Representing Geometry

* See Watt & Watt chapter 3

ie CSG
Polygons
Parametric ← Sub set of
Implicit Surfaces
Subdivision Surfaces

Each has its own strengths & weaknesses

- ease of use for design
- ease/speed for rendering
- simplicity
- smoothness
- colision detection
- flexibility
- suitablity for FEM
- etc...

* No one of these is best at everything.

Parametric Representations

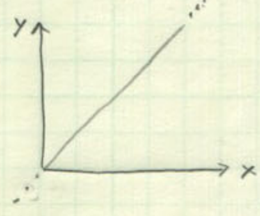
Curve: $x = x(u)$ $\quad x \in R^n, u \in R^1$

Surface: $x = x(u, v)$ $\quad$ w/ u & $v \in R^1$
or $x = x(u)$ $\quad$ w/ $u \in R^2$

Volume: $x(u, v, w)$ or $u \in R^3$
↳ and so on...

Parametric rep is not unique

ie



$x = [u, u]$
or $x = [2u, 2u]$
or $x = [u^3, u^3]$

* Dif. Geo. formulae assume normalize param or they include normaization

ie surface normal: $\hat{n} = \dfrac{\partial_u x \times \partial_v x}{\|\partial_u x\| \, \|\partial_v x\|}$

if x is any possible curve/surface

→ hard to reprent

? How many parameters?
↳ uncountable

Being Practical:

Step 1 - Pick reasonable/usefull subspace

Step 2 - Pick reasonable/usefull basis Fns.

$$x(u) = \sum_{i=0}^{\infty} c_i \, \phi_i(u)$$

Still inf # of parameters but countable

Step 3 - Truncate sum after finite # of terms

Note: Could also pick something that is non-linear in the C_i, but that makes life hard so we won't do that. (Think about NURBS later on...)

Examples

Fourier Series → the ϕ_i are cos/sin
Polynomials → the Φ_i are u^i

But $x \in \cancel{\mathbb{R}^n}\ \mathbb{R}^3$ (b.c. we care about $\mathbb{R}^3$ and its hard to stay generic.)

so let
$C_i \in \cancel{\mathbb{R}^n}\ \mathbb{R}^3$

A Closer look at Polynomials :

$$x(u) = \sum_{i=0}^{d} c_i u^i = c \cdot p^d$$

where $c = [c_0, c_1, c_2, \ldots, c_d]$

$p^d = [1, u, u^2, u^3, \ldots, u^d]$

$\boxed{\text{ie} \quad \phi_i(u) = u^i}$

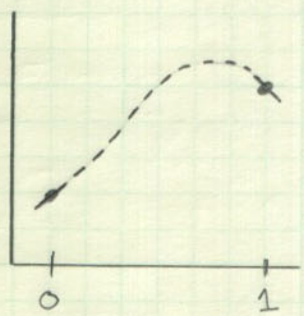
"Power Basis"

- Elements of p^d are linearly independent
  - $\hookrightarrow$ ie no good approx of u^k w/ $\sum_{j \neq k} u^j$

② Why use $u^0, \ldots, u^d$?
Why not
$u^j \quad j = 0, 2, 4, 8 \ldots 2^d$
or $u^j \quad j =$ First d primes

Task: Pick c_i to generate some usefull curve

Imagine we know $X(0)$, $X(1)$, $X'(0)$ & $X'(1)$
& $d=3$ (cubic polys)



$\boxed{\text{② is this curve in } R^1 \text{ or } R^2}$

Note: $X(0) = c_0 \qquad X(1) = \sum c_i$
$X'(0) = c_1 \qquad X'(1) = \sum i c_i$

$$\begin{bmatrix} X(0) \\ X(1) \\ X'(0) \\ X'(1) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 3 \end{bmatrix} \cdot \begin{bmatrix} C_0 \\ C_1 \\ C_2 \\ C_3 \end{bmatrix}$$

call p call B_H

$$p = B_H c \implies c = B_H^{-1} p$$

$$c = \beta_H p$$

$$\beta_H = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -3 & 3 & -2 & -1 \\ 2 & -2 & 1 & 1 \end{bmatrix}$$

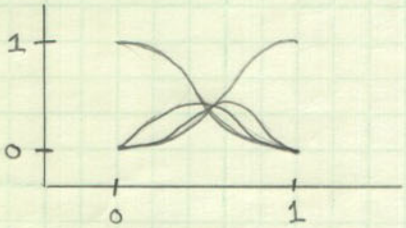
So

$$X(u) = P^3 \cdot c = \underbrace{P^3 \beta_H}\, p$$

$$= \begin{bmatrix} 1 + 0u - 3u^2 + 2u^3 \\ 0 + 0u + 3u^2 - 2u^3 \\ 0 + 1u - 2u^2 + 1u^3 \\ 0 + 0u - 1u^2 + 1u^3 \end{bmatrix} \cdot p$$

$$= \sum_{i=0}^{3} p_i\, b_i(u)$$

↳ look Familiar?



Ok, I drew these poorly but you can plot them yourself to see what they look like...

PS These $b_i(u)$ are known as the Hermite Basis

Cubic Bézier

Note: Bézier are related to Bernstein polys, but well talk about that later

Constraints: $x(0) = p_0 \qquad x'(0) = 3(p_1 - p_0)$
$x(1) = p_3 \qquad x'(1) = 3(p_3 - p_2)$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -3 & 3 & 0 & 0 \\ 0 & 0 & -3 & 3 \end{bmatrix} p = \begin{matrix} x(0) \\ x(1) \\ x'(0) \\ x'(1) \end{matrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

$$\Rightarrow \quad c = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 3 & 0 & 0 \\ 3 & -6 & 3 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix} p$$

β_Z

<u>Note</u> $\quad c = \beta_H p_H \qquad c = \beta_Z p_Z$

$$\Rightarrow \beta_H p_H = \beta_Z p_Z \Rightarrow p_Z = \beta_Z^{-1} \beta_H p_H$$

* Bézier^3, Power Basis^3, & Hermite all span the same space

- Think diff axes in $\mathbb{R}^3$ ...

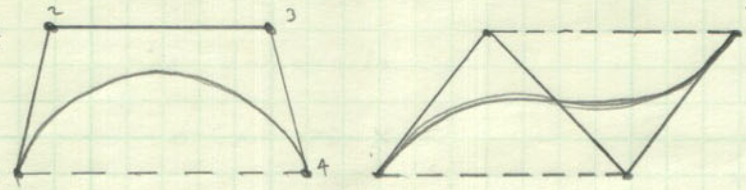
Usefull Properties of a Basis

* Convex Hull

$\sum_i b_i(u) = 1$ & $b_i(u) \geq 0$, $u \in \Omega$

Bézier Has this property :

eg



Hermite & Power donot.

? Why is this a nice property?

* Invariance under some class of transform

$$x(u) = \sum p_i b_i(u) \iff XF(x(u)) = \sum XF(p_i)\, b_i(u)$$

Bézier invariant under afine XF, but not prj.

Hermite not inv. under either afine or prj.

NURBs are inv under Prj

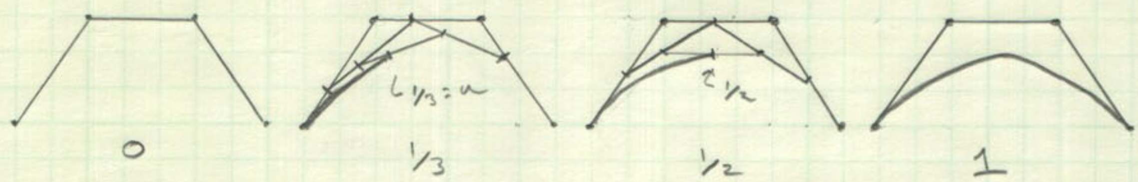
? Why nice property?

* Others

- Local support
- Nice subdivision rules
  ] we'll see in Few minutes
- Orthogonality ie Fourier
- Fast evaluation scheme
- Interpolate -vs- approximate

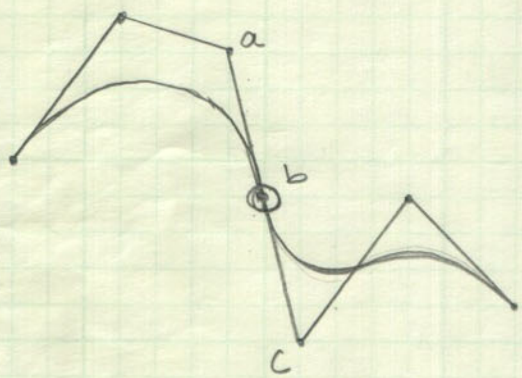
Example of "nice" evaluation scheme for Bézier

De Casteljau Eval.



* Spend 10 minutes & write a program that does this.

Joining



For

$c' \Rightarrow b - a = c - b$

$G' \Rightarrow \dfrac{b-a}{\|b-a\|} = \dfrac{c-b}{\|c-b\|}$

* If you change a, b, or c need to change one of the others as well

* But if you change a, b, or c you <u>don't</u> need to change anything that is <u>not</u> a, b, or c

⊛ Local support

Tensor Product Surfaces

Surface is the result of sweeping a curve through space

* replace control points p_i w/ curves in v

ie
$$x(u,v) = \sum_i \cancel{p_i}\, b_i(u)$$
$$= \sum_i q_i(v)\, b_i(u)$$

Since $q_i(v)$ is curve:

$$q_i(v) = \sum_j p_{ij}\ b_j(v)$$

* So $\quad x(u,v) = \sum_{j,i} p_{ij}\ b_j(v)\ b_i(u)$

$\boxed{\text{* No diff if done } v \text{ then } u \text{ or } u \text{ then } v}$

But $\ b_{ij}(u,v) = b_i(u)\, b_j(v)$

is a 2D Function

* $b_{ij}(u,v)$ are basis Fn of surface!

** $$x(u,v) = \sum_{i,j} p_{ij}\, b_{ij}(u,v)$$

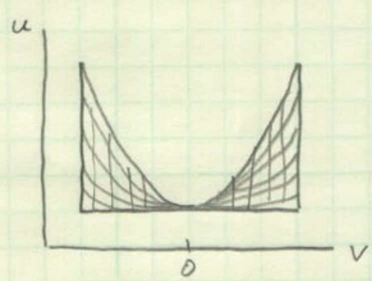
Tangent vectors

$$t_u = \frac{\partial x(u,v)}{\partial u}, \quad t_v = \frac{\partial x(u,v)}{\partial v}$$

$$n = t_u \times t_v$$

Bad things can happen:

$$x(u,v) = \{u, |u| v^2, 0\}$$



$\boxed{t_v = 0 \ @ \ v = 0}$