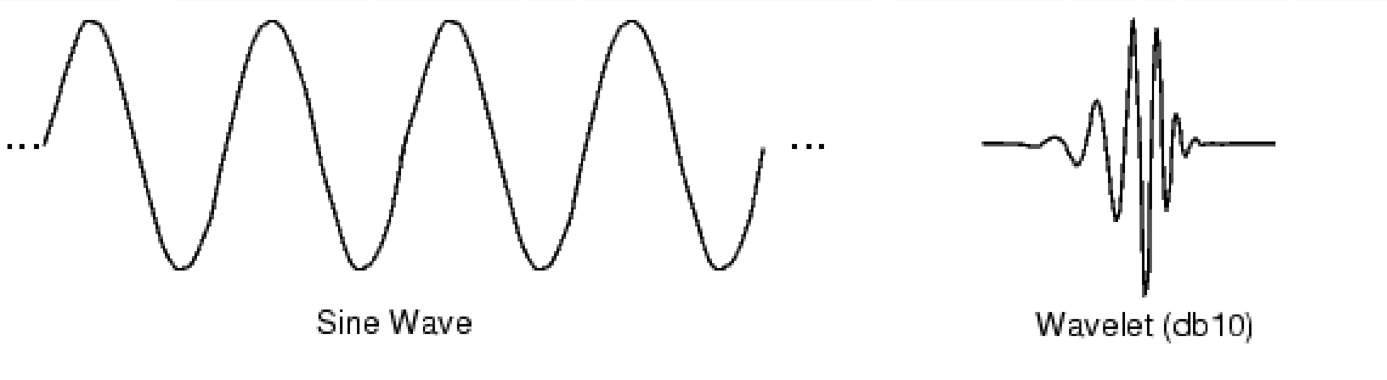
Wavelet Transform

Wavelet Definition

"The wavelet transform is a tool that cuts up data, functions or operators into different frequency components, and then studies each component with a resolution matched to its scale"

Dr. Ingrid Daubechies, Lucent, Princeton U.



Fourier vs. Wavelet

- FFT, basis functions: sinusoids
- Wavelet transforms: small waves, called wavelet
- FFT can only offer frequency information
- Wavelet: frequency + temporal information
- Fourier analysis doesn’t work well on discontinuous, “bursty” data
  - music, video, power, earthquakes,…

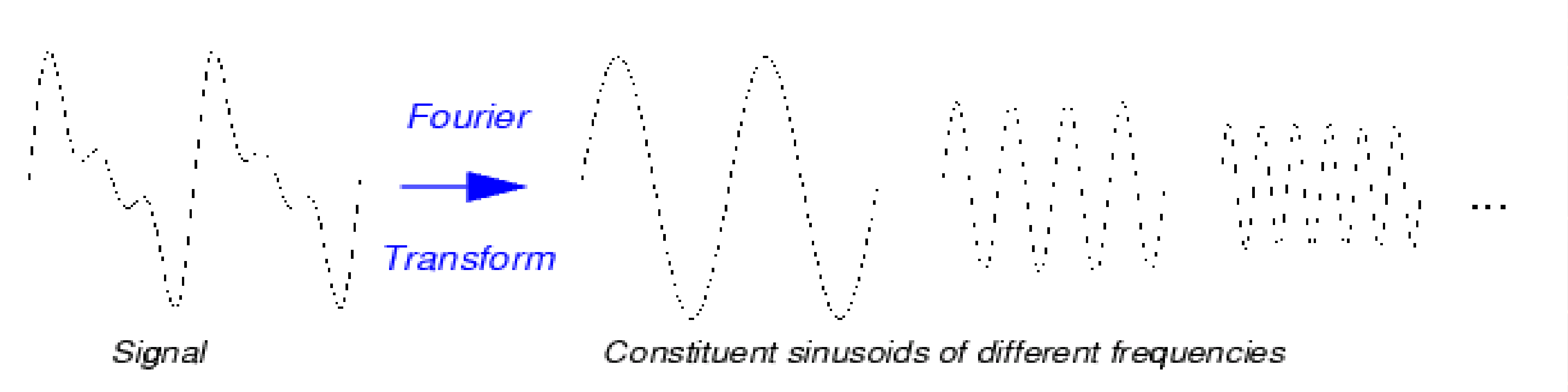
Fourier vs. Wavelet

- Fourier
  - Loses time (location) coordinate completely
  - Analyses the ***whole*** signal
  - Short pieces lose “frequency” meaning
- Wavelets
  - Localized time-frequency analysis
  - Short signal pieces also have significance
  - *Scale = Frequency band*

Fourier transform

Fourier transform:

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$$



Wavelet Transform

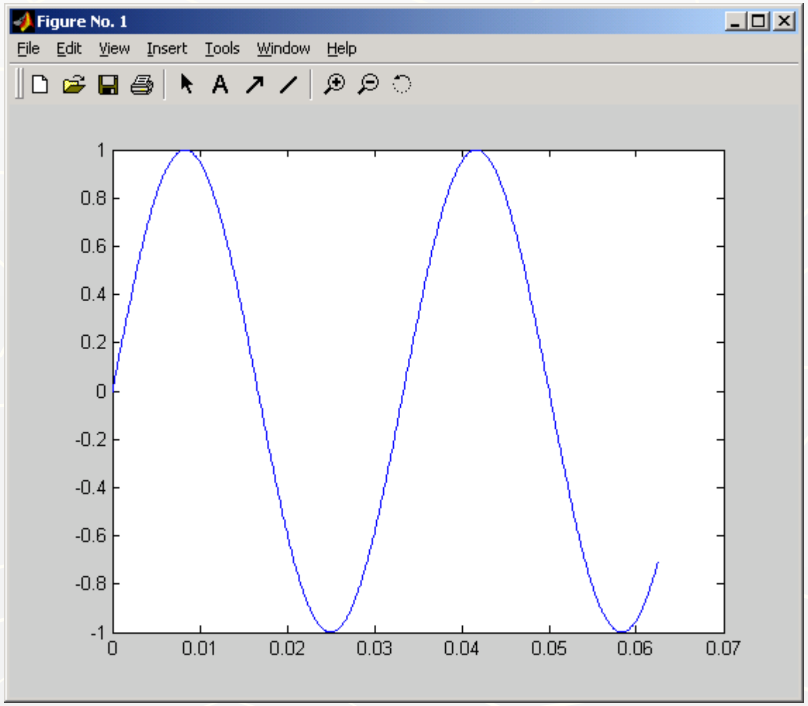
- Scale and shift original waveform
- Compare to a wavelet
- Assign a coefficient of similarity

Scaling-- value of “stretch”

- Scaling a wavelet simply means stretching (or compressing) it.

$$f(t) = \sin(t)$$

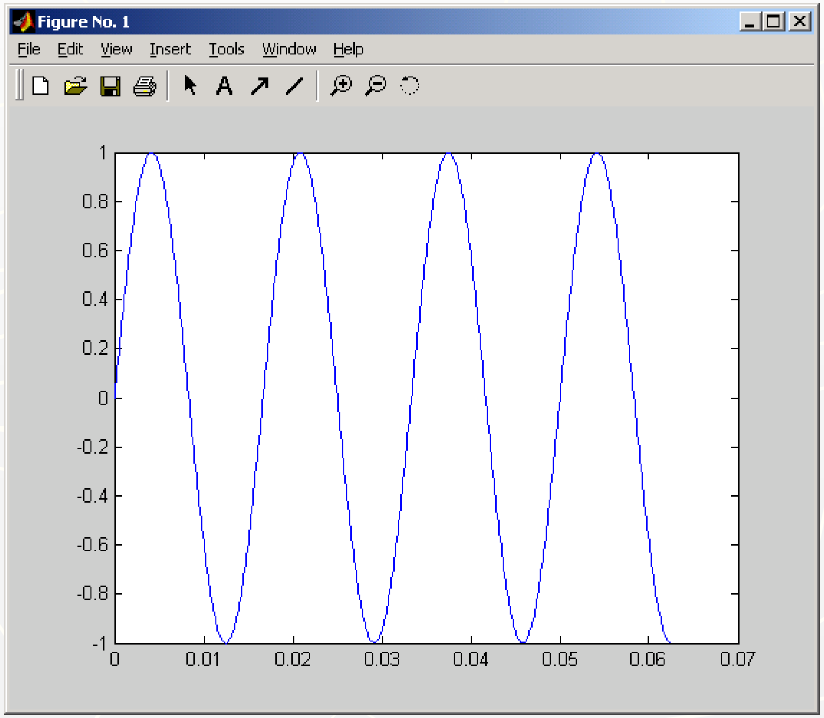
scale factor1



Scaling-- value of "stretch"

- Scaling a wavelet simply means stretching (or compressing) it.

$f(t) = \sin(2t)$
scale factor 2

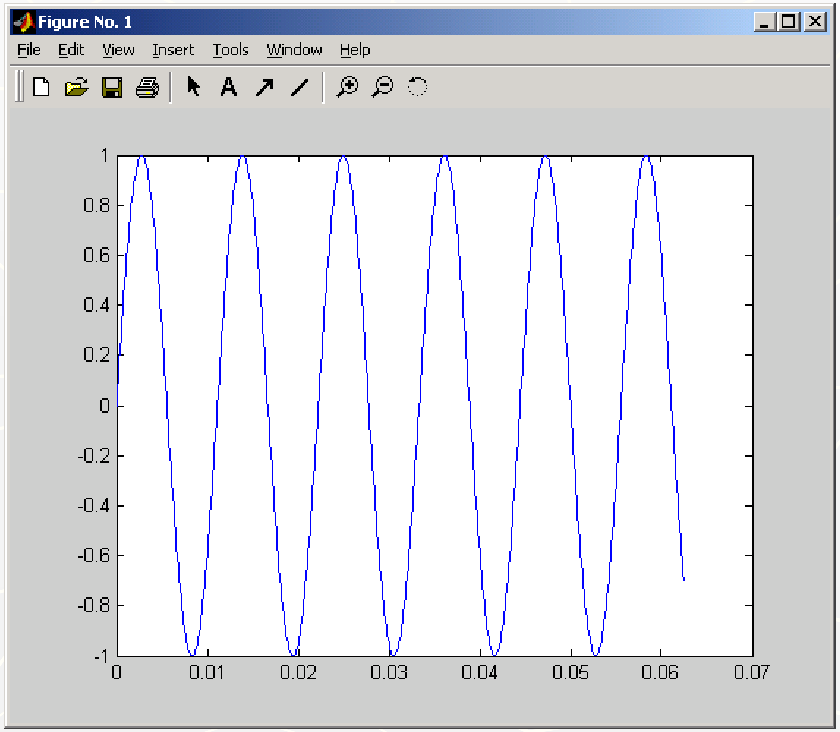


Scaling-- value of "stretch"

- Scaling a wavelet simply means stretching (or compressing) it.

$f(t) = \sin(3t)$

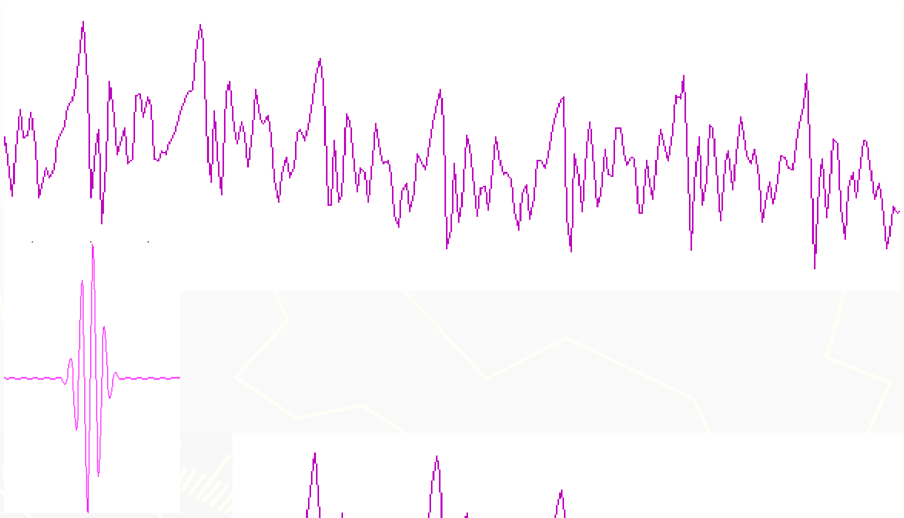
scale factor 3



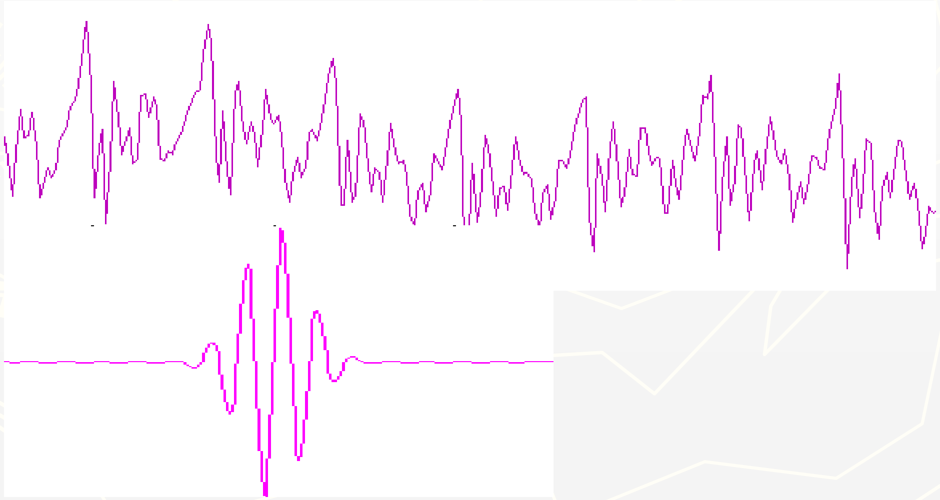
More on scaling

- It lets you either narrow down the frequency band of interest, or determine the frequency content in a narrower time interval
- Scaling = frequency band
- Good for non-stationary data
- Low scale→*a* Compressed wavelet→ Rapidly changing details→High frequency
- High scale →*a* Stretched wavelet → Slowly changing, coarse features → Low frequency

Scale is (sort of) like frequency

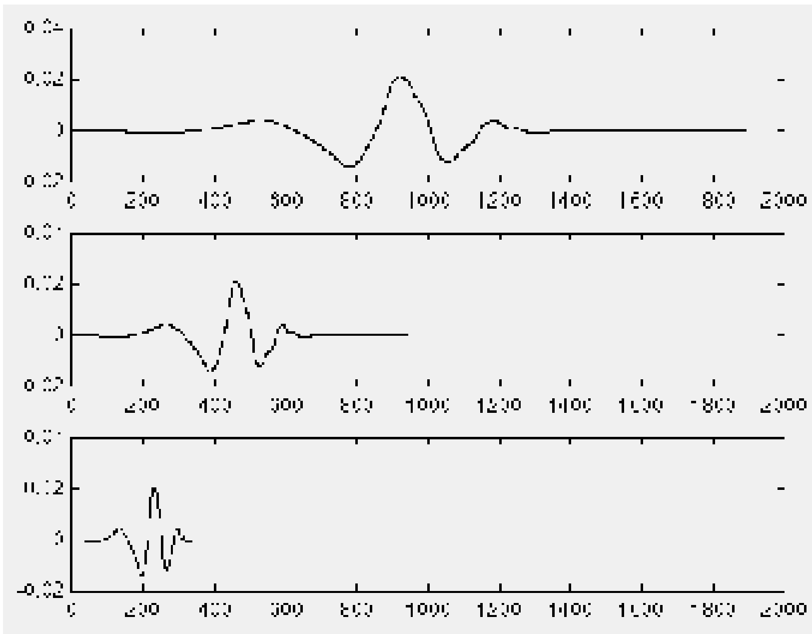


Small scale
-Rapidly changing details,
-Like high frequency



Large scale
-Slowly changing
 details
-Like low frequency

Scale is (sort of) like frequency



$$f(t) = \psi(t) \quad ; \quad a = 1$$

$$f(t) = \psi(2t) \; ; \quad a = \frac{1}{2}$$

$$f(t) = \psi(4t) \; ; \quad a = \frac{1}{4}$$

The scale factor works exactly the same with wavelets. The smaller the scale factor, the more "compressed" the wavelet.

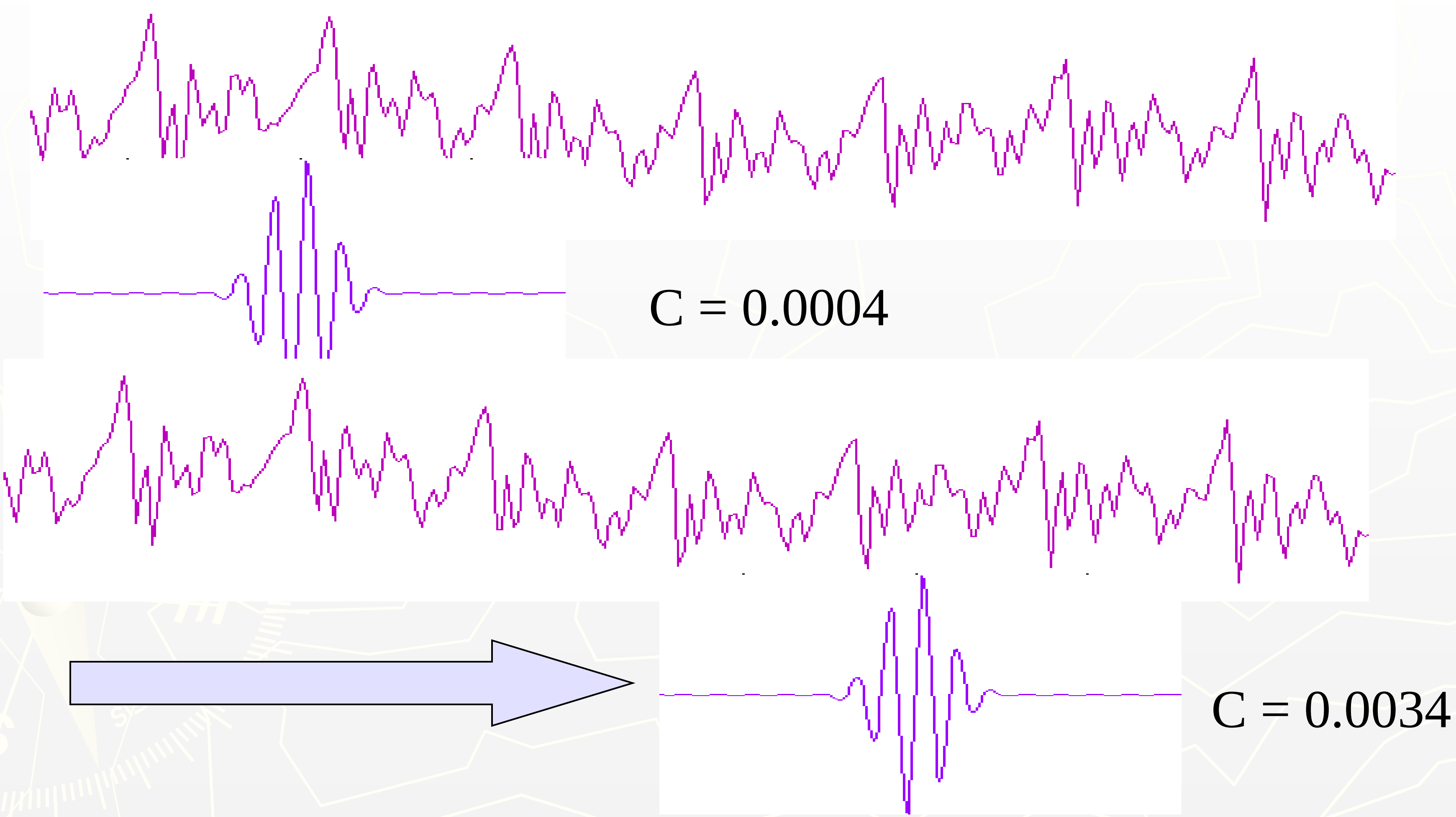
Shifting

Shifting a wavelet simply means delaying (or hastening) its onset. Mathematically, delaying a function f(t) by *k* is represented by f(t-k)

Wavelet function
$\psi(t)$

Shifted wavelet function
$\psi(t-k)$

Shifting

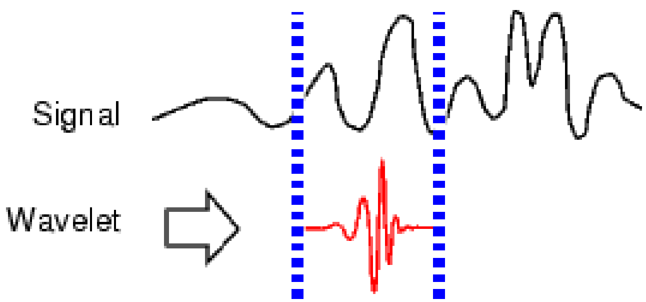


Five Easy Steps to a Continuous Wavelet Transform

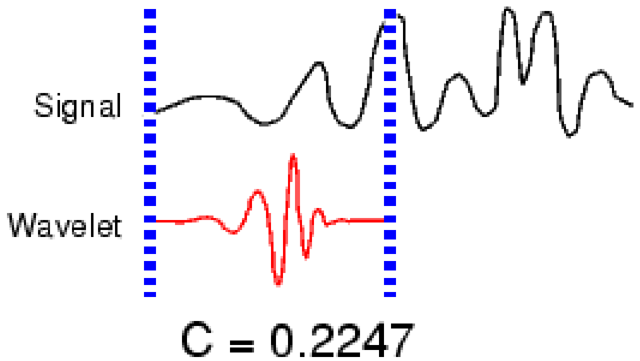
1. Take a wavelet and compare it to a section at the start of the original signal.
2. Calculate a correlation coefficient c

Five Easy Steps to a Continuous Wavelet Transform

3. Shift the wavelet to the right and repeat steps 1 and 2 until you've covered the whole signal.

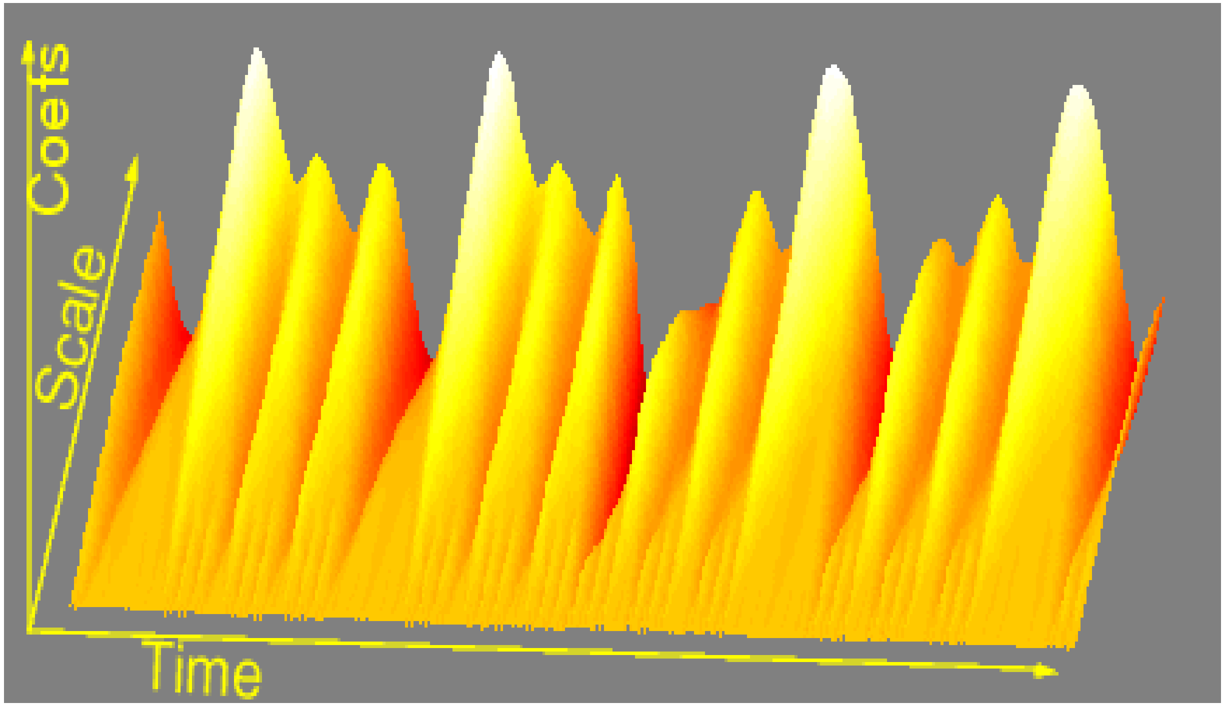


4. Scale (stretch) the wavelet and repeat steps 1 through 3.



5. Repeat steps 1 through 4 for all scales.

Coefficient Plots

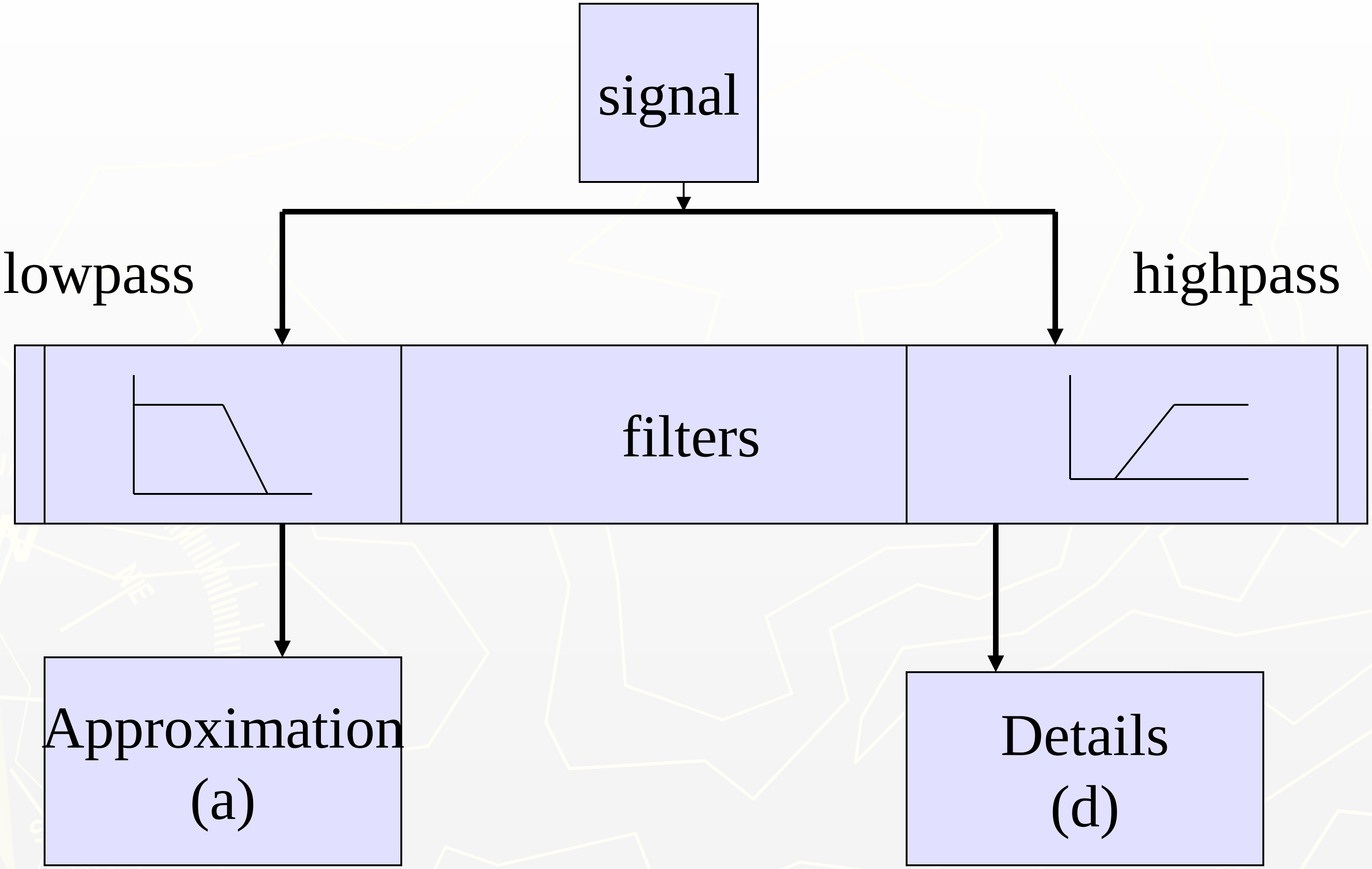


Discrete Wavelet Transform

- "Subset" of scale and position based on power of two
  - rather than every "possible" set of scale and position in continuous wavelet transform
- Behaves like a filter bank: signal in, coefficients out
- Down-sampling necessary

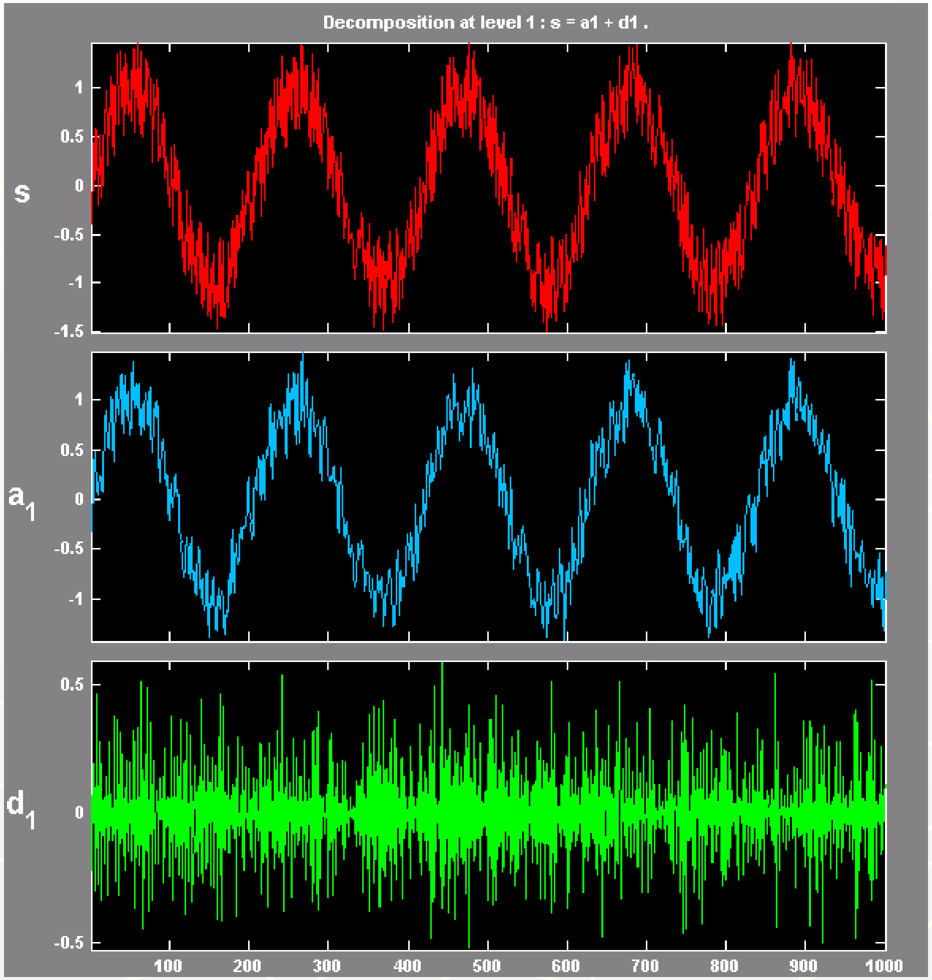
(twice as much data as original signal)

Discrete Wavelet transform

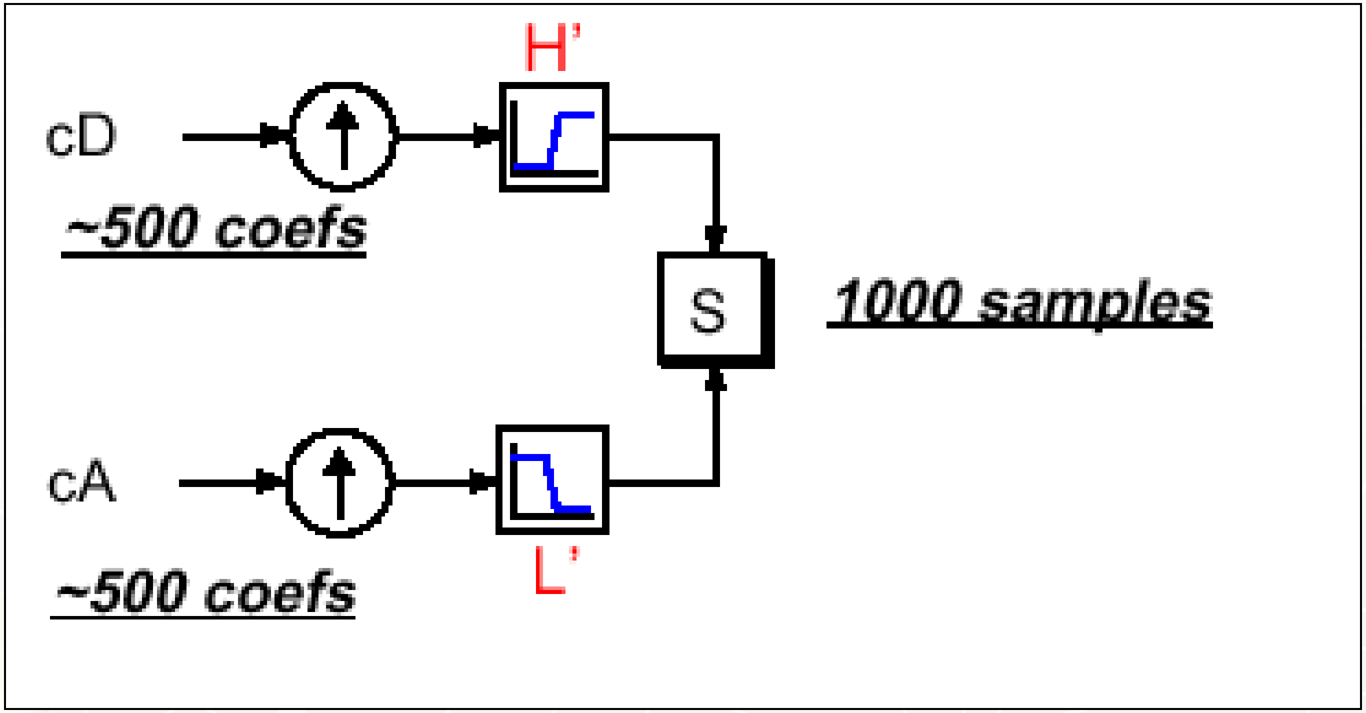


Results of wavelet transform — approximation and details

- Low frequency:
  - approximation (a)
- High frequency
  - details (d)
- "Decomposition" can be performed iteratively



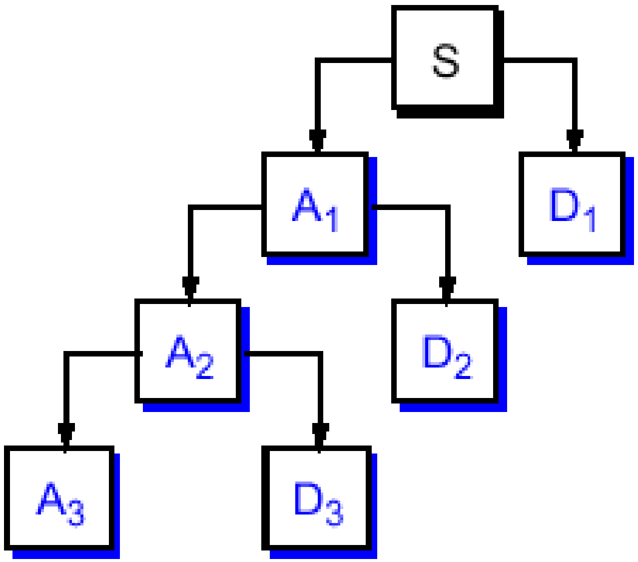
Wavelet synthesis



- Re-creates signal from coefficients
- Up-sampling required

Multi-level Wavelet Analysis

Multi-level wavelet decomposition tree



Reassembling original signal

$$S = A_1 + D_1$$

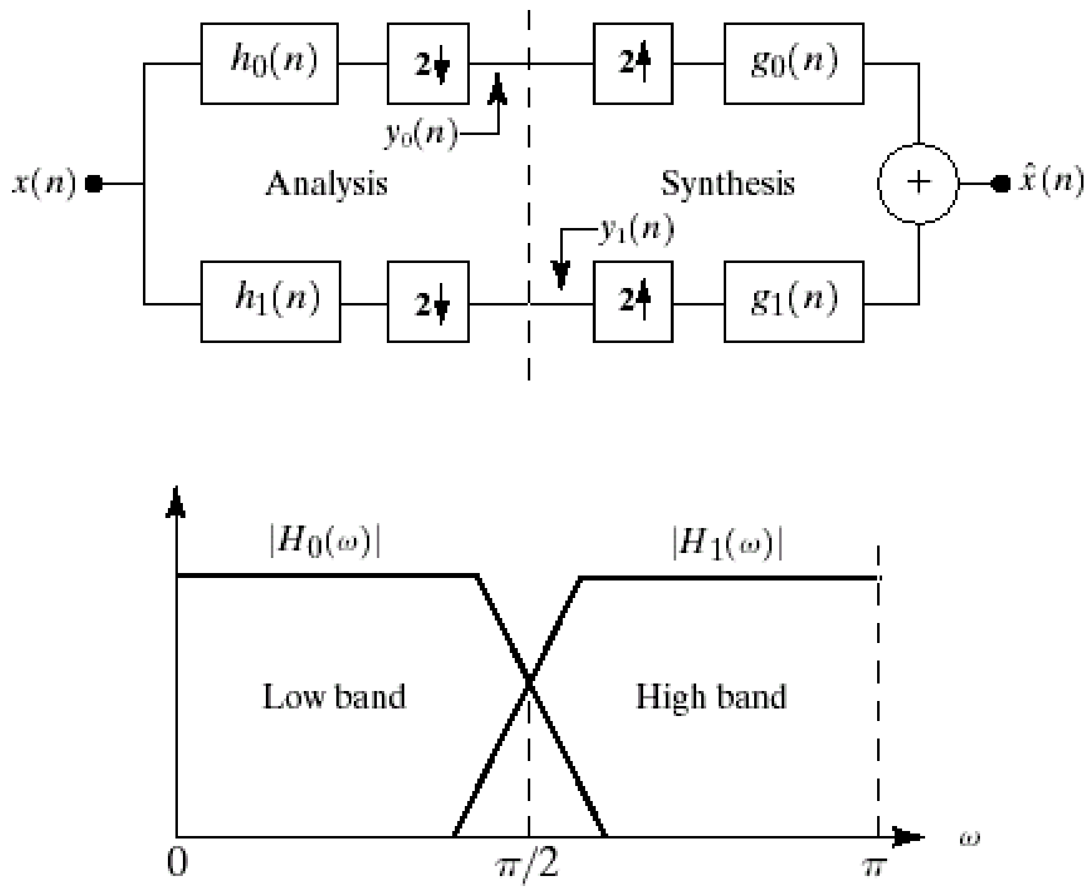
$$= A_2 + D_2 + D_1$$

$$= A_3 + D_3 + D_2 + D_1$$

Subband Coding

a
b

FIGURE 7.4 (a) A two-band filter bank for one-dimensional subband coding and decoding, and (b) its spectrum splitting properties.



2-D 4-band filter bank

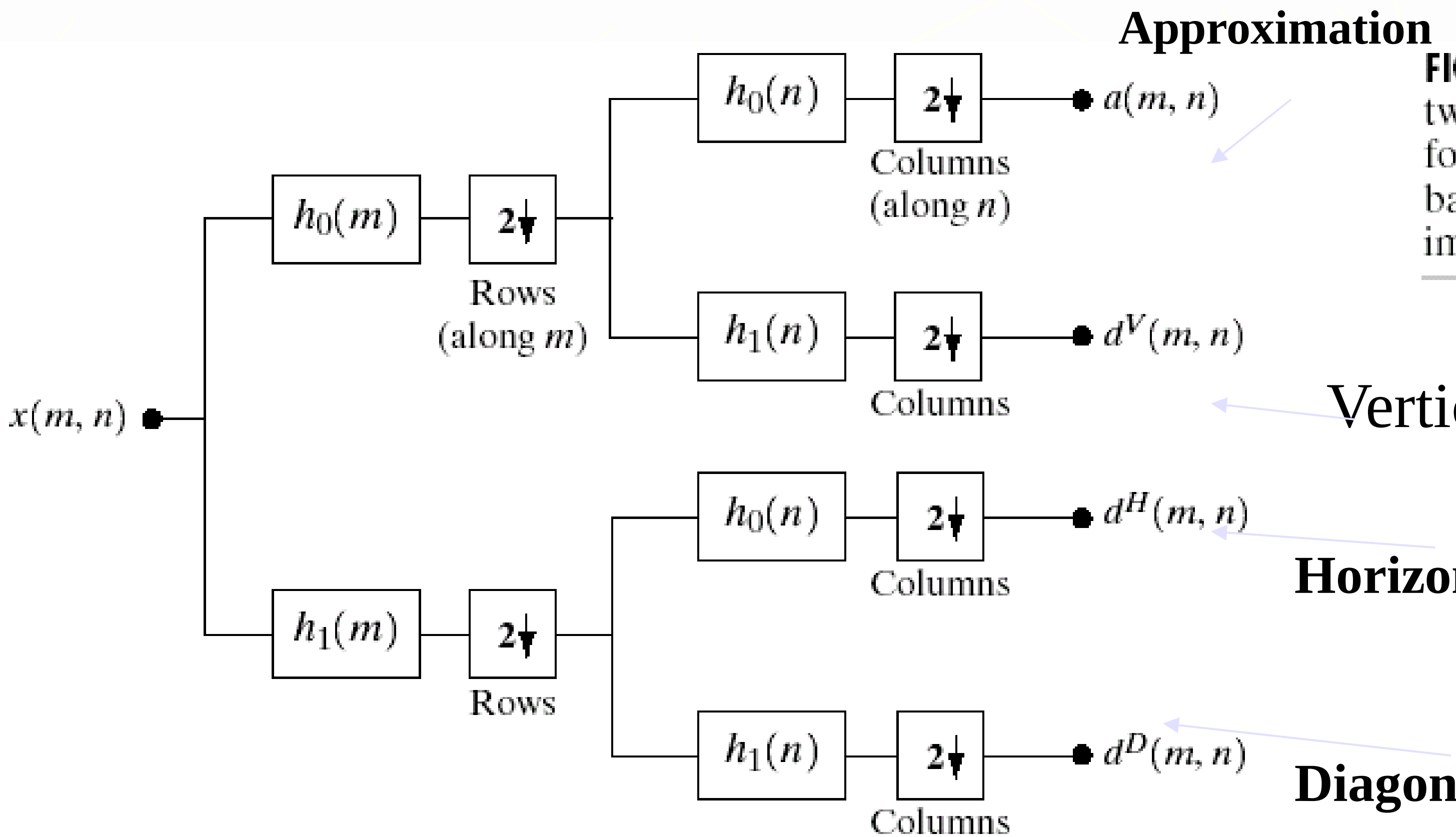
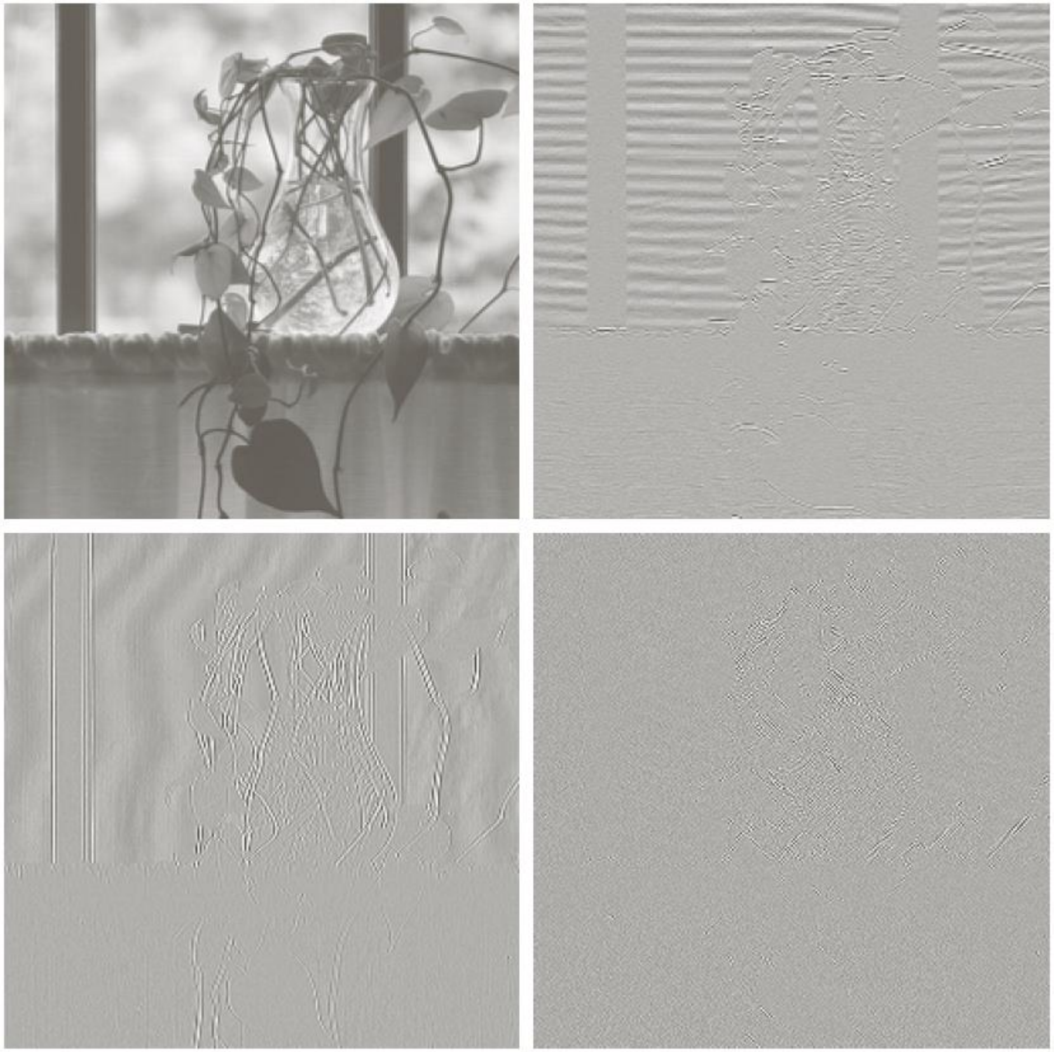


FIGURE 7.5 A two-dimensional, four-band filter bank for subband image coding.

An Example of One-level Decomposition



a	b
c	d

FIGURE 7.9 A four-band split of the vase in Fig. 7.1 using the subband coding system of Fig. 7.7. The four subbands that result are the (a) approximation, (b) horizontal detail, (c) vertical detail, and (d) diagonal detail subbands.

An Example of Multi-level Decomposition

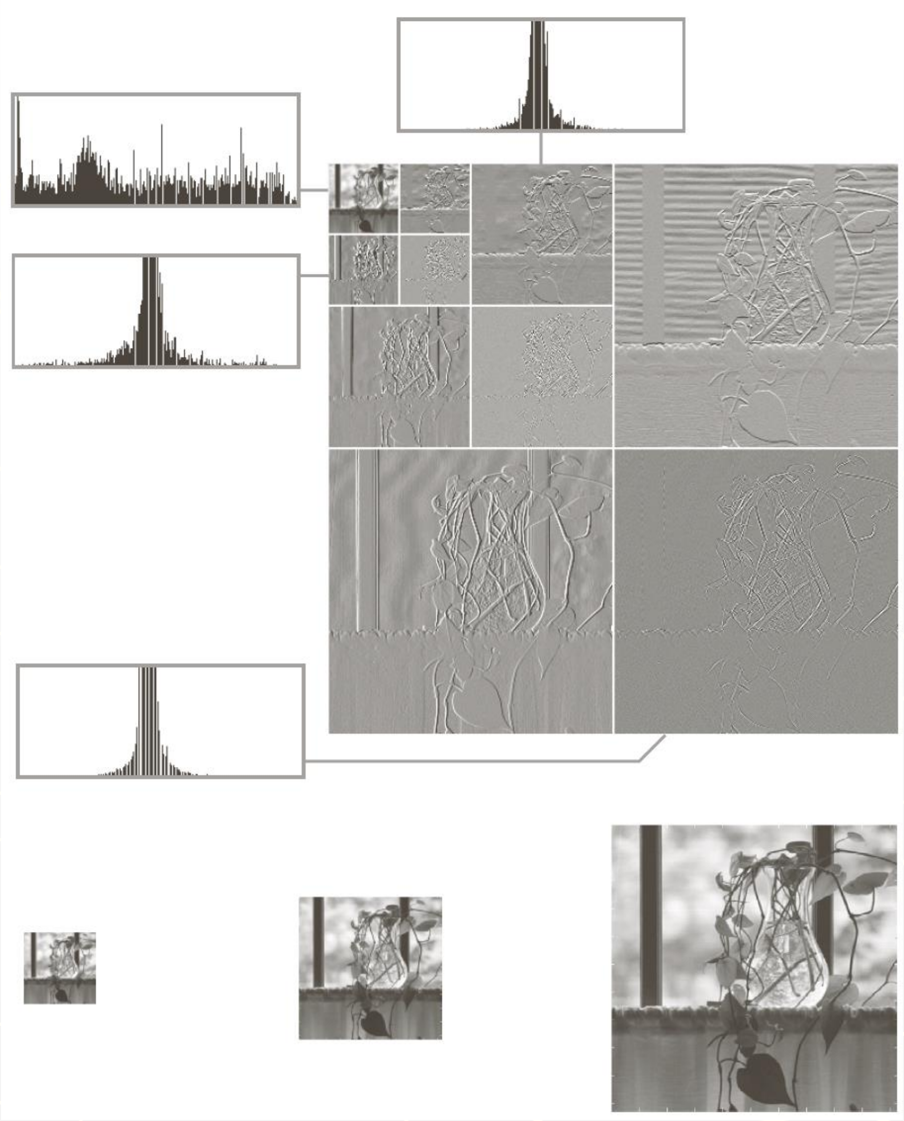


FIGURE 7.10 (a) A discrete wavelet transform using Haar $\mathbf{H}_2$ basis functions. Its local histogram variations are also shown. (b)–(d) Several different approximations (64×64, 128×128, and 256×256) that can be obtained from (a).

Wavelet Series Expansions

Wavelet series expansion of function $f(x) \in L^2(\mathbb{R})$ relative to wavelet $\psi(x)$ and scaling function $\varphi(x)$

$$f(x) = \sum_k c_{j_0}(k)\varphi_{j_0,k}(x) + \sum_{j=j_0}^{\infty} \sum_k d_j(k)\psi_{j,k}(x)$$

where ,

$c_{j_0}(k)$: approximation and/or scaling coefficients

$d_j(k)$: detail and/or wavelet coefficients

Wavelet Series Expansions

$$c_{j_0}(k) = \left\langle f(x), \varphi_{j_0,k}(x) \right\rangle = \int f(x)\varphi_{j_0,k}(x)dx$$

and

$$d_j(k) = \left\langle f(x), \psi_{j,k}(x) \right\rangle = \int f(x)\psi_{j,k}(x)dx$$

Wavelet Transforms in Two Dimensions

$$\varphi(x,y)=\varphi(x)\varphi(y)$$

$$\psi^{H}(x,y)=\psi(x)\varphi(y) \qquad \varphi_{j,m,n}(x,y)=2^{j/2}\varphi(2^{j}x-m,2^{j}y-n)$$

$$\psi^{V}(x,y)=\varphi(x)\psi(y) \qquad \psi^{i}{}_{j,m,n}(x,y)=2^{j/2}\psi^{i}(2^{j}x-m,2^{j}y-n)$$

$$\psi^{D}(x,y)=\psi(x)\psi(y) \qquad i=\{H,V,D\}$$

$$W_{\varphi}(j_0,m,n)=\frac{1}{\sqrt{MN}}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1}f(x,y)\varphi_{j_0,m,n}(x,y)$$

$$W^{i}{}_{\psi}(j,m,n)=\frac{1}{\sqrt{MN}}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1}f(x,y)\psi^{i}{}_{j,m,n}(x,y)$$

$$i=\left\{H,V,D\right\}$$

Inverse Wavelet Transforms in Two Dimensions

$$f(x,y)=\frac{1}{\sqrt{MN}}\sum_{m}\sum_{n}W_{\varphi}(j_0,m,n)\varphi_{j_0,m,n}(x,y)$$

$$+\frac{1}{\sqrt{MN}}\sum_{i=H,V,D}\sum_{j=j_0}^{\infty}\sum_{m}\sum_{n}W^{i}_{\psi}(j,m,n)\psi^{i}_{j,m,n}(x,y)$$