Lecture 18 - Reductions

Nov 5, 2020

Bipartite Matching Problem (BM)

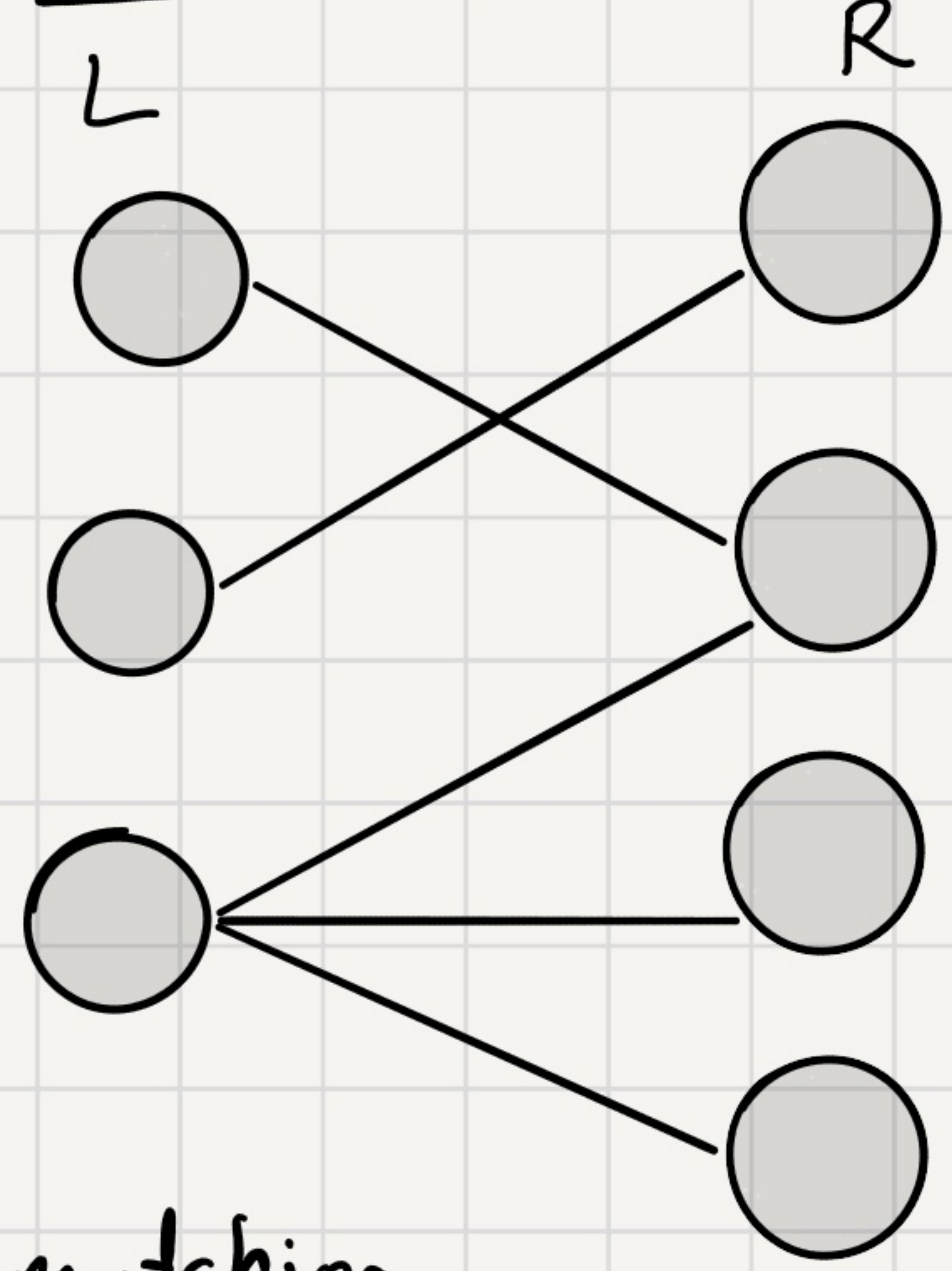
Input: Bipartite Graph $G = (L, R, E)$

$E \subseteq L \times R$

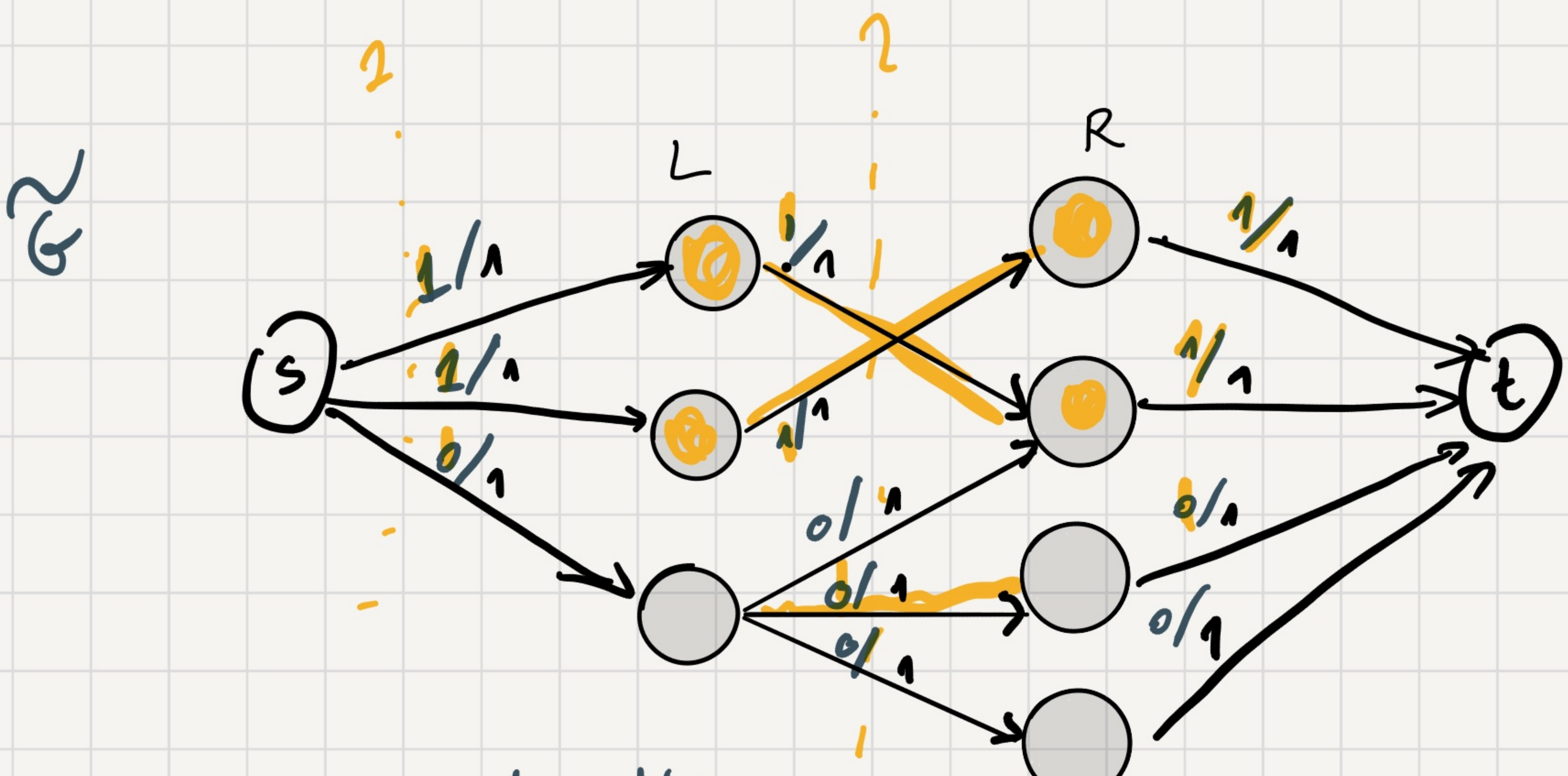
Def'n: A **matching** is a set of edges $M \subseteq E$ s.t. no pair of edges in M touches the same vertex.

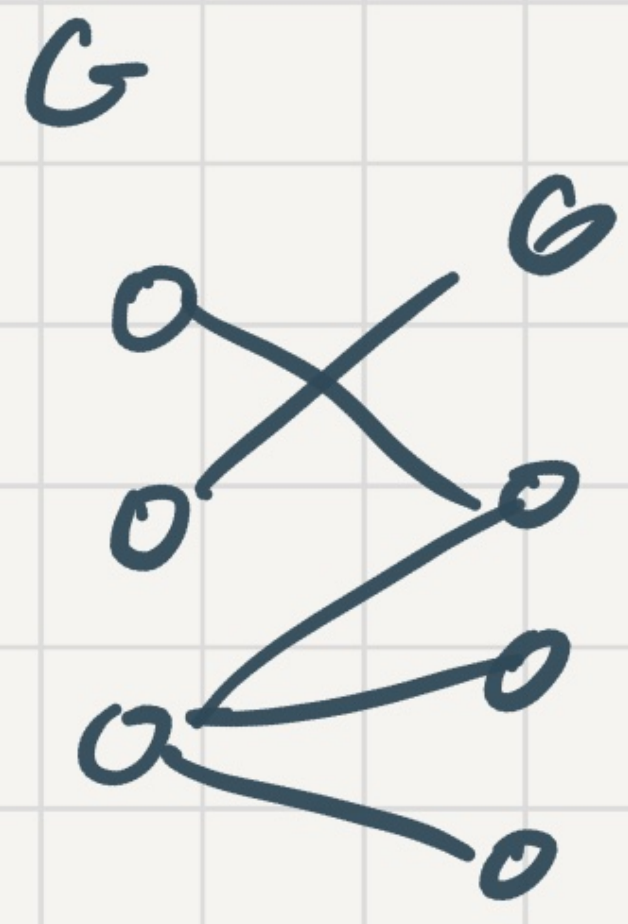
Goal: find maximum matching, i.e., $\max |M|$ s.t. $M \subseteq E$ is a matching.

Example:



We show how to solve BM using an algorithm for MaxFlow.





what if a flow gets split?

Fact: If all capacities are integers then there a integer max flow [Ford-Falkerson]

$|M|$ matching M on a bipartite graph G

$\shortparallel$ $\updownarrow$

$Val(f)$ integral flow f on network $\tilde{G}$

Claim: Suppose M is a matching in G.
Then $\exists$ integral flow on $\tilde{G}$ with
$$\mathrm{val}(f) = |M|.$$

Proof:

Push 1 unit of flow on edges in M.

$L(M)$ = vertices in L touching M
$R(M)$ = " " R " M.

Push 1 unit of flow from s to v, $\forall v \in L(M)$.
Push 1 unit of flow from u to t, $\forall u \in R(M)$.

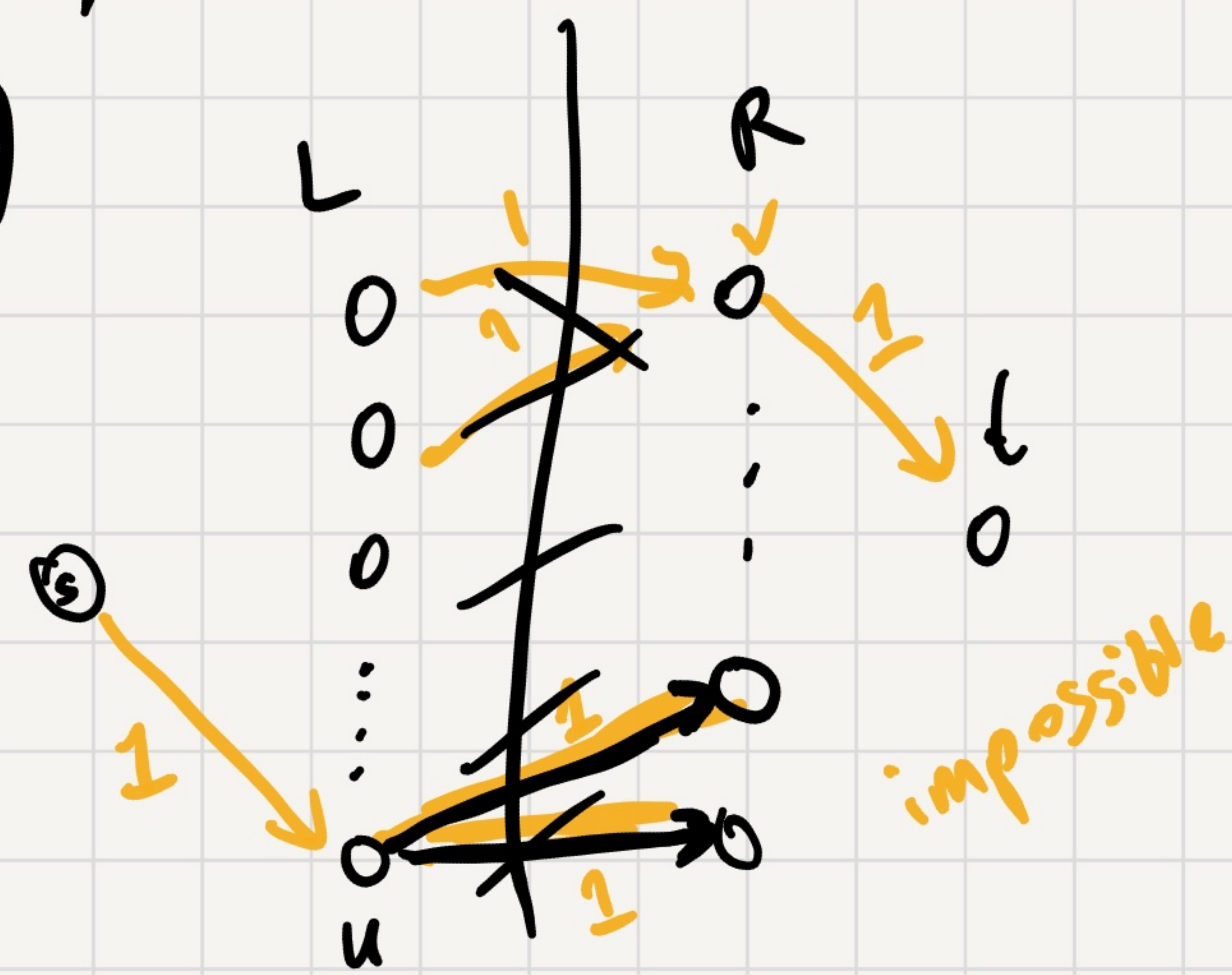
$\mathrm{val}(f) = |L(M)| = |M|$. $\square$

Claim: f is an integral flow on $\tilde{G}$
Then, $\exists$ a matching M on G s.t. $|M| = val(f)$.

Proof: Since capacities in $\tilde{G}$ are all 1, the flow on each edge could be either 0/1.

$$M = \{ (u,v) : u \in L, v \in R, f_{u,v} = 1 \}.$$

$|M| = val(f)$

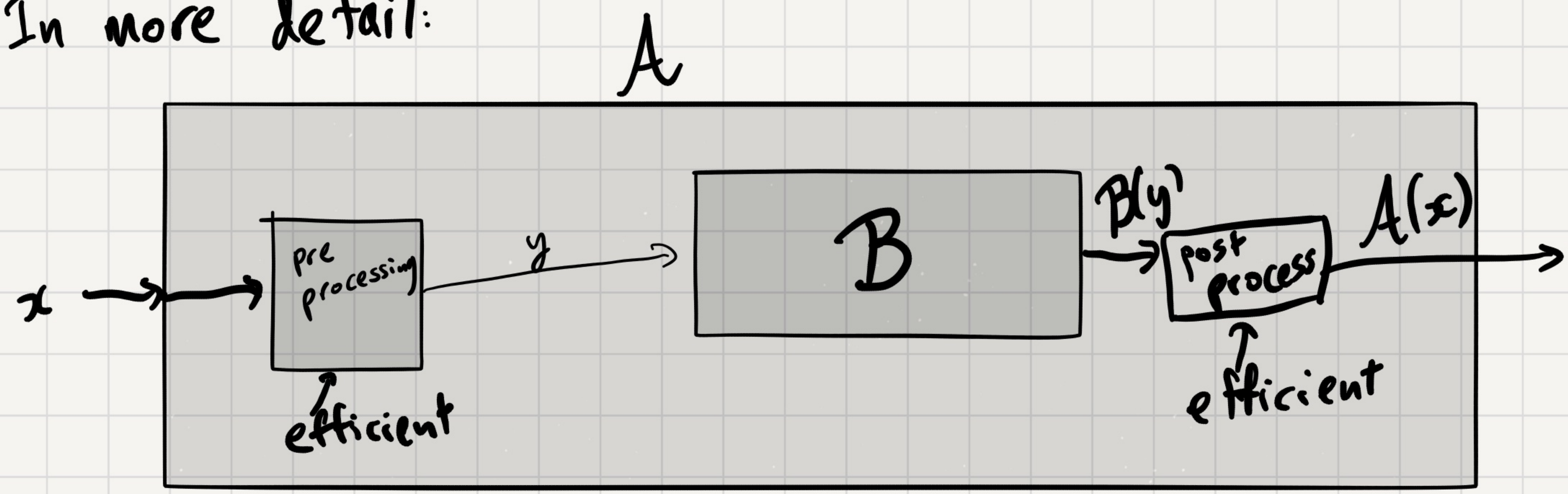


M is a matching because f is a flow & every vertex in L can get at most 1 unit of flow in & every v in R can push at most 1 flow unit tot.

The notion of a reduction

"a problem A reduces to a problem B if any subroutine to solve B can be used to solve A"

In more detail:

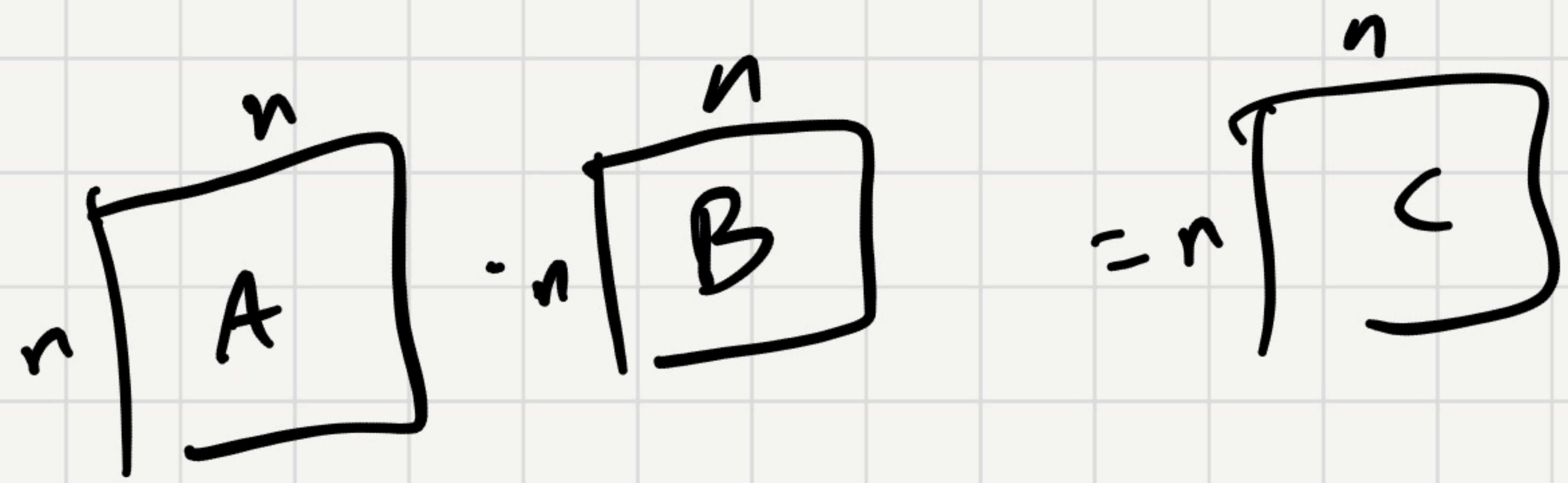


(*) an efficient alg for $B \Rightarrow$ an efficient alg for A.

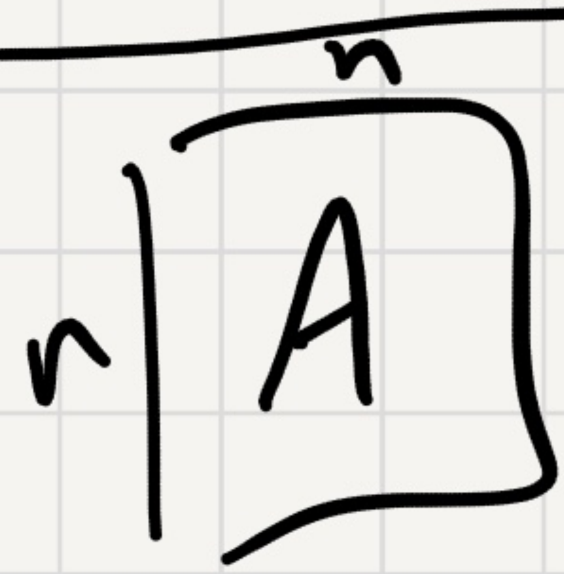
A reduction = pre-processing + post-processing.

(*) $\neg\exists$ an efficient alg for $B \Leftarrow \neg\exists$ an efficient alg for A

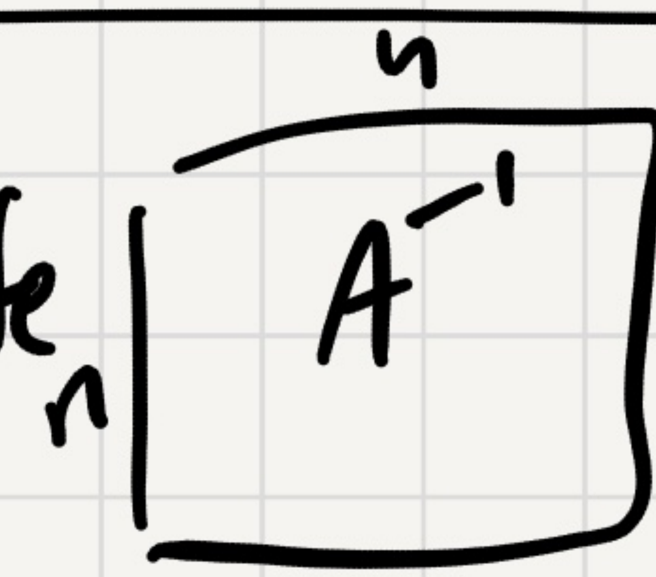
Matrix Multiplication Strassen $7^{\log_2 n} = n^{\log_2 7} \approx n^{2.8}$



$n^{2.37}$



want to compute



$A \cdot A^{-1} = I$.

Matrix Mult $\rightarrow$ Matrix Inverse

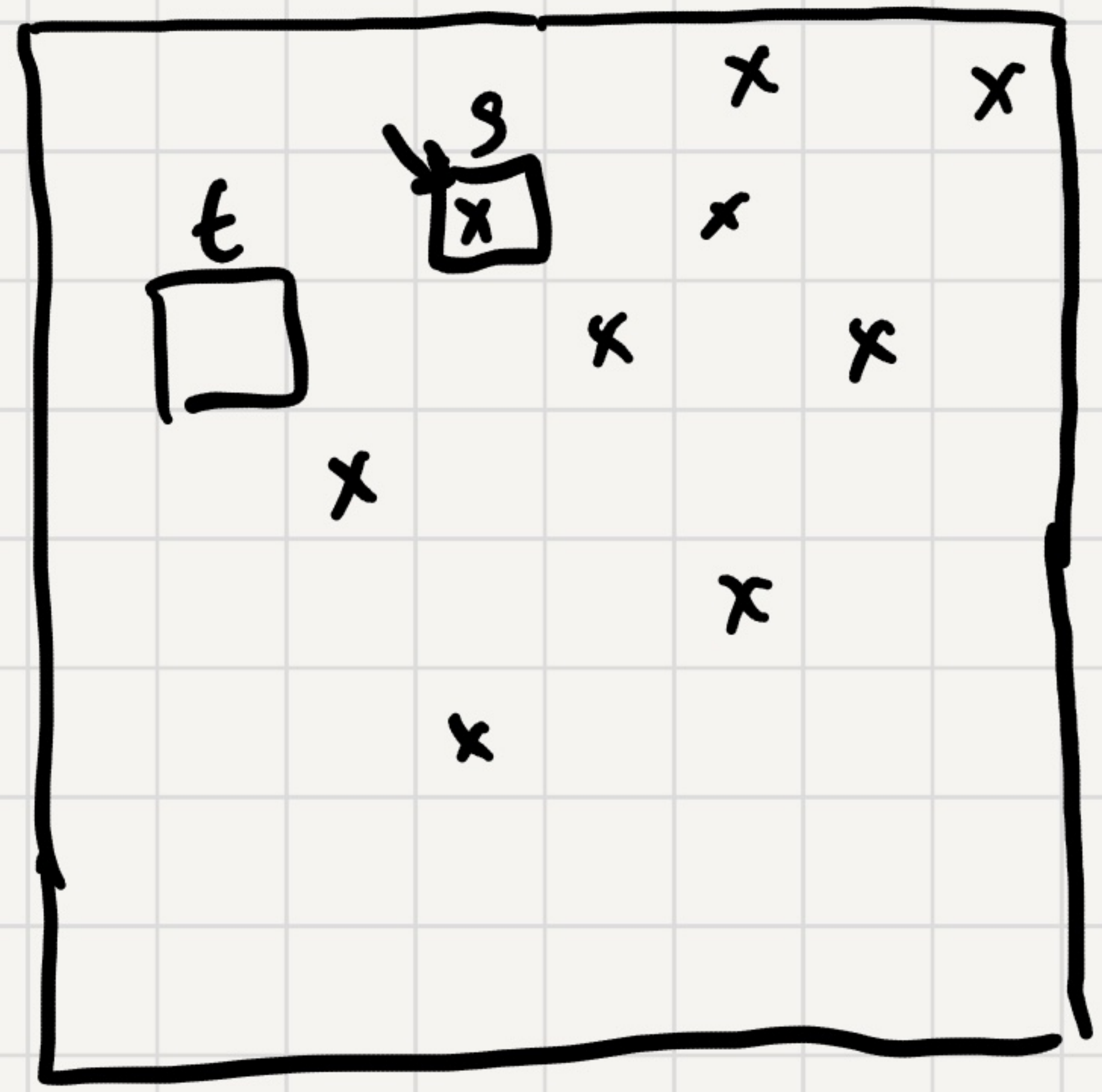
$\underset{n\times n}{A}, \underset{n\times n}{B}$

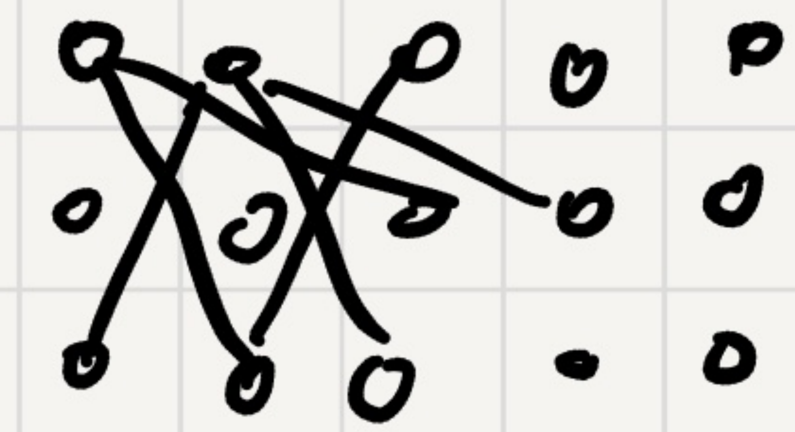
$$\begin{array}{cc} & 3n \\ 3n & \begin{pmatrix} I_n & A & 0 \\ 0 & I_n & B \\ 0 & 0 & I_n \end{pmatrix}^{-1} \end{array} = \begin{pmatrix} I & -A & AB \\ 0 & I & -B \\ 0 & 0 & I \end{pmatrix}$$

Rudrata Cycle

$G=(V,E)$ undirected

find a cycle that visits all vertices exactly once.



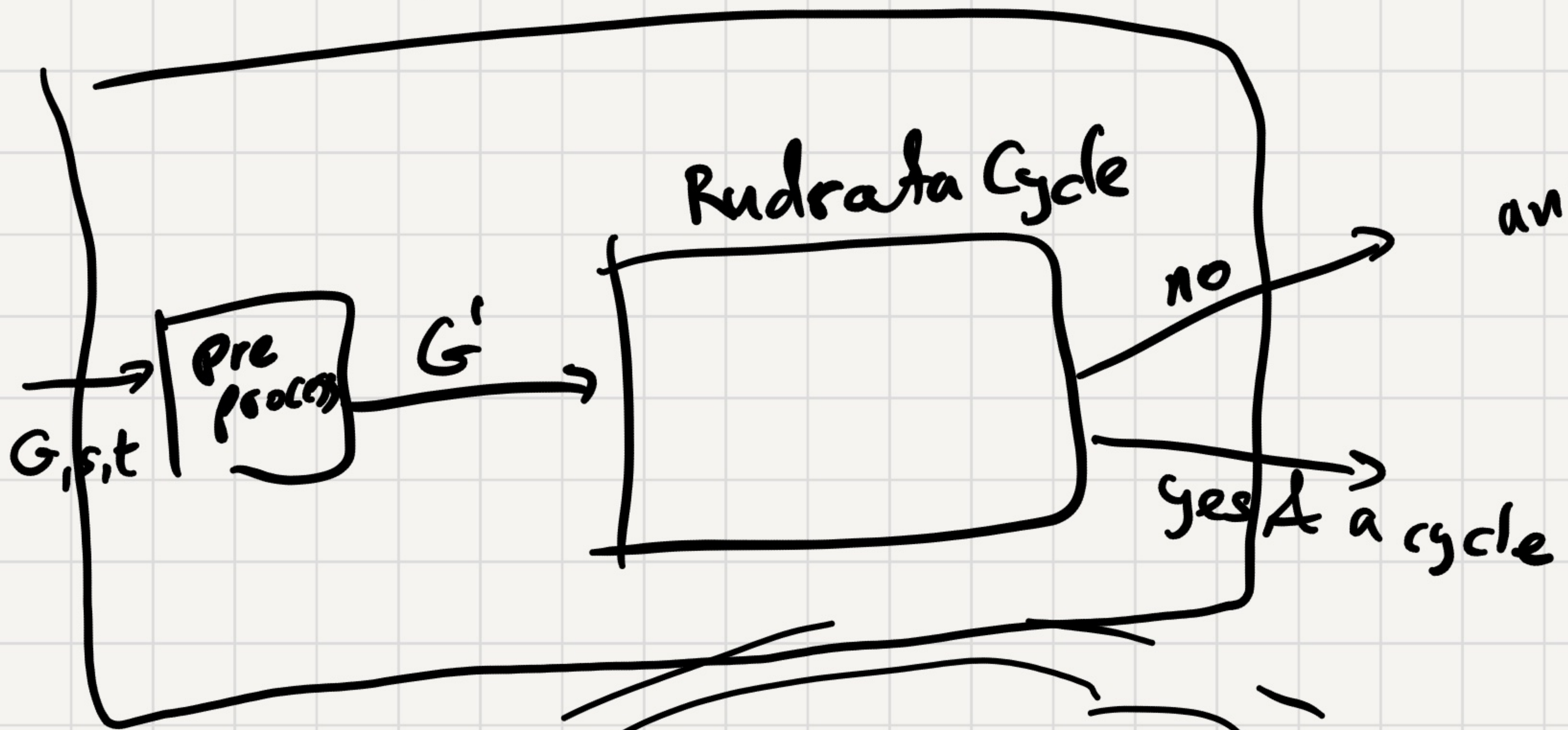


Rudrata path

$G=(V,E)$ undirected source s, target t

Goal: find a path from s to t that visit all vertices exactly once.

G , s , t   Rudrata Path



answer no.

answer yes
remove the edges
(x,s) & (t,x)
from the cycle
⇒ path.

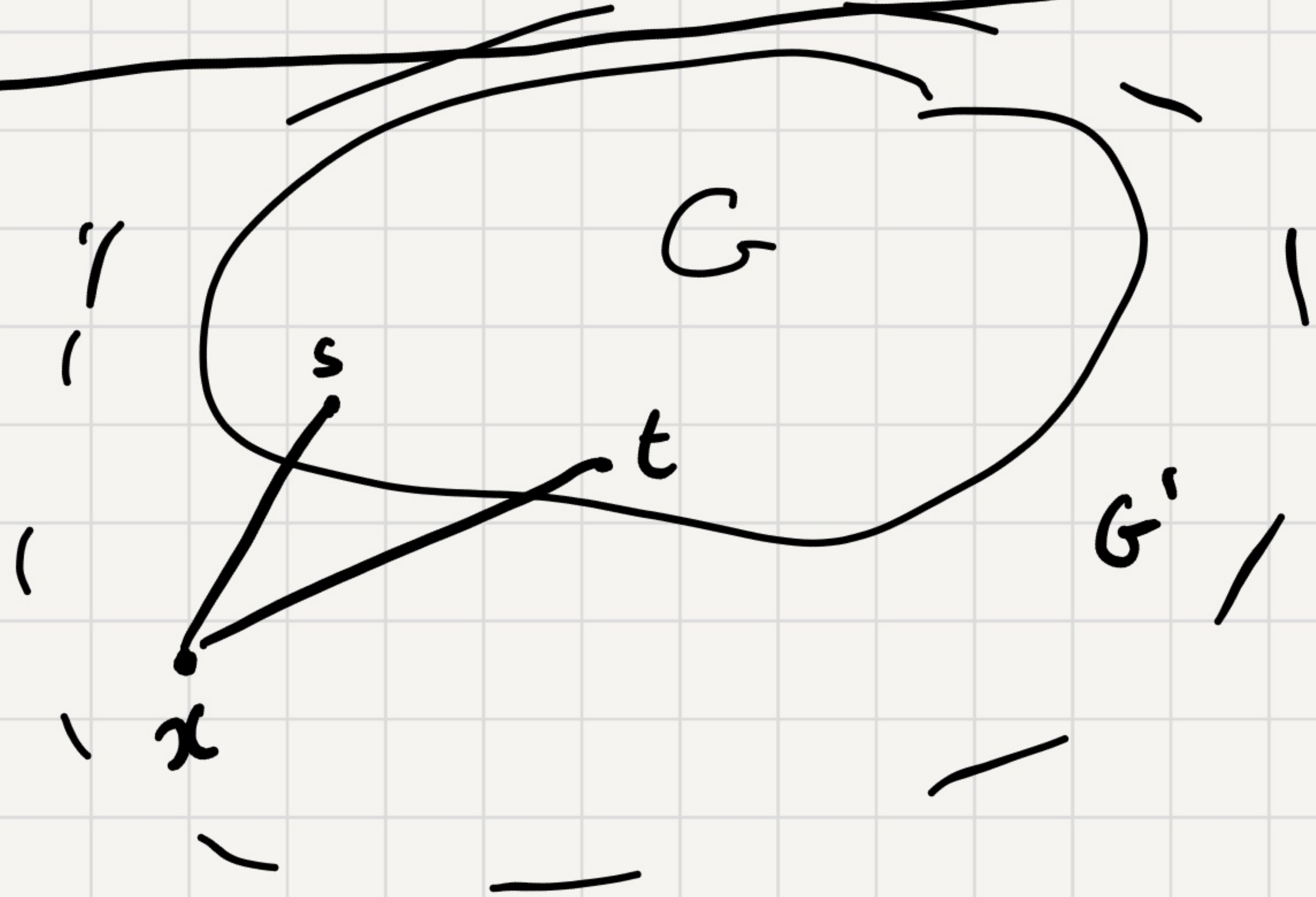
If G has a
Rudrata path
from s to t

⇓

G' has a Rudrata
cycle.

⇓

G has a Rudrata
path from s to t.



Rudrata Path

↓

Rudrata Cycle.

x — s — ... — t — x

s — .... — t