Foundations of Computer Graphics (Spring 2012)

CS 184, Lectures 19: Sampling and Reconstruction

http://inst.eecs.berkeley.edu/~cs184

Acknowledgements: Thomas Funkhouser and Pat Hanrahan

Outline

- *Basic ideas of sampling, reconstruction, aliasing*
- Signal processing and Fourier analysis
- Implementation of digital filters

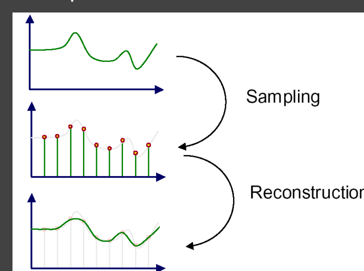
- Section 14.10 of FvDFH (you really should read)
- Post-spring break lectures more advanced topics
  - No programming assignment
  - But can be tested (at high level) in final

Some slides courtesy Tom Funkhouser

HW 3 Demos and Return Midterm

Sampling and Reconstruction

- An image is a 2D array of samples
- Discrete samples from real-world continuous signal



Sampling and Reconstruction

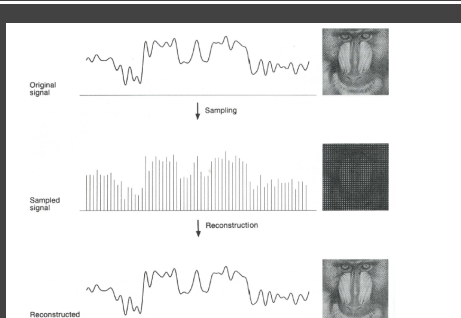
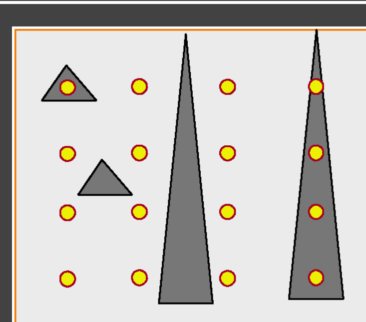


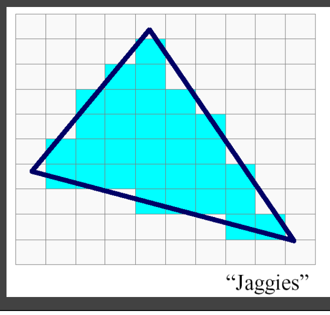
Figure 19.9 FvDFH

(Spatial) Aliasing



(Spatial) Aliasing

- Jaggies probably biggest aliasing problem



Sampling and Aliasing

- Artifacts due to undersampling or poor reconstruction
- Formally, high frequencies masquerading as low
- E.g. high frequency line as low freq jaggies

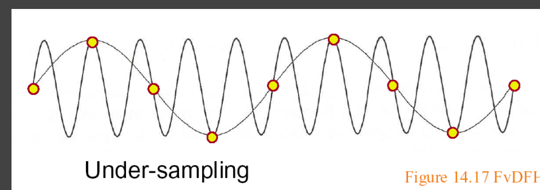
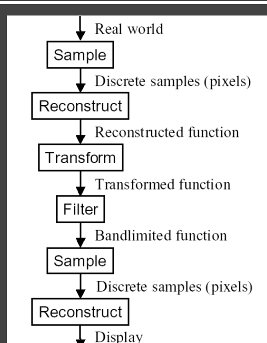


Image Processing pipeline



Outline

- Basic ideas of sampling, reconstruction, aliasing
- *Signal processing and Fourier analysis*
- Implementation of digital filters

- Section 14.10 of FvDFH

Motivation

- Formal analysis of sampling and reconstruction
- Important theory (signal-processing) for graphics
- Also relevant in rendering, modeling, animation

Ideas

- Signal (function of time generally, here of space)
- Continuous: defined at all points; discrete: on a grid
- High frequency: rapid variation; Low Freq: slow variation
- Images are converting continuous to discrete. Do this sampling as best as possible.
- Signal processing theory tells us how best to do this
- Based on concept of frequency domain Fourier analysis

Sampling Theory

Analysis in the frequency (not spatial) domain
- Sum of sine waves, with possibly different offsets (phase)
- Each wave different frequency, amplitude

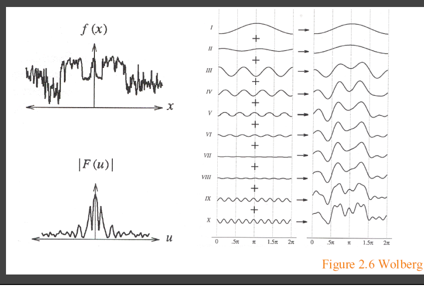


Figure 2.6 Wolberg

Fourier Transform

- Tool for converting from spatial to frequency domain
- Or vice versa
- One of most important mathematical ideas
- Computational algorithm: Fast Fourier Transform
  - One of 10 great algorithms scientific computing
  - Makes Fourier processing possible (images etc.)
  - Not discussed here, but look up if interested

Fourier Transform

- Simple case, function sum of sines, cosines

$$f(x) = \sum_{u=-\infty}^{+\infty} F(u) e^{2\pi i u x}$$

$$F(u) = \int_0^{2\pi} f(x) e^{-2\pi i u x} dx$$

- Continuous infinite case

Forward Transform: $F(u) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i u x} dx$

Inverse Transform: $f(x) = \int_{-\infty}^{+\infty} F(u) e^{2\pi i u x} du$

Fourier Transform

- Simple case, function sum of sines, cosines

$$f(x) = \sum_{u=-\infty}^{+\infty} F(u) e^{2\pi i u x}$$

$$F(u) = \int_0^{2\pi} f(x) e^{-2\pi i u x} dx$$

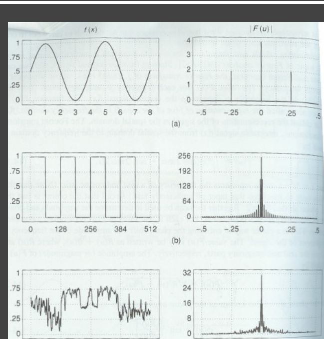
- Discrete case

$$F(u) = \sum_{x=0}^{x=N-1} f(x)\left[\cos(2\pi ux/N) - i\sin(2\pi ux/n)\right], \qquad 0 \le u \le N-1$$

$$f(x) = \frac{1}{N}\sum_{u=0}^{u=N-1} F(u)\left[\cos(2\pi ux/N) + i\sin(2\pi ux/n)\right], \qquad 0 \le x \le N-1$$

Fourier Transform: Examples 1

Single sine curve
(+constant DC term)



$$f(x) = \sum_{u=-\infty}^{+\infty} F(u) e^{2\pi i u x}$$

$$F(u) = \int_0^{2\pi} f(x) e^{-2\pi i u x} dx$$

Fourier Transform Examples 2

Forward Transform: $F(u) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i u x} dx$

Inverse Transform: $f(x) = \int_{-\infty}^{+\infty} F(u) e^{2\pi i u x} du$

- Common examples

$f(x)$	$F(u)$
$\delta(x - x_0)$	$e^{-2\pi i u x_0}$
1	$\delta(u)$
e^{-ax^2}	$\sqrt{\pi/a}\, e^{-\pi^2 u^2 / a}$

Fourier Transform Properties

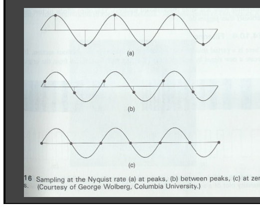
Forward Transform: $F(u) = \int_{-\infty}^{\infty} f(x) e^{-2\pi iux} dx$

Inverse Transform: $f(x) = \int_{-\infty}^{+\infty} F(u) e^{2\pi iux} du$

- Common properties
  - Linearity: $F(af(x) + bg(x)) = aF(f(x)) + bF(g(x))$
  - Derivatives: [integrate by parts] $F(f'(x)) = \int_{-\infty}^{\infty} f'(x) e^{-2\pi iux} dx = 2\pi iuF(u)$
  - 2D Fourier Transform
    Forward Transform: $F(u,v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y) e^{-2\pi iux} e^{-2\pi ivy} dxdy$
- Convolution (next)
  Inverse Transform: $f(x,y) = \int_{-\infty}^{\infty} \int_{-\infty}^{+\infty} F(u,v) e^{2\pi iux} e^{2\pi ivy} dudv$

Sampling Theorem, Bandlimiting

- A signal can be reconstructed from its samples, if the original signal has no frequencies above half the sampling frequency – Shannon
- The minimum sampling rate for a bandlimited function is called the Nyquist rate



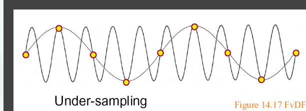


Figure 14.17 FvDFH

Sampling Theorem, Bandlimiting

- A signal can be reconstructed from its samples, if the original signal has no frequencies above half the sampling frequency – Shannon
- The minimum sampling rate for a bandlimited function is called the Nyquist rate
- A signal is bandlimited if the highest frequency is bounded. This frequency is called the bandwidth
- In general, when we transform, we want to filter to bandlimit before sampling, to avoid aliasing

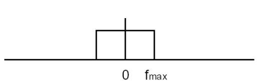
Antialiasing

- Sample at higher rate
  - Not always possible
  - Real world: lines have infinitely high frequencies, can't sample at high enough resolution
- Prefilter to bandlimit signal
  - Low-pass filtering (blurring)
  - Trade blurriness for aliasing

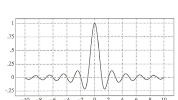
Ideal bandlimiting filter

- Formal derivation is homework exercise

• Frequency domain



• Spatial domain



$Sinc(x) = \frac{\sin \pi x}{\pi x}$

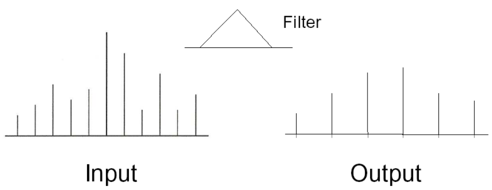
Figure 4.5 Wolberg

Outline

- Basic ideas of sampling, reconstruction, aliasing
- *Signal processing and Fourier analysis*
  - *Convolution*
- Implementation of digital filters
- Section 14.10 of FvDFH

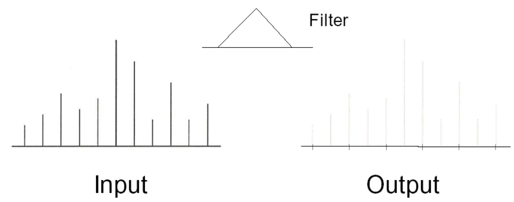
Convolution 1

- Spatial domain: output pixel is weighted sum of pixels in neighborhood of input image
  - Pattern of weights is the “filter”



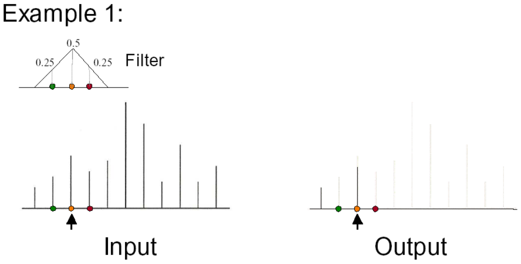
Convolution 2

- Example 1:



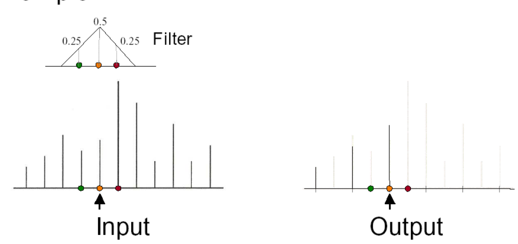
Convolution 3

- Example 1:



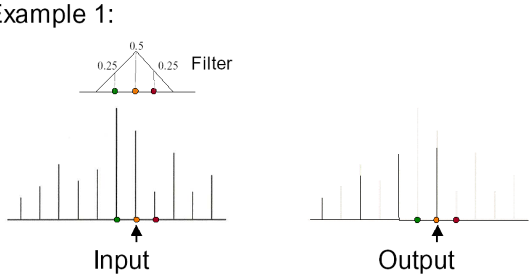
Convolution 4

- Example 1:



Convolution 5

- Example 1:



Convolution in Frequency Domain

Forward Transform: $F(u) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i u x} dx$

Inverse Transform: $f(x) = \int_{-\infty}^{+\infty} F(u) e^{2\pi i u x} du$

- Convolution (f is signal ; g is filter [or vice versa])

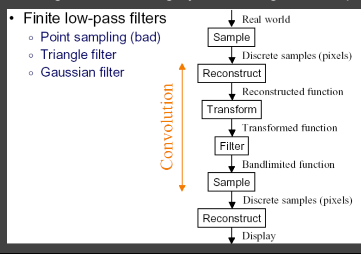
$$h(y) = \int_{-\infty}^{+\infty} f(x) g(y-x) dx = \int_{-\infty}^{+\infty} g(x) f(y-x) dx$$

$$h = f * g \quad or \quad f \otimes g$$

- Fourier analysis (frequency domain multiplication) $H(u) = F(u)G(u)$

Practical Image Processing

- Discrete convolution (in spatial domain) with filters for various digital signal processing operations
- Easy to analyze, understand effects in frequency domain
  - E.g. blurring or bandlimiting by convolving with low pass filter



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- Basic ideas of sampling, reconstruction, aliasing
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- *Implementation of digital filters*
- Section 14.10 of FvDFH

Discrete Convolution

- Previously: Convolution as mult in freq domain
  - But need to convert digital image to and from to use that
  - Useful in some cases, but not for small filters
- Previously seen: Sinc as ideal low-pass filter
  - But has infinite spatial extent, exhibits spatial ringing
  - In general, use frequency ideas, but consider implementation issues as well
- Instead, use simple discrete convolution filters e.g.
  - Pixel gets sum of nearby pixels weighted by filter/mask

2	0	-7
5	4	9
1	-6	-2

Implementing Discrete Convolution

- Fill in each pixel new image convolving with old
  - Not really possible to implement it in place

$$I_{new}(a,b) = \sum_{x=a-width}^{a+width} \sum_{y=b-width}^{b+width} f(x-a, y-b) I_{old}(x,y)$$

  - More efficient for smaller kernels/filters f
- Normalization
  - If you don't want overall brightness change, entries of filter must sum to 1. You may need to normalize by dividing
- Integer arithmetic
  - Simpler and more efficient
  - In general, normalization outside, round to nearest int

Outline

- Implementation of digital filters
  - Discrete convolution in spatial domain
  - *Basic image-processing operations*
  - Antialiased shift and resize

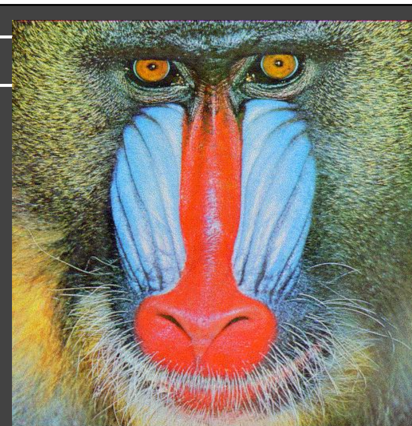
Basic Image Processing

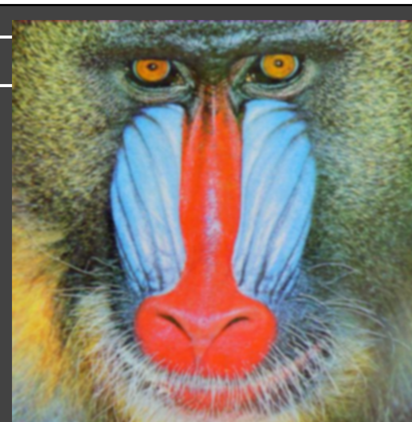
- *Blur*
- Sharpen
- Edge Detection

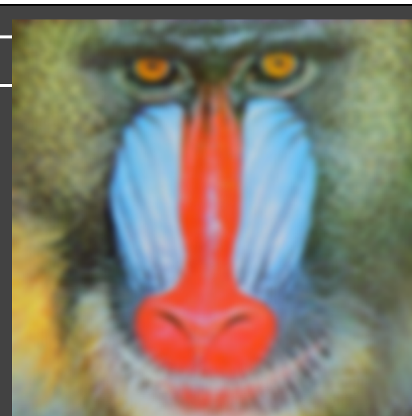
All implemented using convolution with different filters

Blurring

- Used for softening appearance
- Convolve with gaussian filter
  - Same as mult. by gaussian in freq. domain, so reduces high-frequency content
  - Greater the spatial width, smaller the Fourier width, more blurring occurs and vice versa
- How to find blurring filter?







Blurring Filter

- In general, for symmetry f(u,v) = f(u) f(v)
  - You might want to have some fun with asymmetric filters
- We will use a Gaussian blur
  - Blur width sigma depends on kernel size n (3,5,7,11,13,19)

$$f(u) = \frac{1}{\sqrt{2\pi}\sigma}\exp\left[\frac{-u^2}{2\sigma^2}\right] \qquad \sigma = floor(n/2)/2$$

Discrete Filtering, Normalization

- Gaussian is infinite
  - In practice, finite filter of size n (much less energy beyond 2 sigma or 3 sigma).
  - Must renormalize so entries add up to 1
- Simple practical approach
  - Take smallest values as 1 to scale others, round to integers
  - Normalize. E.g. for n = 3, sigma = ½

$$f(u,v) = \frac{1}{2\pi\sigma^2}\exp\left[-\frac{u^2+v^2}{2\sigma^2}\right] = \frac{2}{\pi}\exp\left[-2\left(u^2+v^2\right)\right]$$

$$\approx \begin{pmatrix} 0.012 & 0.09 & 0.012 \\ 0.09 & 0.64 & 0.09 \\ 0.012 & 0.09 & 0.012 \end{pmatrix} \approx \frac{1}{86}\begin{pmatrix} 1 & 7 & 1 \\ 7 & 54 & 7 \\ 1 & 7 & 1 \end{pmatrix}$$

Basic Image Processing

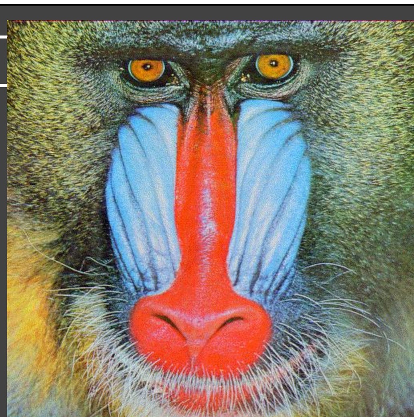
- Blur
- *Sharpen*
- Edge Detection

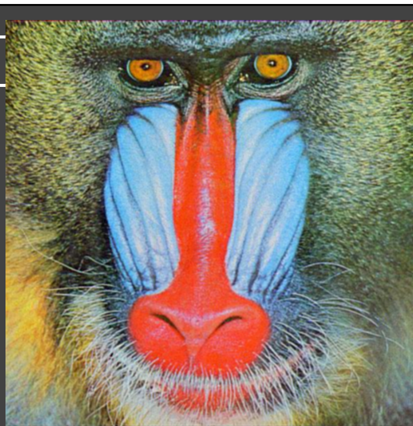
All implemented using convolution with different filters

Sharpening Filter

- Unlike blur, want to accentuate high frequencies
- Take differences with nearby pixels (rather than avg)

$$f(x,y) = \frac{1}{7}\begin{pmatrix} -1 & -2 & -1 \\ -2 & 19 & -2 \\ -1 & -2 & -1 \end{pmatrix}$$





Basic Image Processing

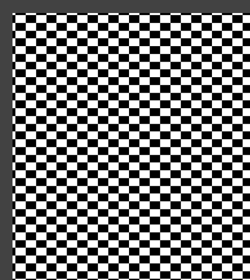
- Blur
- Sharpen
- *Edge Detection*

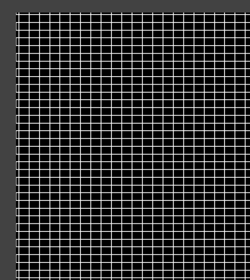
All implemented using convolution with different filters

Edge Detection

- Complicated topic: subject of many PhD theses
- Here, we present one approach (Sobel edge detector)
- Step 1: Convolution with gradient (Sobel) filter
  - Edges occur where image gradients are large
  - Separately for horizontal and vertical directions
- Step 2: Magnitude of gradient
  - Norm of horizontal and vertical gradients
- Step 3: Thresholding
  - Threshold to detect edges

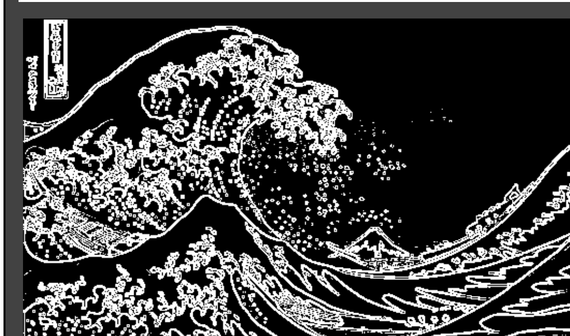
Edge Detection





Edge Detection

Edge Detection



Details

- Step 1: Convolution with gradient (Sobel) filter
  - Edges occur where image gradients are large
  - Separately for horizontal and vertical directions

$$f_{horiz}(x,y) = \begin{pmatrix} -1 & 0 & 1 \\ -2 & 0 & 2 \\ -1 & 0 & 1 \end{pmatrix} \quad f_{vert}(x,y) = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ -1 & -2 & -1 \end{pmatrix}$$

- Step 2: Magnitude of gradient
  - Norm of horizontal and vertical gradients

$$G = \sqrt{|G_x|^2 + |G_y|^2}$$

- Step 3: Thresholding

Outline

- Implementation of digital filters
  - Discrete convolution in spatial domain
  - Basic image-processing operations
  - *Antialiased shift and resize*

Antialiased Shift

Shift image based on (fractional) s_x and s_y
- Check for integers, treat separately
- Otherwise convolve/resample with kernel/filter h:

$$u = x - s_x \qquad v = y - s_y$$

$$I(x,y) = \sum_{u',v'} h(u'-u, v'-v) I(u',v')$$

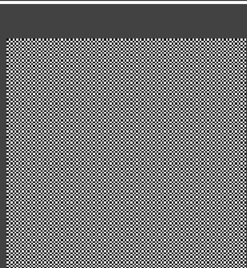
Antialiased Scale Magnification

Magnify image (scale s or $\gamma > 1$)
- Interpolate between orig. samples to evaluate frac vals
- Do so by convolving/resampling with kernel/filter:
- *Treat the two image dimensions independently (diff scales)*

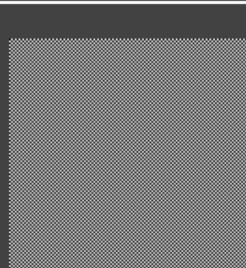
$$u = \frac{x}{\gamma}$$

$$I(x) = \sum_{u'=u/\gamma-\text{width}}^{u/\gamma+\text{width}} h(u'-u) I(u')$$

Antialiased Scale Minification



checkerboard.bmp 300x300: point sample



checkerboard.bmp 300x300: Mitchell

Antialiased Scale Minification

Minify (reduce size of) image
- Similar in some ways to mipmapping for texture maps
- We use *fat* pixels of size $1/\gamma$, with new size γ*orig size (γ is scale factor < 1).
- Each fat pixel must integrate over corresponding region in original image using the filter kernel.

$$u = \frac{x}{\gamma} \qquad I(x) = \sum_{u'=u-\text{width}/\gamma}^{u+\text{width}/\gamma} h(\gamma(u'-u)) I(u') = \sum_{u'=u-\text{width}/\gamma}^{u+\text{width}/\gamma} h(\gamma u' - x) I(u')$$