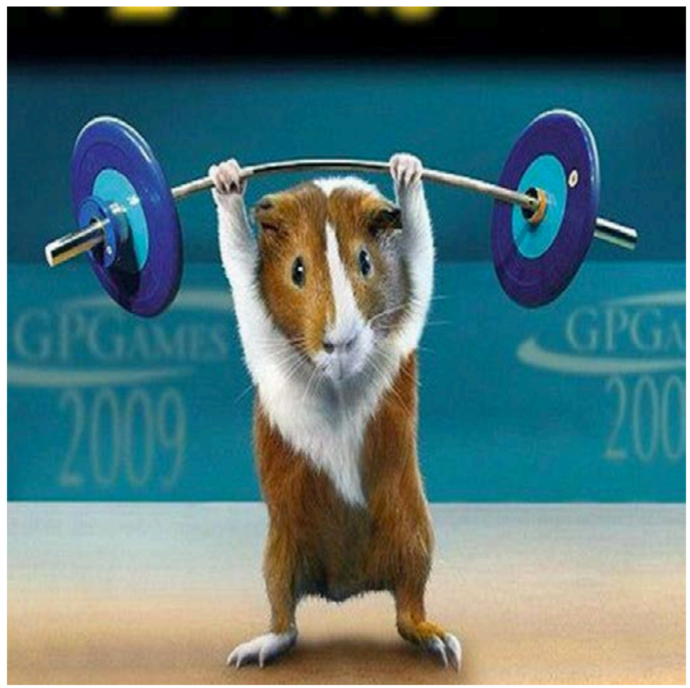
But wait … Do we really need general signal?

- What does general signal mean (say a general image)?
  - With high probability it's something like this …



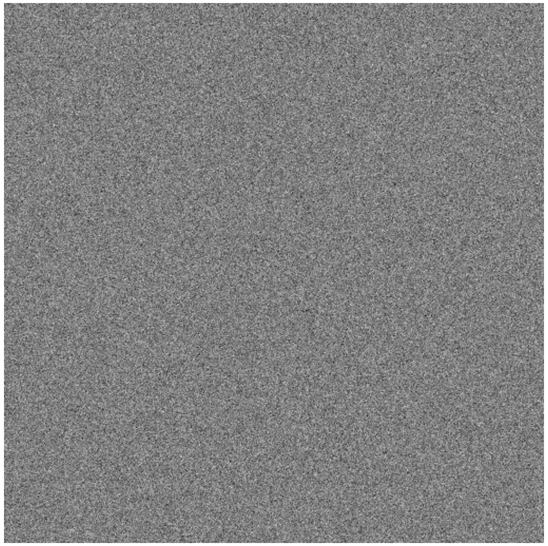
But wait … Do we really need general signal?

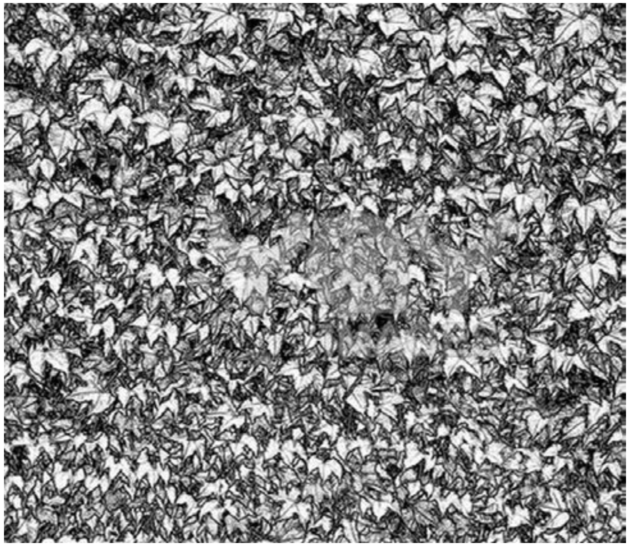
- What does general signal mean (say a general image)?
  - But usually we are just interested in the structured signal like this …



From general signal to structured signal

- We want to make more assumption (rational?) about our signal in order to lower the limit of sampling rate, let compare more between our previous assumption and the reality:



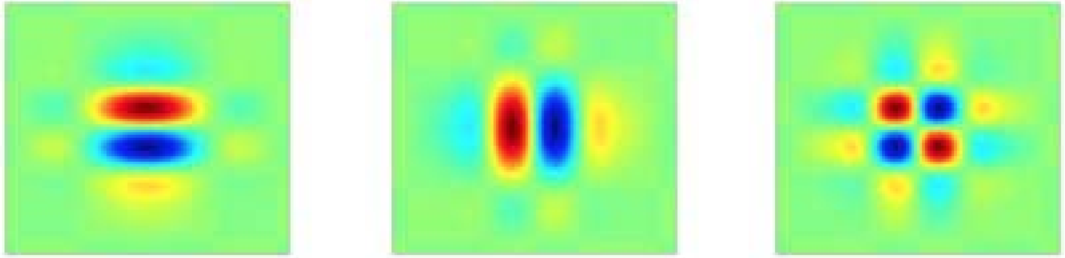


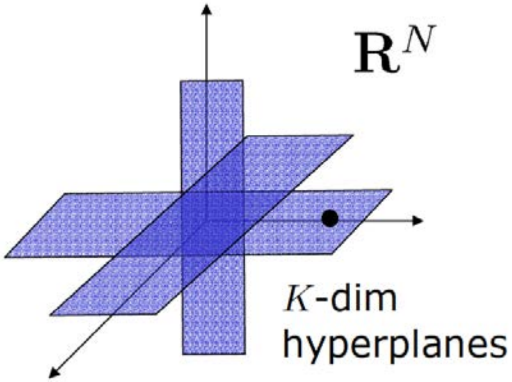
- A lot of high/low level structures, e.g. edges, continuation.

Sparse Representation

- Inspired by the statistics of some transforms (DCT, Wavelet …), we assume our signal is sparse in some domain.

$$x = \sum_{i=1}^{N} \alpha_i \psi_i \qquad \alpha_i = \langle x, \psi_i \rangle$$





- K-sparse: K large coefficients, where K << N

$$x \approx \sum_{K \ll N \ \text{largest terms}} \alpha_i \psi_i$$

Transform Coding and Its Inefficiency

- The previous sample-then-compress framework:
  - K largest coefficients are located and the rest N-K smallest coefficients are discarded.
  - The K values and locations of the largest coefficients are encoded.

- Major three inefficiencies:
  - N maybe large even K is small
  - N coefficients need to be computed even N-K of them will be discarded
  - Encoding of locations introduce an overhead

The Compressive Sensing Problem

- A general M-dimensional linear measurement:

$$\mathbf{y} = \Phi\mathbf{x} = \Phi\Psi\mathbf{s} = \Theta\mathbf{s}$$

- Φ is not adaptive and it's stable – the salient information in any K-sparse or compressive signal is not damaged by this dimension reduction process.
- Then we recover the signal based on y,Φ,Ψ

The Solution to CS

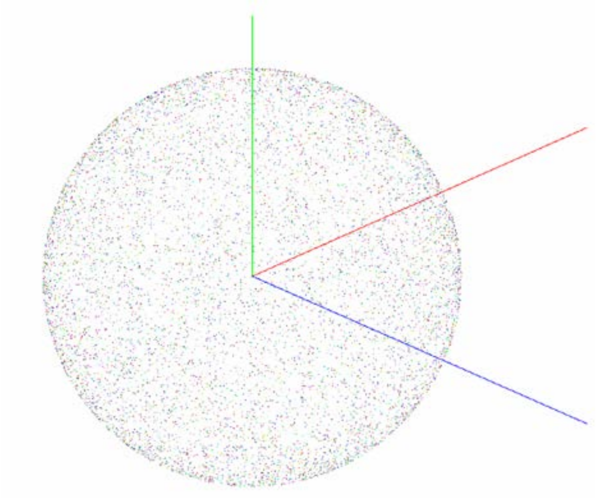
- If the signal is not compressive, the problem is ill-conditioned
- If the K locations are know, a necessary and sufficient condition for well-conditioning is:
  - The matrixΘpreserves the lengths of these K-sparse vectors.

$$1 - \epsilon \leq \frac{\|\Theta \mathbf{v}\|_2}{\|\mathbf{v}\|_2} \leq 1 + \epsilon$$

- A sufficient condition for a stable solution for both K-sparse and compressible signals isΘsatisfies the above condition for an arbitrary 3K-sparse vector v. This is referred to as restricted isometry property (RIP).
- Incoherence: rows of Φdo not sparsely represent Ψ.

The Solution to CS: Sampling

- However fortunately such a Φ is easy to construct as a random matrix and the previous condition is satisfied with high probability.
- For instance: let each elements of Φbe i.i.d Gaussian random variables ~ $N(0, 1/M)$.
  - Quick verification by expectation and Chebyshev inequality …

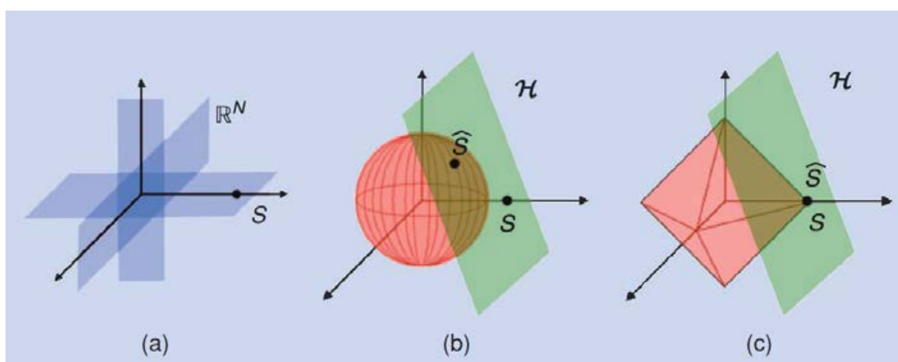


The Solution to CS: Reconstruction

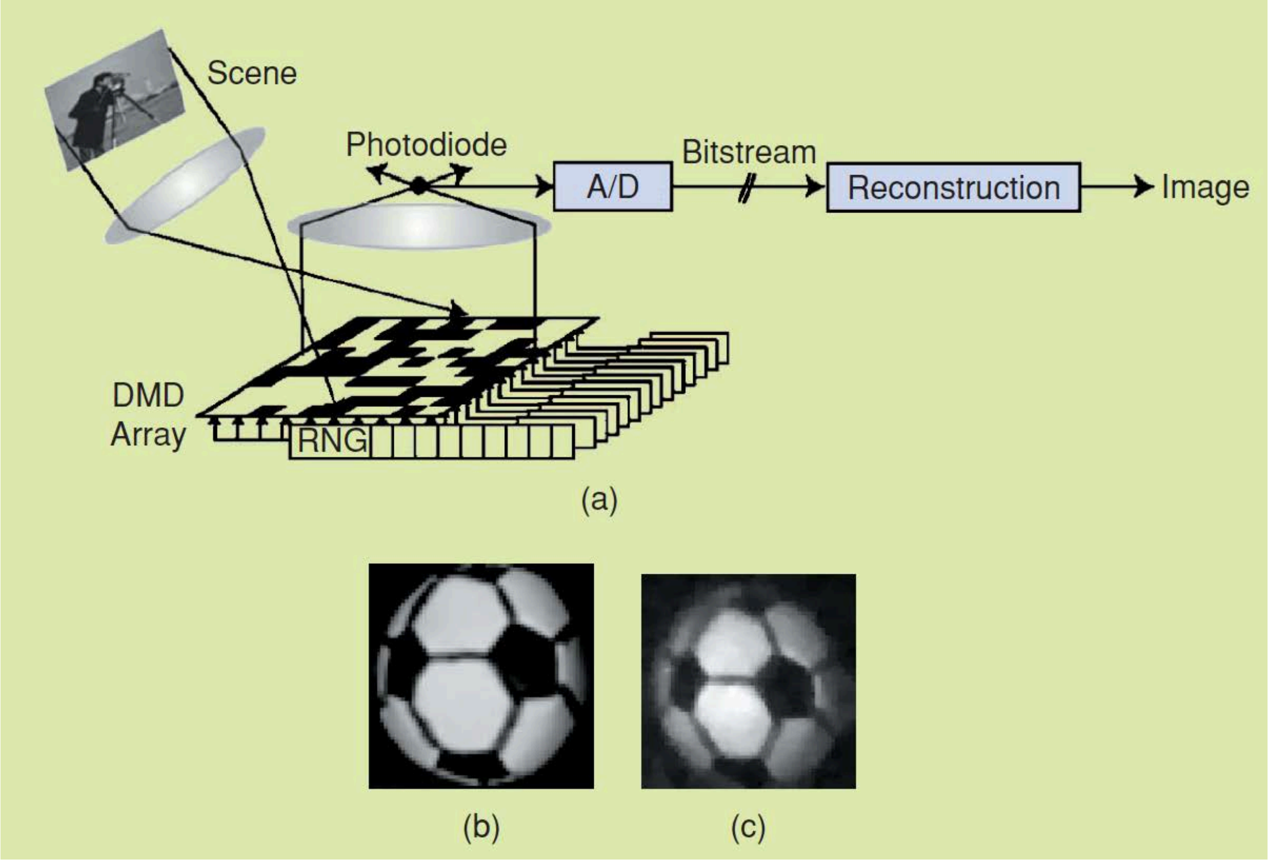
- E.g. Lasso Regression

$$\min_{s} \|y - \Phi \Psi s\|_2^2 + \lambda \|s\|_1$$

- The basic idea is to preserve the information and also seek sparse representation.
- Other similar convex optimization based on this idea can also be formulated, say, make the L2 norm a convex constraint.

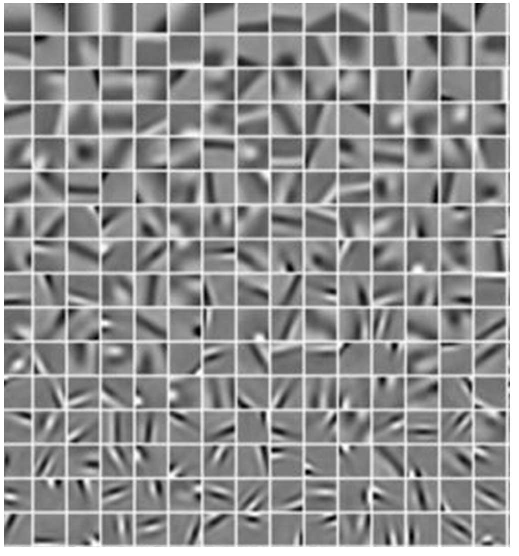


An Image Filling-in Case: Single Pixel Camera



Further Relaxation

- Non-orthogonal basis map
- Redundant basis map
- Dictionary learning: Sparse coding, Independent component analysis (infomax etc.)



Conclusion of Compressive Sensing Approach

- Transform Sparsity
- Non-coherency
- Non-linear Construction (Optimization)

Thanks!