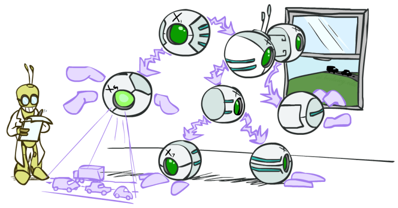
# CS 188: Artificial Intelligence

## Bayes' Nets: Inference



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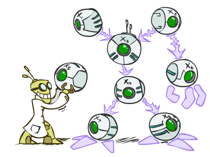
# Bayes' Net Representation

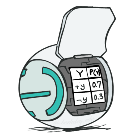
- A directed, acyclic graph, one node per random variable
- A conditional probability table (CPT) for each node
  - A collection of distributions over X, one for each combination of parents' values

$$P(X|a_1 \ldots a_n)$$

- Bayes' nets implicitly encode joint distributions
  - As a product of local conditional distributions
  - To see what probability a BN gives to a full assignment, multiply all the relevant conditionals together:

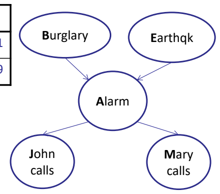
$$P(x_1, x_2, \ldots x_n) = \prod_{i=1}^{n} P(x_i | parents(X_i))$$





# Example: Alarm Network

| B | P(B) |
|---|---|
| +b | 0.001 |
| -b | 0.999 |




| E | P(E) |
|---|---|
| +e | 0.002 |
| -e | 0.998 |

| A | J | P(J\|A) |
|---|---|---|
| +a | +j | 0.9 |
| +a | -j | 0.1 |
| -a | +j | 0.05 |
| -a | -j | 0.95 |

| A | M | P(M\|A) |
|---|---|---|
| +a | +m | 0.7 |
| +a | -m | 0.3 |
| -a | +m | 0.01 |
| -a | -m | 0.99 |

| B | E | A | P(A\|B,E) |
|---|---|---|---|
| +b | +e | +a | 0.95 |
| +b | +e | -a | 0.05 |
| +b | -e | +a | 0.94 |
| +b | -e | -a | 0.06 |
| -b | +e | +a | 0.29 |
| -b | +e | -a | 0.71 |
| -b | -e | +a | 0.001 |
| -b | -e | -a | 0.999 |

[Demo: BN Applet]

# Example: Alarm Network

| B | P(B) |
|---|---|
| +b | 0.001 |
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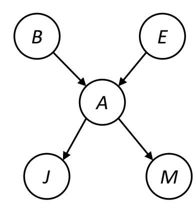


| E | P(E) |
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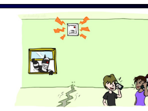

| A | J | P(J\|A) |
|---|---|---|
| +a | +j | 0.9 |
| +a | -j | 0.1 |
| -a | +j | 0.05 |
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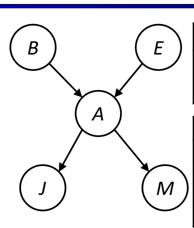
| -a | +m | 0.01 |
| -a | -m | 0.99 |

| B | E | A | P(A\|B,E) |
|---|---|---|---|
| +b | +e | +a | 0.95 |
| +b | +e | -a | 0.05 |
| +b | -e | +a | 0.94 |
| +b | -e | -a | 0.06 |
| -b | +e | +a | 0.29 |
| -b | +e | -a | 0.71 |
| -b | -e | +a | 0.001 |
| -b | -e | -a | 0.999 |

$$P(+b, -e, +a, -j, +m) =$$
$$P(+b)P(-e)P(+a|+b,-e)P(-j|+a)P(+m|+a) =$$

# Example: Alarm Network

| B | P(B) |
|---|---|
| +b | 0.001 |
| -b | 0.999 |




| E | P(E) |
|---|---|
| +e | 0.002 |
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| A | J | P(J\|A) |
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| +a | +j | 0.9 |
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| -a | +j | 0.05 |
| -a | -j | 0.95 |

| A | M | P(M\|A) |
|---|---|---|
| +a | +m | 0.7 |
| +a | -m | 0.3 |
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| -a | -m | 0.99 |

| B | E | A | P(A\|B,E) |
|---|---|---|---|
| +b | +e | +a | 0.95 |
| +b | +e | -a | 0.05 |
| +b | -e | +a | 0.94 |
| +b | -e | -a | 0.06 |
| -b | +e | +a | 0.29 |
| -b | +e | -a | 0.71 |
| -b | -e | +a | 0.001 |
| -b | -e | -a | 0.999 |

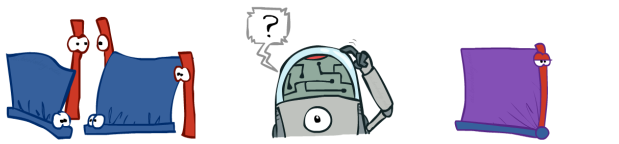
$$P(+b, -e, +a, -j, +m) =$$
$$P(+b)P(-e)P(+a|+b,-e)P(-j|+a)P(+m|+a) =$$
$$0.001 \times 0.998 \times 0.94 \times 0.1 \times 0.7$$

# Bayes' Nets

- ✔ Representation
- ✔ Conditional Independences
- Probabilistic Inference
  - Enumeration (exact, exponential complexity)
  - Variable elimination (exact, worst-case exponential complexity, often better)
  - Inference is NP-complete
  - Sampling (approximate)
- Learning Bayes' Nets from Data

# Inference

- Inference: calculating some useful quantity from a joint probability distribution
- Examples:
  - Posterior probability
    $$P(Q|E_1 = e_1, \dots E_k = e_k)$$
  - Most likely explanation:
    $$\text{argmax}_q \; P(Q = q|E_1 = e_1 \dots)$$



# Inference by Enumeration

- General case:
  - Evidence variables: $E_1 \dots E_k = e_1 \dots e_k$
  - Query* variable: $Q$
  - Hidden variables: $H_1 \dots H_r$

  $\left.\right\}$ $X_1, X_2, \dots X_n$ *All variables*

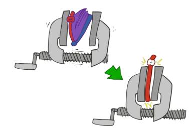
- We want:

  ** Works fine with multiple query variables, too*

  $$P(Q|e_1 \dots e_k)$$

- Step 1: Select the entries consistent with the evidence

- Step 2: Sum out H to get joint of Query and evidence



$$P(Q, e_1 \dots e_k) = \sum_{h_1 \dots h_r} \underbrace{P(Q, h_1 \dots h_r, e_1 \dots e_k)}_{X_1, X_2, \dots X_n}$$

- Step 3: Normalize

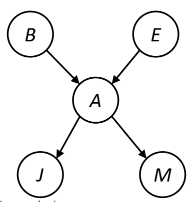
$$\times \frac{1}{Z}$$

$$Z = \sum_q P(Q, e_1 \cdots e_k)$$

$$P(Q|e_1 \cdots e_k) = \frac{1}{Z} P(Q, e_1 \cdots e_k)$$

# Inference by Enumeration in Bayes' Net

- Given unlimited time, inference in BNs is easy
- Reminder of inference by enumeration by example:



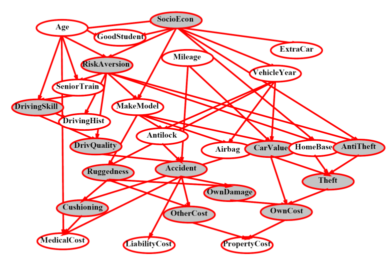

$$P(B \mid +j, +m) \;\; \propto_B \; P(B, +j, +m)$$

$$= \sum_{e,a} P(B, e, a, +j, +m)$$

$$= \sum_{e,a} P(B)P(e)P(a|B,e)P(+j|a)P(+m|a)$$

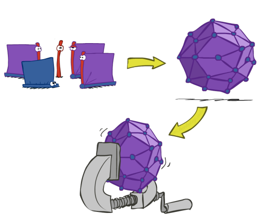
$$= P(B)P(+e)P(+a|B,+e)P(+j|+a)P(+m|+a) + P(B)P(+e)P(-a|B,+e)P(+j|-a)P(+m|-a)$$
$$P(B)P(-e)P(+a|B,-e)P(+j|+a)P(+m|+a) + P(B)P(-e)P(-a|B,-e)P(+j|-a)P(+m|-a)$$

# Inference by Enumeration?

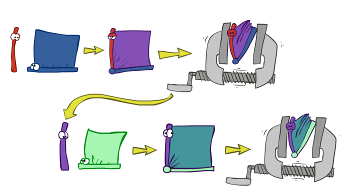



# Inference by Enumeration vs. Variable Elimination

- Why is inference by enumeration so slow?
  - You join up the whole joint distribution before you sum out the hidden variables

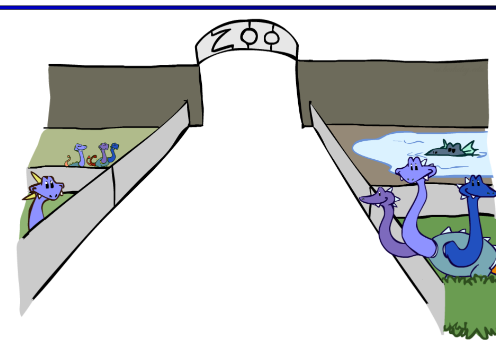


- Idea: interleave joining and marginalizing!
  - Called "Variable Elimination"
  - Still NP-hard, but usually much faster than inference by enumeration



  - First we'll need some new notation: factors

# Factor Zoo



## Factor Zoo I

- Joint distribution: P(X,Y)
  - Entries P(x,y) for all x, y
  - Sums to 1

$P(T,W)$

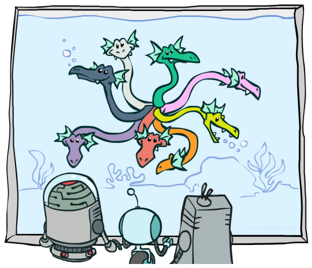
| T | W | P |
|---|---|---|
| hot | sun | 0.4 |
| hot | rain | 0.1 |
| cold | sun | 0.2 |
| cold | rain | 0.3 |

- Selected joint: P(x,Y)
  - A slice of the joint distribution
  - Entries P(x,y) for fixed x, all y
  - Sums to P(x)

$P(cold,W)$

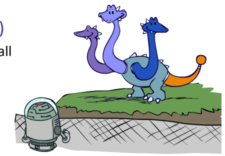
| T | W | P |
|---|---|---|
| cold | sun | 0.2 |
| cold | rain | 0.3 |

- Number of capitals = dimensionality of the table



## Factor Zoo II

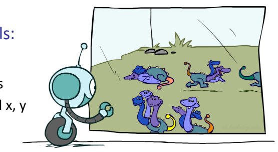
- Single conditional: P(Y | x)
  - Entries P(y | x) for fixed x, all y
  - Sums to 1



$P(W|cold)$

| T | W | P |
|---|---|---|
| cold | sun | 0.4 |
| cold | rain | 0.6 |

- Family of conditionals: P(Y | X)
  - Multiple conditionals
  - Entries P(y | x) for all x, y
  - Sums to |X|



$P(W|T)$

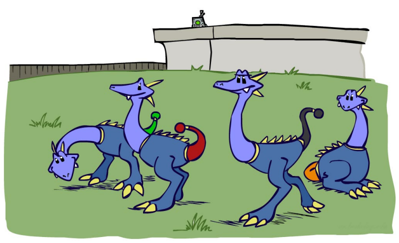
| T | W | P | |
|---|---|---|---|
| hot | sun | 0.8 | $P(W\|hot)$ |
| hot | rain | 0.2 | |
| cold | sun | 0.4 | $P(W\|cold)$ |
| cold | rain | 0.6 | |

## Factor Zoo III

- Specified family: P( y | X )
  - Entries P(y | x) for fixed y, but for all x
  - Sums to … who knows!

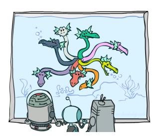
$P(rain|T)$

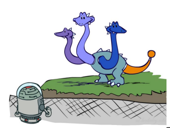
| T | W | P | |
|---|---|---|---|
| hot | rain | 0.2 | $P(rain\|hot)$ |
| cold | rain | 0.6 | $P(rain\|cold)$ |

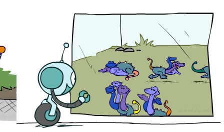


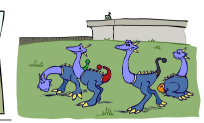
## Factor Zoo Summary

- In general, when we write $P(Y_1 \ldots Y_N \mid X_1 \ldots X_M)$
  - It is a "factor," a multi-dimensional array
  - Its values are $P(y_1 \ldots y_N \mid x_1 \ldots x_M)$
  - Any assigned (=lower-case) X or Y is a dimension missing (selected) from the array



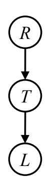






## Example: Traffic Domain

- Random Variables
  - R: Raining
  - T: Traffic
  - L: Late for class!



$$P(L) = ?$$

$$= \sum_{r,t} P(r,t,L)$$

$$= \sum_{r,t} P(r)P(t|r)P(L|t)$$

$P(R)$

| | |
|---|---|
| +r | 0.1 |
| -r | 0.9 |

$P(T|R)$

| | | |
|---|---|---|
| +r | +t | 0.8 |
| +r | -t | 0.2 |
| -r | +t | 0.1 |
| -r | -t | 0.9 |

$P(L|T)$

| | | |
|---|---|---|
| +t | +l | 0.3 |
| +t | -l | 0.7 |
| -t | +l | 0.1 |
| -t | -l | 0.9 |

## Inference by Enumeration: Procedural Outline

- Track objects called factors
- Initial factors are local CPTs (one per node)

$P(R)$

| | |
|---|---|
| +r | 0.1 |
| -r | 0.9 |

$P(T|R)$

| | | |
|---|---|---|
| +r | +t | 0.8 |
| +r | -t | 0.2 |
| -r | +t | 0.1 |
| -r | -t | 0.9 |

$P(L|T)$

| | | |
|---|---|---|
| +t | +l | 0.3 |
| +t | -l | 0.7 |
| -t | +l | 0.1 |
| -t | -l | 0.9 |

- Any known values are selected
  - E.g. if we know $L = +\ell$, the initial factors are

$P(R)$

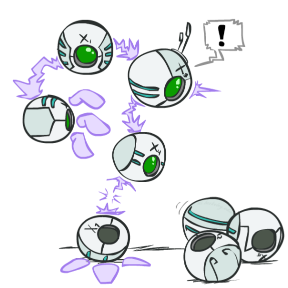
| | |
|---|---|
| +r | 0.1 |
| -r | 0.9 |

$P(T|R)$

| | | |
|---|---|---|
| +r | +t | 0.8 |
| +r | -t | 0.2 |
| -r | +t | 0.1 |
| -r | -t | 0.9 |

$P(+\ell|T)$

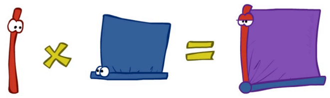
| | | |
|---|---|---|
| +t | +l | 0.3 |
| -t | +l | 0.1 |



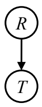
- Procedure: Join all factors, eliminate all hidden variables, normalize

# Operation 1: Join Factors

- First basic operation: joining factors
- Combining factors:
  - Just like a database join
  - Get all factors over the joining variable
  - Build a new factor over the union of the variables involved



- Example: Join on R



$P(R)$

| | |
|---|---|
| +r | 0.1 |
| -r | 0.9 |

$\times$ $P(T|R)$

| | | |
|---|---|---|
| +r | +t | 0.8 |
| +r | -t | 0.2 |
| -r | +t | 0.1 |
| -r | -t | 0.9 |

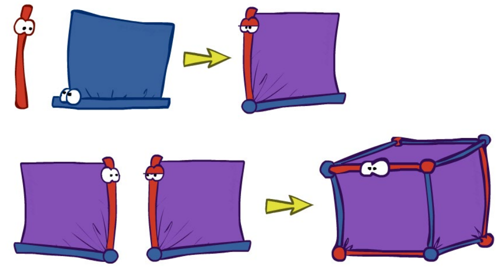
$\rightarrow$ $P(R,T)$

| | | |
|---|---|---|
| +r | +t | 0.08 |
| +r | -t | 0.02 |
| -r | +t | 0.09 |
| -r | -t | 0.81 |

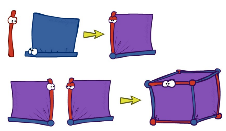
$R,T$

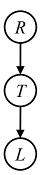
- Computation for each entry: pointwise products $\forall r,t: \quad P(r,t) = P(r) \cdot P(t|r)$

# Example: Multiple Joins



# Example: Multiple Joins





$P(R)$

| | |
|---|---|
| +r | 0.1 |
| -r | 0.9 |

$P(T|R)$

| | | |
|---|---|---|
| +r | +t | 0.8 |
| +r | -t | 0.2 |
| -r | +t | 0.1 |
| -r | -t | 0.9 |

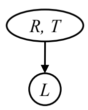
$P(L|T)$

| | | |
|---|---|---|
| +t | +l | 0.3 |
| +t | -l | 0.7 |
| -t | +l | 0.1 |
| -t | -l | 0.9 |

Join R

$P(R,T)$

| | | |
|---|---|---|
| +r | +t | 0.08 |
| +r | -t | 0.02 |
| -r | +t | 0.09 |
| -r | -t | 0.81 |



$P(L|T)$

| | | |
|---|---|---|
| +t | +l | 0.3 |
| +t | -l | 0.7 |
| -t | +l | 0.1 |
| -t | -l | 0.9 |

Join T

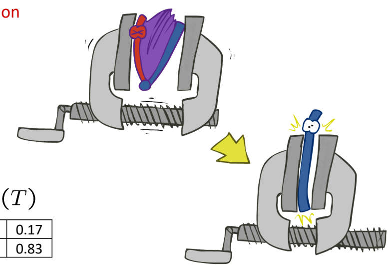
$R, T, L$

$P(R,T,L)$

| | | | |
|---|---|---|---|
| +r | +t | +l | 0.024 |
| +r | +t | -l | 0.056 |
| +r | -t | +l | 0.002 |
| +r | -t | -l | 0.018 |
| -r | +t | +l | 0.027 |
| -r | +t | -l | 0.063 |
| -r | -t | +l | 0.081 |
| -r | -t | -l | 0.729 |

# Operation 2: Eliminate

- Second basic operation: marginalization
- Take a factor and sum out a variable
  - Shrinks a factor to a smaller one
  - A projection operation
- Example:



$P(R,T)$

| | | |
|---|---|---|
| +r | +t | 0.08 |
| +r | -t | 0.02 |
| -r | +t | 0.09 |
| -r | -t | 0.81 |

sum $R$ $\rightarrow$ $P(T)$

| | |
|---|---|
| +t | 0.17 |
| -t | 0.83 |

# Multiple Elimination

$R, T, L$

$P(R,T,L)$

| | | | |
|---|---|---|---|
| +r | +t | +l | 0.024 |
| +r | +t | -l | 0.056 |
| +r | -t | +l | 0.002 |
| +r | -t | -l | 0.018 |
| -r | +t | +l | 0.027 |
| -r | +t | -l | 0.063 |
| -r | -t | +l | 0.081 |
| -r | -t | -l | 0.729 |

Sum out R

$T, L$

$P(T,L)$

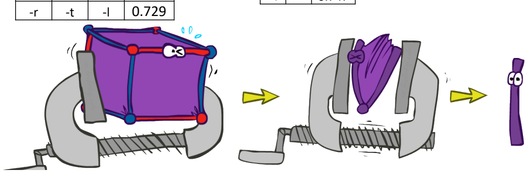
| | | |
|---|---|---|
| +t | +l | 0.051 |
| +t | -l | 0.119 |
| -t | +l | 0.083 |
| -t | -l | 0.747 |

Sum out T

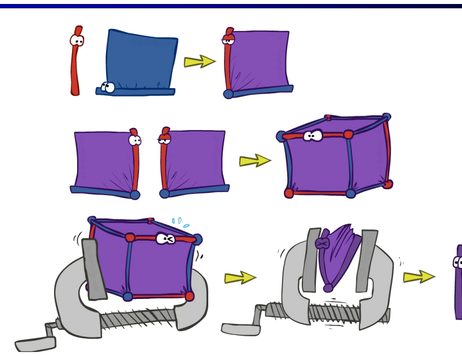
$L$

$P(L)$

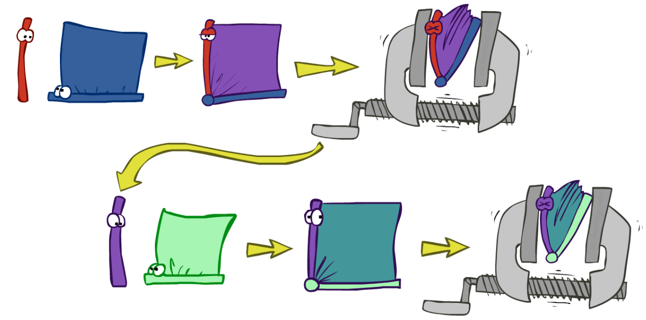
| | |
|---|---|
| +l | 0.134 |
| -l | 0.886 |



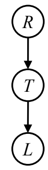
# Thus Far: Multiple Join, Multiple Eliminate (= Inference by Enumeration)



# Marginalizing Early (= Variable Elimination)



# Traffic Domain



$P(L) = ?$

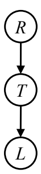
- Inference by Enumeration

$$=\underbrace{\sum_t \underbrace{\sum_r \underbrace{P(L|t)\underbrace{P(r)P(t|r)}_{\text{Join on r}}}_{\text{Join on t}}}_{\text{Eliminate r}}}_{\text{Eliminate t}}$$

- Variable Elimination

$$=\underbrace{\sum_t \underbrace{P(L|t)\underbrace{\sum_r \underbrace{P(r)P(t|r)}_{\text{Join on r}}}_{\text{Eliminate r}}}_{\text{Join on t}}}_{\text{Eliminate t}}$$

# Marginalizing Early! (aka VE)



$P(R)$

| | |
|---|---|
| +r | 0.1 |
| -r | 0.9 |

$P(T|R)$

| | | |
|---|---|---|
| +r | +t | 0.8 |
| +r | -t | 0.2 |
| -r | +t | 0.1 |
| -r | -t | 0.9 |

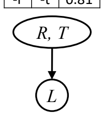
$P(L|T)$

| | | |
|---|---|---|
| +t | +l | 0.3 |
| +t | -l | 0.7 |
| -t | +l | 0.1 |
| -t | -l | 0.9 |

Join R

$P(R, T)$

| | | |
|---|---|---|
| +r | +t | 0.08 |
| +r | -t | 0.02 |
| -r | +t | 0.09 |
| -r | -t | 0.81 |



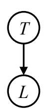
$P(L|T)$

| | | |
|---|---|---|
| +t | +l | 0.3 |
| +t | -l | 0.7 |
| -t | +l | 0.1 |
| -t | -l | 0.9 |

Sum out R

$P(T)$

| | |
|---|---|
| +t | 0.17 |
| -t | 0.83 |



$P(L|T)$

| | | |
|---|---|---|
| +t | +l | 0.3 |
| +t | -l | 0.7 |
| -t | +l | 0.1 |
| -t | -l | 0.9 |

Join T

$P(T, L)$

| | | |
|---|---|---|
| +t | +l | 0.051 |
| +t | -l | 0.119 |
| -t | +l | 0.083 |
| -t | -l | 0.747 |

Sum out T

$P(L)$

| | |
|---|---|
| +l | 0.134 |
| -l | 0.866 |

# Evidence

- If evidence, start with factors that select that evidence
  - No evidence uses these initial factors:

$P(R)$

| | |
|---|---|
| +r | 0.1 |
| -r | 0.9 |

$P(T|R)$

| | | |
|---|---|---|
| +r | +t | 0.8 |
| +r | -t | 0.2 |
| -r | +t | 0.1 |
| -r | -t | 0.9 |

$P(L|T)$

| | | |
|---|---|---|
| +t | +l | 0.3 |
| +t | -l | 0.7 |
| -t | +l | 0.1 |
| -t | -l | 0.9 |

  - Computing $P(L|+r)$ the initial factors become:

$P(+r)$

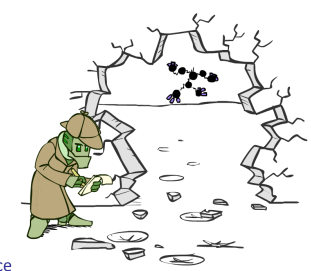
| | |
|---|---|
| +r | 0.1 |

$P(T|+r)$

| | | |
|---|---|---|
| +r | +t | 0.8 |
| +r | -t | 0.2 |

$P(L|T)$

| | | |
|---|---|---|
| +t | +l | 0.3 |
| +t | -l | 0.7 |
| -t | +l | 0.1 |
| -t | -l | 0.9 |



- We eliminate all vars other than query + evidence

# Evidence II

- Result will be a selected joint of query and evidence
  - E.g. for P(L | +r), we would end up with:

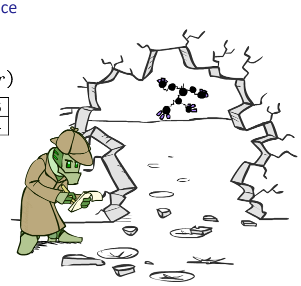
$P(+r, L)$

| | | |
|---|---|---|
| +r | +l | 0.026 |
| +r | -l | 0.074 |

Normalize

$P(L|+r)$

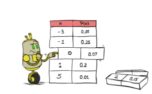
| | |
|---|---|
| +l | 0.26 |
| -l | 0.74 |



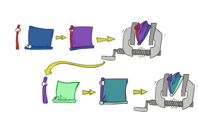
- To get our answer, just normalize this!
- That's it!

# General Variable Elimination

- Query: $P(Q|E_1 = e_1, \dots E_k = e_k)$
- Start with initial factors:
  - Local CPTs (but instantiated by evidence)



- While there are still hidden variables (not Q or evidence):
  - Pick a hidden variable H
  - Join all factors mentioning H
  - Eliminate (sum out) H

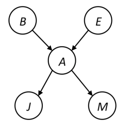


- Join all remaining factors and normalize

$\times \frac{1}{Z}$

# Example

$P(B|j,m) \propto P(B,j,m)$




| $P(B)$ | $P(E)$ | $P(A|B,E)$ | $P(j|A)$ | $P(m|A)$ |
|---|---|---|---|---|

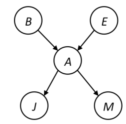
Choose A

$P(A|B,E)$, $P(j|A)$, $P(m|A)$ $\xrightarrow{\times}$ $P(j,m,A|B,E)$ $\xrightarrow{\sum}$ $P(j,m|B,E)$

| $P(B)$ | $P(E)$ | $P(j,m|B,E)$ |
|---|---|---|

# Example

| $P(B)$ | $P(E)$ | $P(j,m|B,E)$ |
|---|---|---|




Choose E

$P(E)$, $P(j,m|B,E)$ $\xrightarrow{\times}$ $P(j,m,E|B)$ $\xrightarrow{\sum}$ $P(j,m|B)$

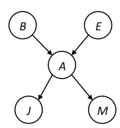
| $P(B)$ | $P(j,m|B)$ |
|---|---|

Finish with B

$P(B)$, $P(j,m|B)$ $\xrightarrow{\times}$ $P(j,m,B)$ $\xrightarrow{\text{Normalize}}$ $P(B|j,m)$

# Same Example in Equations

$P(B|j,m) \propto P(B,j,m)$




| $P(B)$ | $P(E)$ | $P(A|B,E)$ | $P(j|A)$ | $P(m|A)$ |
|---|---|---|---|---|

$$
\begin{aligned}
P(B|j,m) &\propto P(B,j,m) \\
&= \sum_{e,a} P(B,j,m,e,a) && \text{marginal obtained from joint by summing out} \\
&= \sum_{e,a} P(B)P(e)P(a|B,e)P(j|a)P(m|a) && \text{use Bayes' net joint distribution expression} \\
&= \sum_{e} P(B)P(e)\sum_{a} P(a|B,e)P(j|a)P(m|a) && \text{use } x*(y+z) = xy + xz \\
&= \sum_{e} P(B)P(e)f_1(B,e,j,m) && \text{joining on a, and then summing out gives } f_1 \\
&= P(B)\sum_{e} P(e)f_1(B,e,j,m) && \text{use } x*(y+z) = xy + xz \\
&= P(B)f_2(B,j,m) && \text{joining on e, and then summing out gives } f_2
\end{aligned}
$$

**All we are doing is exploiting uwy + uwz + uxy + uxz + vwy + vwz + vxy +vxz = (u+v)(w+x)(y+z) to improve computational efficiency!**

# Another Variable Elimination Example

Query: $P(X_3|Y_1=y_1, Y_2=y_2, Y_3=y_3)$

Start by inserting evidence, which gives the following initial factors:

$$p(Z)p(X_1|Z)p(X_2|Z)p(X_3|Z)p(y_1|X_1)p(y_2|X_2)p(y_3|X_3)$$

Eliminate $X_1$, this introduces the factor $f_1(Z,y_1) = \sum_{x_1} p(x_1|Z)p(y_1|x_1)$, and we are left with:

$$p(Z)f_1(Z,y_1)p(X_2|Z)p(X_3|Z)p(y_2|X_2)p(y_3|X_3)$$

Eliminate $X_2$, this introduces the factor $f_2(Z,y_2) = \sum_{x_2} p(x_2|Z)p(y_2|x_2)$, and we are left with:

$$p(Z)f_1(Z,y_1)f_2(Z,y_2)p(X_3|Z)p(y_3|X_3)$$

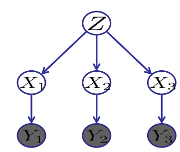
Eliminate $Z$, this introduces the factor $f_3(y_1,y_2,X_3) = \sum_{z} p(z)f_1(z,y_1)f_2(z,y_2)p(X_3|z)$, and we are left:

$$p(y_3|X_3), f_3(y_1,y_2,X_3)$$

No hidden variables left. Join the remaining factors to get:

$$f_4(y_1,y_2,y_3,X_3) = P(y_3|X_3)f_3(y_1,y_2,X_3).$$

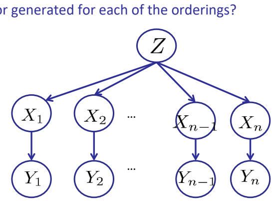
Normalizing over $X_3$ gives $P(X_3|y_1,y_2,y_3)$.




Computational complexity critically depends on the largest factor being generated in this process. Size of factor = number of entries in table. In example above (assuming binary) all factors generated are of size 2 --- as they all only have one variable (Z, Z, and $X_3$ respectively).

# Variable Elimination Ordering

- For the query $P(X_n|y_1,\ldots,y_n)$ work through the following two different orderings as done in previous slide: Z, $X_1$, …, $X_{n-1}$ and $X_1$, …, $X_{n-1}$, Z. What is the size of the maximum factor generated for each of the orderings?




- Answer: $2^{n+1}$ versus $2^2$ (assuming binary)
- In general: the ordering can greatly affect efficiency.

# VE: Computational and Space Complexity

- The computational and space complexity of variable elimination is determined by the largest factor
- The elimination ordering can greatly affect the size of the largest factor.
  - E.g., previous slide's example $2^n$ vs. 2
- Does there always exist an ordering that only results in small factors?
  - No!

# Worst Case Complexity?

- CSP:

$(x_1 \vee x_2 \vee \neg x_3) \wedge (\neg x_1 \vee x_3 \vee \neg x_4) \wedge (x_2 \vee \neg x_2 \vee x_4) \wedge (\neg x_3 \vee \neg x_4 \vee \neg x_5) \wedge (x_2 \vee x_5 \vee x_7) \wedge (x_4 \vee x_5 \vee x_6) \wedge (\neg x_5 \vee x_6 \vee \neg x_7) \wedge (\neg x_5 \vee \neg x_6 \vee x_7)$

$P(X_i = 0) = P(X_i = 1) = 0.5$

$Y_1 = X_1 \vee X_2 \vee \neg X_3$

...

$Y_8 = \neg X_5 \vee X_6 \vee X_7$

$Y_{1,2} = Y_1 \wedge Y_2$

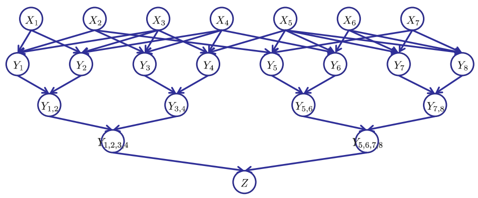
...

$Y_{7,8} = Y_7 \wedge Y_8$

$Y_{1,2,3,4} = Y_{1,2} \wedge Y_{3,4}$

$Y_{5,6,7,8} = Y_{5,6} \wedge Y_{7,8}$

$Z = Y_{1,2,3,4} \wedge Y_{5,6,7,8}$




- If we can answer P(z) equal to zero or not, we answered whether the 3-SAT problem has a solution.
- Hence inference in Bayes' nets is NP-hard. No known efficient probabilistic inference in general.

# Polytrees

- A polytree is a directed graph with no undirected cycles
- For poly-trees you can always find an ordering that is efficient
  - Try it!!
- Cut-set conditioning for Bayes' net inference
  - Choose set of variables such that if removed only a polytree remains
  - Exercise: Think about how the specifics would work out!

# Bayes' Nets

- ✔ Representation
- ✔ Conditional Independences
- Probabilistic Inference
  - ✔ Enumeration (exact, exponential complexity)
  - ✔ Variable elimination (exact, worst-case exponential complexity, often better)
  - ✔ Inference is NP-complete
  - Sampling (approximate)
- Learning Bayes' Nets from Data