Lecture #22: Type Inference and Unification
Typing In the Language ML

• Examples from the language ML:

        fun map f [] = []
            | map f (a :: y) = (f a) :: (map f y)
        fun reduce f init [] = init
            | reduce f init (a :: y) = reduce f (f init a) y
        fun count [] = 0
            | count (_ :: y) = 1 + count y
        fun addt [] = 0
            addt ((a,_,c) :: y) = (a+c) :: addt y

• Despite lack of explicit types here, this language is statically typed!

• Compiler will reject the calls \texttt{map 3 [1, 2]} and \texttt{reduce (op +) [] [3, 4, 5]}.

• Does this by \textit{deducing} types from their uses.
Type Inference

- In simple case:

  ```haskell
  fun add [] = 0
  | add (a :: L) = a + add L
  ```

  compiler deduces that \texttt{add} has type \( \text{int list} \rightarrow \text{int} \).

- Uses facts that (a) 0 is an \texttt{int}, (b) \( \texttt{[]} \) and \( \texttt{a::L} \) are lists (\( :: \) is cons), (c) \( + \) yields \texttt{int}.

- More interesting case:

  ```haskell
  fun count [] = 0
  | count (_ :: y) = 1 + count y
  ```

  (\_\_ means “don’t care” or “wildcard”). In this case, compiler deduces that \texttt{count} has type \( \alpha \text{ list} \rightarrow \text{int} \).

- Here, \( \alpha \) is a \textit{type parameter} (we say that \texttt{count} is \textit{polymorphic}).
Doing Type Inference

• Given a definition such as

\[
\text{fun add} \, \, [\,] = 0 \\
\text{\quad| add (a :: L) = a + add L}
\]

• First give each named entity here an unbound type parameter as its type: \( add : \alpha, \, a : \beta, \, L : \gamma \).

• Now use the type rules of the language to give types to everything and to relate the types:
  - \( 0 : \text{int}, \, [\,] : \delta \, \text{list} \).
  - Since \( \text{add} \) is function and applies to \( \text{int} \), must be that \( \alpha = \iota \rightarrow \kappa \), and \( \iota = \delta \, \text{list} \).
  - etc.

• Gives us a large set of type equations, which can be solved to give types.

• Solving involves pattern matching, known formally as type unification.
Type Expressions

• For this lecture, a type expression can be
  - A **primitive type** (int, bool);
  - A **type variable** (today we’ll use ML notation: ’a, ’b, ’c₁, etc.);
  - The **type constructor** \( T \text{ list} \), where \( T \) is a type expression;
  - A **function type** \( D \rightarrow C \), where \( D \) and \( C \) are type expressions.

• Will formulate our problems as systems of **type equations** between pairs of type expressions.

• Need to find the substitution
Solving Simple Type Equations

• Simple example: solve
  
  \`a list = int list

• Easy: \`a = int.

• How about this:
  
  \`a list = \`b list list; \`b list = int list

• Also easy: \`a = int list; \`b = int.

• On the other hand:
  
  \`a list = \`b \rightarrow \`b

  is unsolvable: lists are not functions.

• Also, if we require finite solutions, then
  
  \`a = \`b list; \`b = \`a list

  is unsolvable. However, our algorithm will allow infinite solutions.
Most General Solutions

• Rather trickier:

\[
'a \ list = 'b \ list \ list
\]

• Clearly, there are lots of solutions to this: e.g,

\[
'a = \text{int list}; \quad 'b = \text{int}
\]
\[
'a = (\text{int} \to \text{int}) \ list; \quad 'b = \text{int} \to \text{int}
\]

etc.

• But prefer a most general solution that will be compatible with any possible solution.

• Any substitution for \(a\) must be some kind of list, and \(b\) must be the type of element in \(a\), but otherwise, no constraints

• Leads to solution

\[
'a = 'b \ list
\]

where \(b\) remains a free type variable.

• In general, our solutions look like a bunch of equations \(a_i = T_i\), where the \(T_i\) are type expressions and none of the \(a_i\) appear in any of the \(T_i\)’s.
Finding Most-General Solution by Unification

• To *unify* two type expressions is to find substitutions for all type variables that make the expressions identical.

• The set of substitutions is called a *unifier*.

• Represent substitutions by giving each type variable, \( \tau \), a *binding* to some type expression.

• The algorithm that follows treats type expressions as objects (so two type expressions may have identical content and still be different objects). All type variables with the same name are represented by the same object.

• It generalizes binding by allowing all type expressions (not just type variables) to be bound to other type expressions.

• Initially, each type expression object is *unbound*.
Unification Algorithm

• For any type expression, define

\[
\text{binding}(T) = \begin{cases} 
\text{binding}(T'), & \text{if } T \text{ is bound to type expression } T' \\
T, & \text{otherwise}
\end{cases}
\]

• Now proceed recursively:

\[
\text{unify}(TA, TB): \quad \begin{align*}
TA &= \text{binding}(TA); \quad TB = \text{binding}(TB); \\
&\text{if } TA \text{ is } TB: \text{ return True; } \quad \# \text{ True if } TA \text{ and } TB \text{ are the same object} \\
&\text{if } TA \text{ is a type variable:} \\
&\quad \text{bind } TA \text{ to } TB; \text{ return True} \\
&\text{bind } TB \text{ to } TA; \quad \# \text{ Prevents infinite recursion} \\
&\text{if } TB \text{ is a type variable:} \\
&\quad \text{return True} \\
&\# \text{ Now check that binding } TB \text{ to } TA \text{ was really OK.} \\
&\text{if } TA \text{ is } C(TA_1, TA_2, \ldots, TA_n) \text{ and } TB \text{ is } C(TB_1, \ldots, TB_n): \\
&\quad \text{return } \text{unify}(TA_1, TB_1) \text{ and } \text{unify}(TA_2, TB_2 \text{ and } \ldots \\
&\quad \# \text{ where } C \text{ is some type constructor} \\
&\text{else: return False}
\end{align*}
\]
Example of Unification I

- Try to solve $A = B$, where
  
  $A = 'a \rightarrow \text{int}; B = 'b \text{ list} \rightarrow 'b$

  by computing $\text{unify}(A, B)$.

Dashed arrows are bindings
Red items are current TA and TB

\[
\begin{array}{c}
'a \\
\rightarrow \\
\text{int} \\
\rightarrow \\
\text{list} \\
\rightarrow \\
'b
\end{array}
\]
Example of Unification I

- Try to solve $A = B$, where
  
  $A = 'a \rightarrow \text{int}; B = 'b \text{ list} \rightarrow 'b$

  by computing $\text{unify}(A, B)$.

Dashed arrows are bindings

Red items are current TA and TB
Example of Unification I

- Try to solve $A = B$, where
  
  $A = \text{'a } \rightarrow \text{int}; B = \text{'b list } \rightarrow \text{'b}$

by computing $\text{unify}(A, B)$.

Dashed arrows are bindings
Red items are current TA and TB
Example of Unification I

• Try to solve \( A = B \), where

\[
A = 'a \rightarrow \text{int}; \quad B = 'b \text{ list} \rightarrow 'b
\]

by computing \( \text{unify}(A, B) \).

Dashed arrows are bindings

Red items are current TA and TB
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- Try to solve $A = B$, where

  $A = 'a \rightarrow \text{int}; B = 'b \text{ list} \rightarrow 'b$

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Dashed arrows are bindings
Red items are current TA and TB
Example of Unification I

• Try to solve $A = B$, where
  
  $A = \text{'a} \rightarrow \text{int}; \ B = \text{'b} \ \text{list} \rightarrow \text{'b}$

  by computing $\text{unify}(A, B)$.

Dashed arrows are bindings
Red items are current TA and TB

So $\text{'a} = \text{int list}$ and $\text{'b} = \text{int}$. 
Example of Unification II

- Try to solve $A = B$, where
  
  \[ A = \text{'a} \rightarrow \text{'c list}; B = \text{'b} \rightarrow \text{'a} \]
  
  by computing $\text{unify}(A, B)$.

```
  \[ \text{unify}(\text{'a} \rightarrow \text{'c list}, \text{'b} \rightarrow \text{'a}) \rightarrow \]
  
  \[ \text{'a} \rightarrow \text{'b} \rightarrow (to \text{'a}) \]
  \]
  \]
  \]
  \]
  \]
  \]
  \]
```
Example of Unification II

- Try to solve $A = B$, where
  
  $A = 'a \rightarrow 'c \text{ list}$; $B = 'b \rightarrow 'a$

  by computing $\text{unify}(A, B)$.
Example of Unification II

• Try to solve $A = B$, where

$A = \text{'a} \rightarrow \text{'c list}; B = \text{'b} \rightarrow \text{'a}$

by computing $\text{unify}(A, B)$. 

```
→
/   \
|    |
'\text{a}' | \text{list} | 'b' |

\text{(to 'a)}
\text{'c}
```
Example of Unification II

- Try to solve $A = B$, where
  
  $A = 'a \rightarrow 'c$ list; $B = 'b \rightarrow 'a$

by computing $\text{unify}(A, B)$.
Example of Unification II

- Try to solve $A = B$, where
  
  $A = \text{'a} \rightarrow \text{'c list}$; $B = \text{'b} \rightarrow \text{'a}$

by computing $\text{unify}(A, B)$. 

![Diagram showing the unification process]
Example of Unification II

- Try to solve \( A = B \), where
  \[
  A = \text{'a} \rightarrow \text{'c list}; \quad B = \text{'b} \rightarrow \text{'a}
  \]

by computing \( \text{unify}(A, B) \).
Example of Unification II

- Try to solve $A = B$, where

  $A = \text{'a} \to \text{'c list}$; $B = \text{'b} \to \text{'a}$

by computing $\text{unify}(A, B)$.
Example of Unification II

- Try to solve \( A = B \), where
  \[
  A = 'a \rightarrow 'c \text{ list}; \quad B = 'b \rightarrow 'a
  \]

by computing \( \text{unify}(A, B) \).

So \( 'a = 'b = 'c \text{ list} \) and \( 'c \text{ is free} \).
Example of Unification III: Simple Recursive Type

• Introduce a new type constructor: (‘h, ’t) pair, which is intended to model typed Lisp cons-cells (or nil). The car of such a pair has type ’h, and the cdr has type ’t.

• Try to solve \( A = B \), where

\[
A = \text{’a}; \quad B = (\text{’b}, \text{’a})\text{ pair}
\]

by computing \( \text{unify}(A, B) \).

• This one is very easy:
Example of Unification III: Simple Recursive Type

- Introduce a new type constructor: (’h, ’t) pair, which is intended to model typed Lisp cons-cells (or nil). The car of such a pair has type ’h, and the cdr has type ’t.

- Try to solve \( A = B \), where
  
  \[
  A = ’a; B = (’b, ’a) \text{ pair}
  \]

  by computing \text{unify}(A, B).

- This one is very easy:
Example of Unification III: Simple Recursive Type

• Introduce a new type constructor: (’h, ’t) pair, which is intended to model typed Lisp cons-cells (or nil). The car of such a pair has type ’h, and the cdr has type ’t.

• Try to solve $A = B$, where
  
  $A = ’a; B = (’b, ’a)$ pair

  by computing $\text{unify}(A, B)$.

• This one is very easy:

  So $’a = (’b, ’a)$ pair; ’b is free.
Example of Unification IV: Another Recursive Type

• This time, consider solving $A = B$, $C = D$, $A = C$, where

$A = 'a$; $B = ('b, 'a)$ pair; $C = 'c$; $D = ('d, ('d, 'c)$ pair)$) pair$.

We just did the first one, and the second is almost the same, so we’ll just skip those steps.
Example of Unification IV: Another Recursive Type

- This time, consider solving $A = B$, $C = D$, $A = C$, where
  $$A = 'a;\quad B = ('b, 'a)\text{ pair};\quad C = 'c;\quad D = ('d, ('d, 'c)\text{ pair})\text{ pair}.$$  

We just did the first one, and the second is almost the same, so we'll just skip those steps.
Example of Unification IV: Another Recursive Type

- This time, consider solving $A = B$, $C = D$, $A = C$, where

  $A = 'a$; $B = ('b, 'a)$ pair; $C = 'c$; $D = ('d, ('d, 'c)$ pair) pair.

We just did the first one, and the second is almost the same, so we'll just skip those steps.
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  $A = 'a$; $B = ('b, 'a)$ pair; $C = 'c$; $D = ('d, ('d, 'c)$ pair) pair.

  We just did the first one, and the second is almost the same, so we'll just skip those steps.
Example of Unification IV: Another Recursive Type

- This time, consider solving \( A = B, \ C = D, \ A = C, \) where

\[
A = 'a; \ B = ('b, 'a) \text{ pair}; \ C = 'c; \ D = ('d, ('d, 'c) \text{ pair}) \text{ pair}.
\]

We just did the first one, and the second is almost the same, so we'll just skip those steps.
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- This time, consider solving $A = B$, $C = D$, $A = C$, where
  
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  We just did the first one, and the second is almost the same, so we'll just skip those steps.
Example of Unification IV: Another Recursive Type

- This time, consider solving $A = B$, $C = D$, $A = C$, where
  
  $A = \text{'a}; B = (\text{'b, 'a})\text{ pair}; C = \text{'c}; D = (\text{'d, ('d, 'c})\text{ pair})\text{ pair}.$

  We just did the first one, and the second is almost the same, so we'll just skip those steps.
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- This time, consider solving $A = B$, $C = D$, $A = C$, where
  
  $A = 'a$; $B = ('b, 'a)$ pair; $C = 'c$; $D = ('d, ('d, 'c)$ pair)$ pair$.

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- This time, consider solving $A = B$, $C = D$, $A = C$, where
  
  $A = 'a$; $B = (b, a)$ pair; $C = c$; $D = (d, (d, c)$ pair$)$ pair.

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  $A = 'a$; $B = (b, 'a) \text{pair}$; $C = 'c$; $D = (d, (d, 'c) \text{pair}) \text{pair}$.

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Example of Unification IV: Another Recursive Type

• This time, consider solving $A = B$, $C = D$, $A = C$, where
  
  $A = 'a$; $B = ('b, 'a)$ pair; $C = 'c$; $D = ('d, ('d, 'c)$ pair) pair.

  We just did the first one, and the second is almost the same, so we'll just skip those steps.

  So $'a = 'c = ('d, 'a)$ pair; $'b$ = $'d$; $'d$ is free.
Example of Unification V

• Try to solve

  'b list = 'a list; 'a → 'b = 'c;
  'c → bool = (bool → bool) → bool

• We unify both sides of each equation (in any order), keeping the bindings from one unification to the next.

  'a:

  'b:

  'c:
Example of Unification V

• Try to solve

\[ \text{'b list} = \text{'a list}; \text{'a} \rightarrow \text{'b} = \text{'c}; \]
\[ \text{'c} \rightarrow \text{bool} = (\text{bool} \rightarrow \text{bool}) \rightarrow \text{bool} \]

• We unify both sides of each equation (in any order), keeping the bindings from one unification to the next.

\[ \text{'a}: \quad \text{Unify 'b list, 'a list:} \]

\[ \text{'b}: \]

\[ \text{'c}: \]
Example of Unification V

- Try to solve
  
  \[ \text{'b list} = \text{'a list}; \text{'a} \rightarrow \text{'b} = \text{'c}; \]
  
  \[ \text{'c} \rightarrow \text{bool} = (\text{bool} \rightarrow \text{bool}) \rightarrow \text{bool} \]

- We unify both sides of each equation (in any order), keeping the bindings from one unification to the next.

  \[ \text{'a}: \]
  
  Unify \text{'}b list, \text{'a list}:
  
  Unify \text{'b}, \text{'a}
  
  \[ \text{'b}: \text{'a} \]
  
  \[ \text{'c}: \]
Example of Unification V

• Try to solve
  
  \[
  \begin{align*}
  \text{'b list} &= \text{'a list}; \\
  \text{'a} &\rightarrow \text{'b} = \text{'c}; \\
  \text{'c} &\rightarrow \text{bool} = (\text{bool} \rightarrow \text{bool}) \rightarrow \text{bool}
  \end{align*}
  \]

• We unify both sides of each equation (in any order), keeping the bindings from one unification to the next.

\[
\begin{align*}
\text{'a}: \\
\text{'b}: \text{'a} \\
\text{'c}: \text{'a} \rightarrow \text{'b}
\end{align*}
\]

Unify \text{'b list}, \text{'a list}:

Unify \text{'b}, \text{'a}

Unify \text{'a} \rightarrow \text{'b}, \text{'c}
Example of Unification V

• Try to solve

\[ \text{'b list} = \text{'a list; 'a} \rightarrow \text{'b} = \text{'c;} \]
\[ \text{'c} \rightarrow \text{bool} = (\text{bool} \rightarrow \text{bool}) \rightarrow \text{bool} \]

• We unify both sides of each equation (in any order), keeping the bindings from one unification to the next.

\[ \text{'a:} \]
\[ \text{'b: 'a} \]
\[ \text{'c: 'a} \rightarrow \text{'b} \]

Unify \text{'b list, 'a list:}

Unify \text{'b, 'a}

Unify \text{'a} \rightarrow \text{'b, 'c}

Unify \text{'c} \rightarrow \text{bool, (bool} \rightarrow \text{bool)} \rightarrow \text{bool}
Example of Unification V

• Try to solve

  ’b list = ’a list; ’a→ ’b = ’c;
  ’c → bool = (bool → bool) → bool

• We unify both sides of each equation (in any order), keeping the
  bindings from one unification to the next.

  ’a:
  Unify ’b list, ’a list:
    Unify ’b, ’a
  ’b: ’a
  Unify ’a→ ’b, ’c
  Unify ’c → bool, (bool → bool) → bool
    Unify ’c, bool → bool:
  ’c: ’a → ’b
Example of Unification V

• Try to solve

  \[ \text{'b list} = \text{'a list}; \text{'a} \to \text{'b} = \text{'c}; \]
  \[ \text{'c} \to \text{bool} = (\text{bool} \to \text{bool}) \to \text{bool} \]

• We unify both sides of each equation (in any order), keeping the bindings from one unification to the next.

  \[
  \text{'a:} \\
  \text{'b:} \text{'a} \\
  \text{'c:} \text{'a} \to \text{'b}
  \]
  Unify \text{'b list}, \text{'a list:}
  Unify \text{'b}, \text{'a}
  Unify \text{'a} \to \text{'b}, \text{'c}
  Unify \text{'c} \to \text{bool}, (\text{bool} \to \text{bool}) \to \text{bool}
  Unify \text{'c}, \text{bool} \to \text{bool:}
  Unify \text{'a} \to \text{'b}, \text{bool} \to \text{bool:}

Example of Unification V

• Try to solve

\[ \text{'b list= 'a list; 'a\rightarrow 'b = 'c; } \]
\[ \text{'c \rightarrow bool= (bool\rightarrow bool) \rightarrow bool} \]

• We unify both sides of each equation (in any order), keeping the bindings from one unification to the next.

\[ \text{'a: bool} \]
\[ \text{Unify 'b list, 'a list:} \]
\[ \text{Unify 'b, 'a} \]
\[ \text{'b: 'a} \]
\[ \text{Unify 'a\rightarrow 'b, 'c} \]
\[ \text{Unify 'c \rightarrow bool, (bool\rightarrow bool) \rightarrow bool} \]
\[ \text{Unify 'c, bool \rightarrow bool:} \]
\[ \text{'c: 'a \rightarrow 'b} \]
\[ \text{Unify 'a \rightarrow 'b, bool \rightarrow bool:} \]
\[ \text{Unify 'a, bool} \]
Example of Unification V

- Try to solve

  \[ \text{'}b\text{ list} = \text{'}a\text{ list}; \text{'}a \rightarrow \text{'}b = \text{'}c; \]
  \[ \text{'}c \rightarrow \text{bool} = (\text{bool} \rightarrow \text{bool}) \rightarrow \text{bool} \]

- We unify both sides of each equation (in any order), keeping the bindings from one unification to the next.

\[
\begin{align*}
\text{'a: } & \text{ bool} & \text{ Unify 'b list, 'a list:} \\
\text{'b: } & \text{ 'a} & \text{ Unify 'a } \rightarrow \text{ 'b, 'c} \\
\text{'c: } & \text{ 'a } \rightarrow \text{ 'b} & \text{ Unify 'c } \rightarrow \text{ bool, (bool } \rightarrow \text{ bool) } \rightarrow \text{ bool:}
\end{align*}
\]
Example of Unification V

• Try to solve

'b list = 'a list; 'a → 'b = 'c;
'c → bool = (bool → bool) → bool

• We unify both sides of each equation (in any order), keeping the bindings from one unification to the next.

'a: bool  Unify 'b list, 'a list:
         Unify 'b, 'a
'b: 'a  Unify 'a → 'b, 'c
Unify 'c → bool, (bool → bool) → bool
         Unify 'c, bool → bool:
'c: 'a → 'b  Unify 'a → 'b, bool → bool:
          Unify 'a, bool
          Unify 'b, bool:
            Unify bool, bool
Example of Unification V

• Try to solve
  
  'b list = 'a list; 'a → 'b = 'c;
  'c → bool = (bool → bool) → bool

• We unify both sides of each equation (in any order), keeping the bindings from one unification to the next.

  'a: bool  
  
  Unify 'b list, 'a list:
  
  'b: 'a  
  
  bool  
  
  Unify 'a → 'b, 'c

  'c: 'a → 'b  
  
  bool → bool  
  
  Unify 'a → 'b, bool → bool:

  Unify 'a, bool
  
  Unify 'b, bool:
  
  Unify bool, bool
  
  Unify bool, bool
<table>
<thead>
<tr>
<th>Construct</th>
<th>Type</th>
<th>Conditions</th>
</tr>
</thead>
<tbody>
<tr>
<td>Integer literal</td>
<td>int</td>
<td></td>
</tr>
<tr>
<td>[]</td>
<td>'a list</td>
<td></td>
</tr>
<tr>
<td>hd (L)</td>
<td>'a</td>
<td>L: 'a list</td>
</tr>
<tr>
<td>tl (L)</td>
<td>'a list</td>
<td>L: 'a list</td>
</tr>
<tr>
<td>E₁+E₂</td>
<td>int</td>
<td>E₁: int, E₂: int</td>
</tr>
<tr>
<td>E₁::E₂</td>
<td>'a list</td>
<td>E₁: 'a, E₂: 'a list</td>
</tr>
<tr>
<td>E₁ = E₂</td>
<td>bool</td>
<td>E₁: 'a, E₂: 'a</td>
</tr>
<tr>
<td>E₁!=E₂</td>
<td>bool</td>
<td>E₁: 'a, E₂: 'a</td>
</tr>
<tr>
<td>if E₁ then E₂ else E₃ fi</td>
<td>'a</td>
<td>E₁: bool, E₂: 'a, E₃: 'a</td>
</tr>
<tr>
<td>E₁ E₂</td>
<td>'b</td>
<td>E₁: 'a → 'b, E₂: 'a</td>
</tr>
<tr>
<td>def f x₁ ...xₙ = E</td>
<td>x₁: 'a₁, ..., xₙ: 'aₙ E:'a₀, f: 'a₁ → ... → 'aₙ → 'a₀.</td>
<td></td>
</tr>
</tbody>
</table>
Using the Type Rules

• Interpret the notation $E : T$, where $E$ is an expression and $T$ is a type, as
  
  $\text{type}(E) = T$

• Seed the process by introducing a set of fresh type variables to describe the types of all the variables used in the program you are attempting to process. For example, given

  \[\text{def } f \ x = x\]

  we might start by saying that

  \[\text{type}(f) = 'a0, \text{type}(x) = 'a1\]

• Apply the type rules to your program to get a bunch of Conditions.

• Whenever two Conditions ascribe a type to the same expression, equate those types.

• Solve the resulting equations.
Aside: Currying

- Writing

  \[
  \text{def } sqr \ x = x \times x;
  \]

  means essentially that \( sqr \) is defined to have the value \( \lambda \ x. \ x \times x \).

- To get more than one argument, write

  \[
  \text{def } f \ x \ y = x + y;
  \]

  and \( f \) will have the value \( \lambda \ x. \ \lambda \ y. \ x + y \).

- Its type will be \text{int} \rightarrow \text{int} \rightarrow \text{int} \ (\text{Note: } \rightarrow \text{ is right associative}).

- So, \( f \ 2 \ 3 = (f \ 2) \ 3 = (\lambda \ y. \ 2 + y) \ (3) = 5 \)

- Zounds! It’s the CS61A substitution model!

- This trick of turning multi-argument functions into one-argument functions is called \textit{currying} (after Haskell Curry).
Example

```plaintext
if p L then init else f init (hd L) fi + 3
```

- Let's initially use 'p, 'L, etc. as the fresh type variables giving the types of identifiers.
- Using the rules then generates equations like this:

  ```plaintext
  'p = 'a0→ 'a1, 'L = 'a0, type(p L) = 'a1  # call rule
  'L = 'a2 list, type(hd L) = 'a2        # hd rule
  'f = 'a3→ 'a4, 'init = 'a3, type(f init) = 'a4
      # call rule
  'a4 = 'a5→ 'a6, 'a2 = 'a5, type(f init (hd L)) = 'a6
      # call rule
  'a1 = bool, 'init = 'a7, 'a6 = 'a7, type(if... fi) = 'a7
      # if rule
  'a7 = int, int = int, type(if... fi+3) = int  # + rule
  ```
Example, contd.

Solve all these equations by sequentially unifying the two sides of each equation, in any order, keeping the bindings as you go.

\[
\begin{align*}
'p &= 'a0 \rightarrow 'a1, 'L = 'a0 \\
'L &= 'a2 \text{ list} \\
'a0 &= 'a2 \text{ list} \\
'f &= 'a3 \rightarrow 'a4, 'init = 'a3 \\
'a4 &= 'a5 \rightarrow 'a6, 'a2 = 'a5 \\
'a1 &= \text{bool, } 'init = 'a7, 'a6 = 'a7 \\
'a3 &= 'a7 \\
'a7 &= \text{int, int} = \text{int}
\end{align*}
\]

So,

\[
\begin{align*}
'p &= 'a5 \text{ list} \rightarrow \text{bool, } 'L = 'a5 \text{ list, } 'init = \text{int, } \\
'f &= \text{int } \rightarrow 'a5 \rightarrow \text{int}
\end{align*}
\]