# CS 188: Artificial Intelligence

## Constraint Satisfaction Problems II


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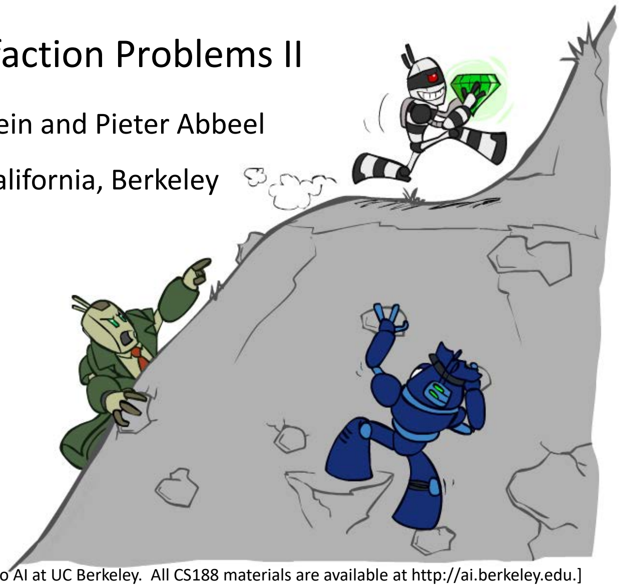


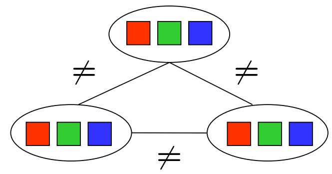


# Today

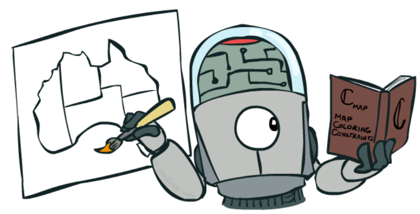
- Efficient Solution of CSPs
- Local Search

# Reminder: CSPs

- CSPs:
  - Variables
  - Domains
  - Constraints
    - Implicit (provide code to compute)
    - Explicit (provide a list of the legal tuples)
    - Unary / Binary / N-ary



- Goals:
  - Here: find any solution
  - Also: find all, find best, etc.



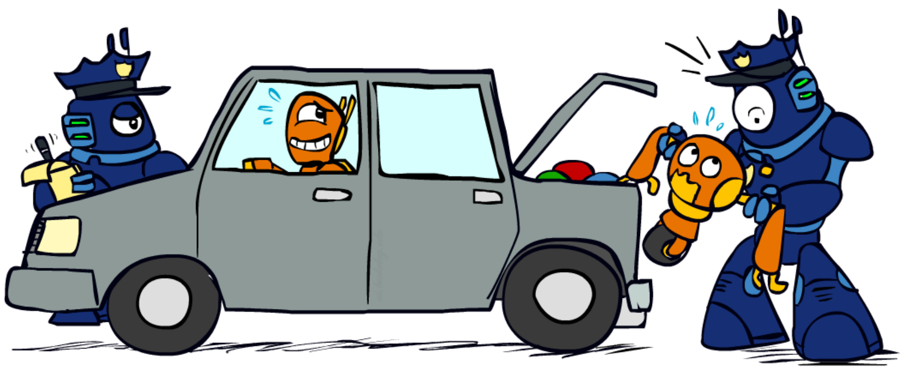
# Backtracking Search

```
function BACKTRACKING-SEARCH(csp) returns solution/failure
   return RECURSIVE-BACKTRACKING({ }, csp)
function RECURSIVE-BACKTRACKING(assignment, csp) returns soln/failure
   if assignment is complete then return assignment
   var ← SELECT-UNASSIGNED-VARIABLE(VARIABLES[csp], assignment, csp)
   for each value in ORDER-DOMAIN-VALUES(var, assignment, csp) do
      if value is consistent with assignment given CONSTRAINTS[csp] then
         add {var = value} to assignment
         result ← RECURSIVE-BACKTRACKING(assignment, csp)
         if result ≠ failure then return result
         remove {var = value} from assignment
   return failure
```

# Improving Backtracking

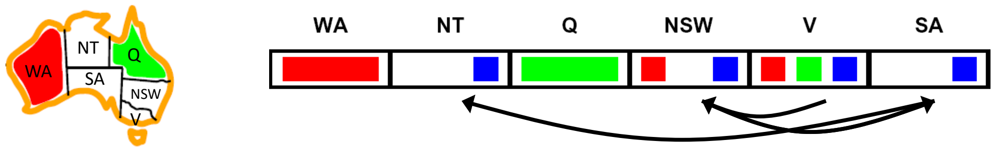
- General-purpose ideas give huge gains in speed
  - … but it's all still NP-hard
- Filtering: Can we detect inevitable failure early?
- Ordering:
  - Which variable should be assigned next?  (MRV)
  - In what order should its values be tried?  (LCV)
- Structure: Can we exploit the problem structure?

# Arc Consistency and Beyond



# Arc Consistency of an Entire CSP

- A simple form of propagation makes sure all arcs are simultaneously consistent:

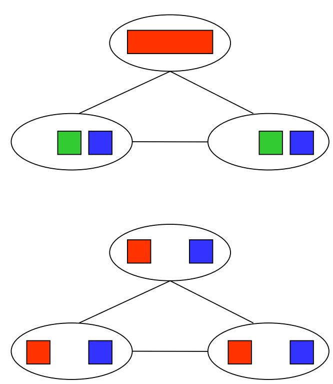



- Arc consistency detects failure earlier than forward checking
- Important: If X loses a value, neighbors of X need to be rechecked!
- Must rerun after each assignment!

*Remember: Delete from the tail!*

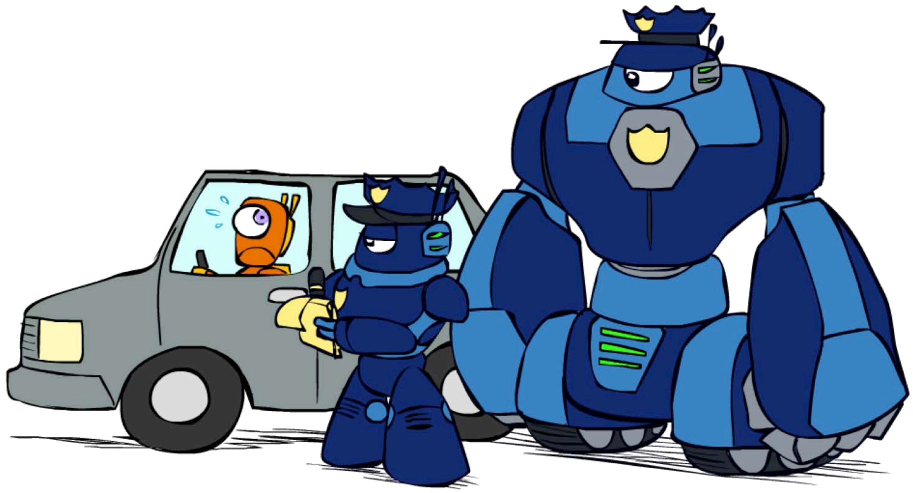
# Limitations of Arc Consistency

- After enforcing arc consistency:
  - Can have one solution left
  - Can have multiple solutions left
  - Can have no solutions left (and not know it)
- Arc consistency still runs inside a backtracking search!



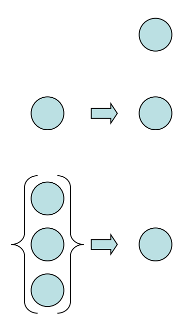
*What went wrong here?*

# K-Consistency

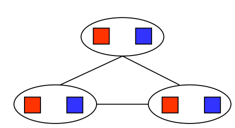


# K-Consistency

- Increasing degrees of consistency
  - 1-Consistency (Node Consistency): Each single node's domain has a value which meets that node's unary constraints
  - 2-Consistency (Arc Consistency): For each pair of nodes, any consistent assignment to one can be extended to the other
  - K-Consistency: For each k nodes, any consistent assignment to k-1 can be extended to the $k^{th}$ node.



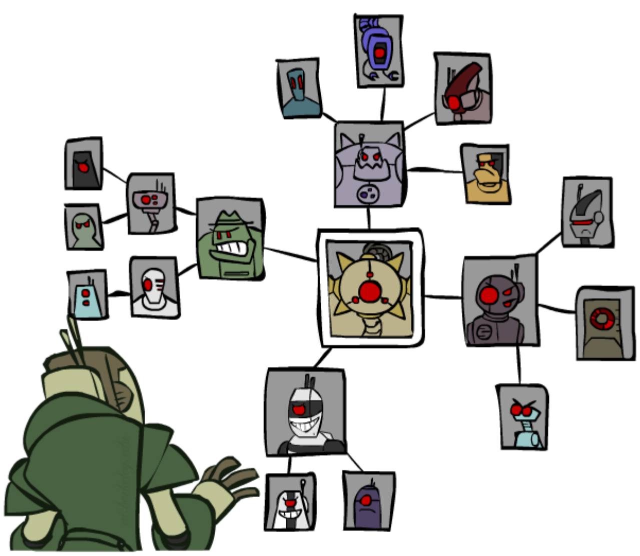
- Higher k more expensive to compute
- (You need to know the k=2 case: arc consistency)



# Strong K-Consistency

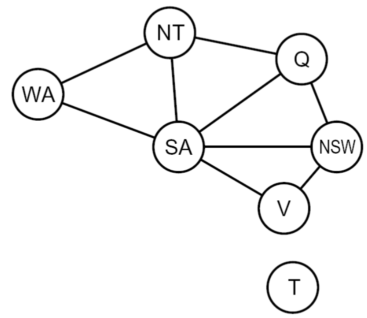
- Strong k-consistency: also k-1, k-2, … 1 consistent
- Claim: strong n-consistency means we can solve without backtracking!
- Why?
  - Choose any assignment to any variable
  - Choose a new variable
  - By 2-consistency, there is a choice consistent with the first
  - Choose a new variable
  - By 3-consistency, there is a choice consistent with the first 2
  - …
- Lots of middle ground between arc consistency and n-consistency! (e.g. k=3, called path consistency)

# Structure

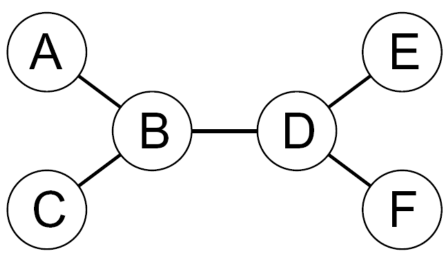


# Problem Structure

- Extreme case: independent subproblems
  - Example: Tasmania and mainland do not interact
- Independent subproblems are identifiable as connected components of constraint graph
- Suppose a graph of n variables can be broken into subproblems of only c variables:
  - Worst-case solution cost is $O((n/c)(d^c))$, linear in n
  - E.g., n = 80, d = 2, c =20
  - $2^{80}$ = 4 billion years at 10 million nodes/sec
  - $(4)(2^{20})$ = 0.4 seconds at 10 million nodes/sec



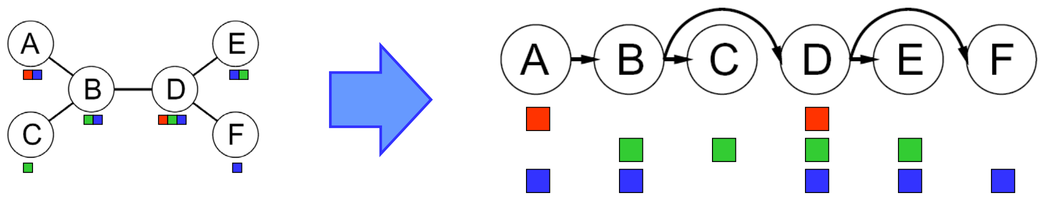

# Tree-Structured CSPs




- Theorem: if the constraint graph has no loops, the CSP can be solved in $O(n\ d^2)$ time
  - Compare to general CSPs, where worst-case time is $O(d^n)$
- This property also applies to probabilistic reasoning (later): an example of the relation between syntactic restrictions and the complexity of reasoning

# Tree-Structured CSPs

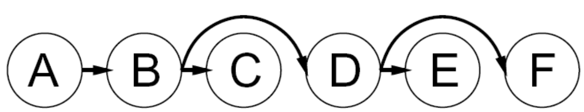
- Algorithm for tree-structured CSPs:
  - Order: Choose a root variable, order variables so that parents precede children




  - Remove backward: For i = n : 2, apply RemoveInconsistent(Parent($X_i$),$X_i$)
  - Assign forward: For i = 1 : n, assign $X_i$ consistently with Parent($X_i$)
- Runtime: $O(n\ d^2)$  (why?)

# Tree-Structured CSPs

- Claim 1: After backward pass, all root-to-leaf arcs are consistent
- Proof: Each $X \rightarrow Y$ was made consistent at one point and Y's domain could not have been reduced thereafter (because Y's children were processed before Y)



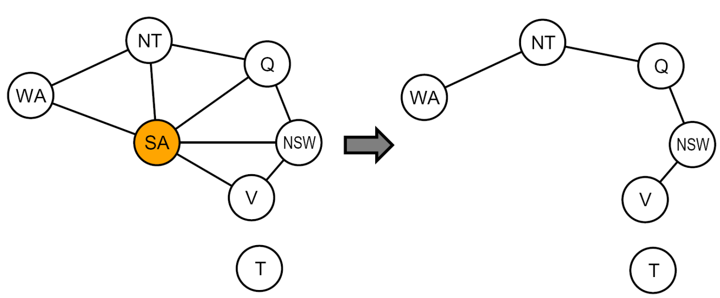

- Claim 2: If root-to-leaf arcs are consistent, forward assignment will not backtrack
- Proof: Induction on position

- Why doesn't this algorithm work with cycles in the constraint graph?

- Note: we'll see this basic idea again with Bayes' nets

# Improving Structure

# Nearly Tree-Structured CSPs




- Conditioning: instantiate a variable, prune its neighbors' domains
- Cutset conditioning: instantiate (in all ways) a set of variables such that the remaining constraint graph is a tree
- Cutset size c gives runtime $O( (d^c) (n-c) d^2 )$, very fast for small c

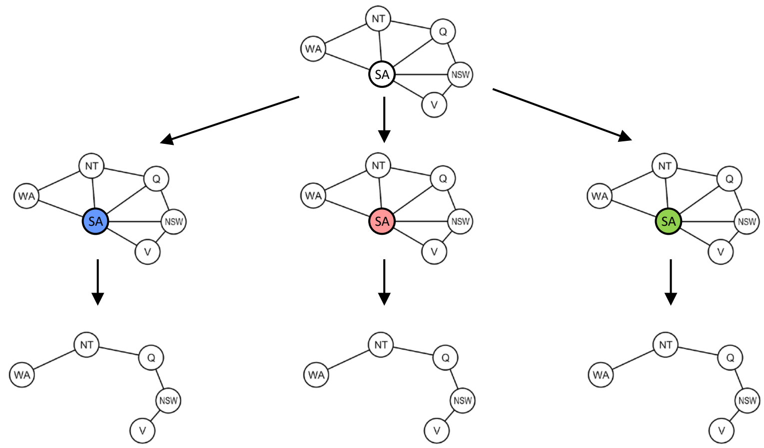
# Cutset Conditioning

Choose a cutset

Instantiate the cutset (all possible ways)

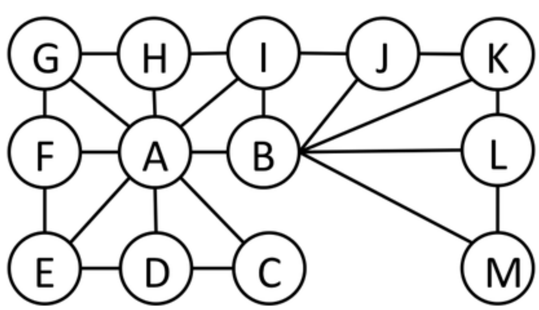
Compute residual CSP for each assignment

Solve the residual CSPs (tree structured)



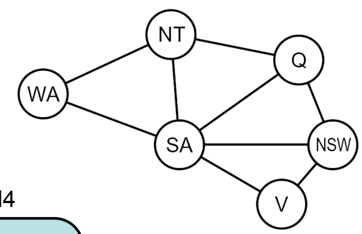

# Cutset Quiz

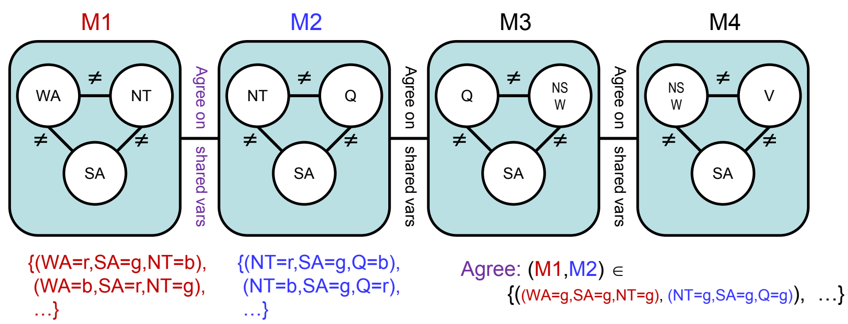
- Find the smallest cutset for the graph below.



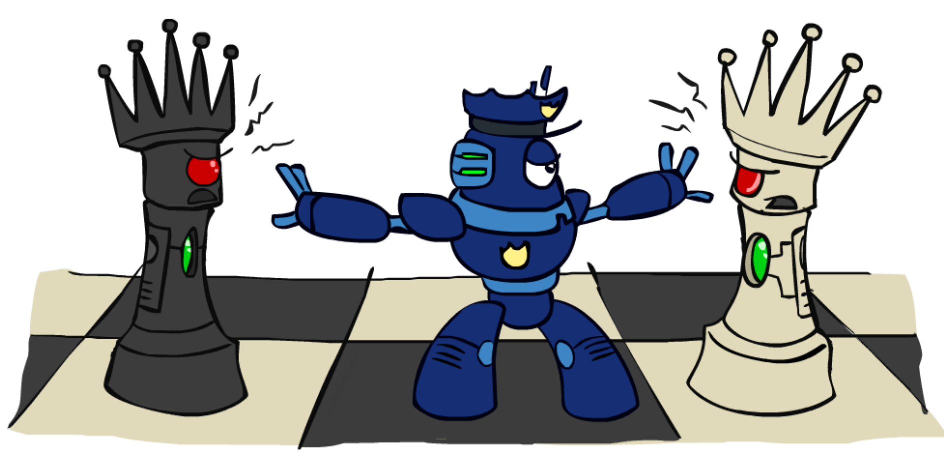
# Tree Decomposition*

- Idea: create a tree-structured graph of mega-variables
- Each mega-variable encodes part of the original CSP
- Subproblems overlap to ensure consistent solutions



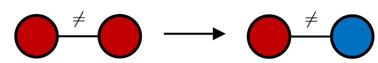




# Iterative Improvement



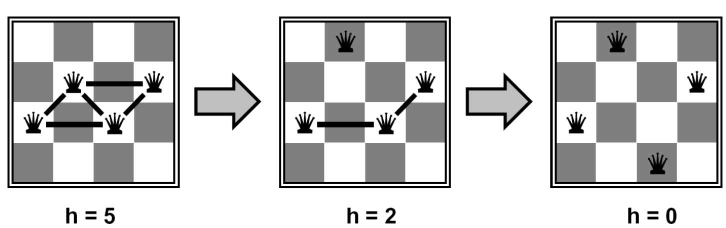
# Iterative Algorithms for CSPs

- Local search methods typically work with "complete" states, i.e., all variables assigned
- To apply to CSPs:
  - Take an assignment with unsatisfied constraints
  - Operators *reassign* variable values
  - No fringe!  Live on the edge.




- Algorithm: While not solved,
  - Variable selection: randomly select any conflicted variable
  - Value selection: min-conflicts heuristic:
    - Choose a value that violates the fewest constraints
    - I.e., hill climb with h(n) = total number of violated constraints

# Example: 4-Queens




- States: 4 queens in 4 columns ($4^4$ = 256 states)
- Operators: move queen in column
- Goal test: no attacks
- Evaluation: c(n) = number of attacks

[Demo: n-queens – iterative improvement (L5D1)]
[Demo: coloring – iterative improvement]

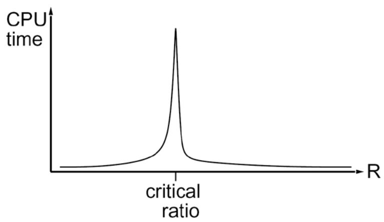
# Video of Demo Iterative Improvement – n Queens

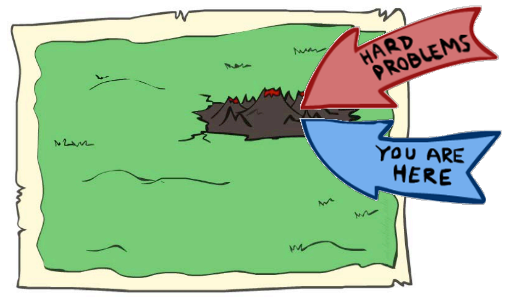
# Video of Demo Iterative Improvement – Coloring

# Performance of Min-Conflicts

- Given random initial state, can solve n-queens in almost constant time for arbitrary n with high probability (e.g., n = 10,000,000)!
- The same appears to be true for any randomly-generated CSP *except* in a narrow range of the ratio

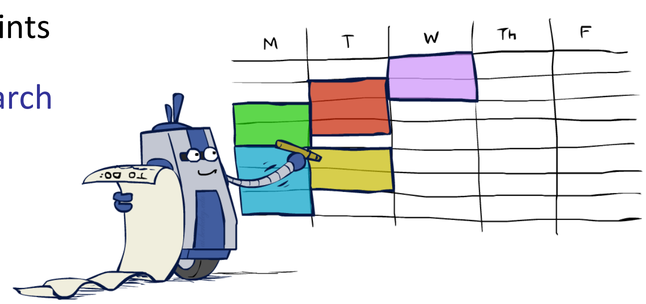
$$R = \frac{\text{number of constraints}}{\text{number of variables}}$$




# Summary: CSPs

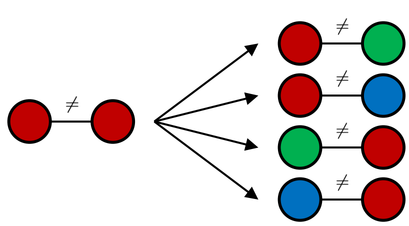
- CSPs are a special kind of search problem:
  - States are partial assignments
  - Goal test defined by constraints
- Basic solution: backtracking search
- Speed-ups:
  - Ordering
  - Filtering
  - Structure
- Iterative min-conflicts is often effective in practice



# Local Search

# Local Search

- Tree search keeps unexplored alternatives on the fringe (ensures completeness)
- Local search: improve a single option until you can’t make it better (no fringe!)
- New successor function: local changes

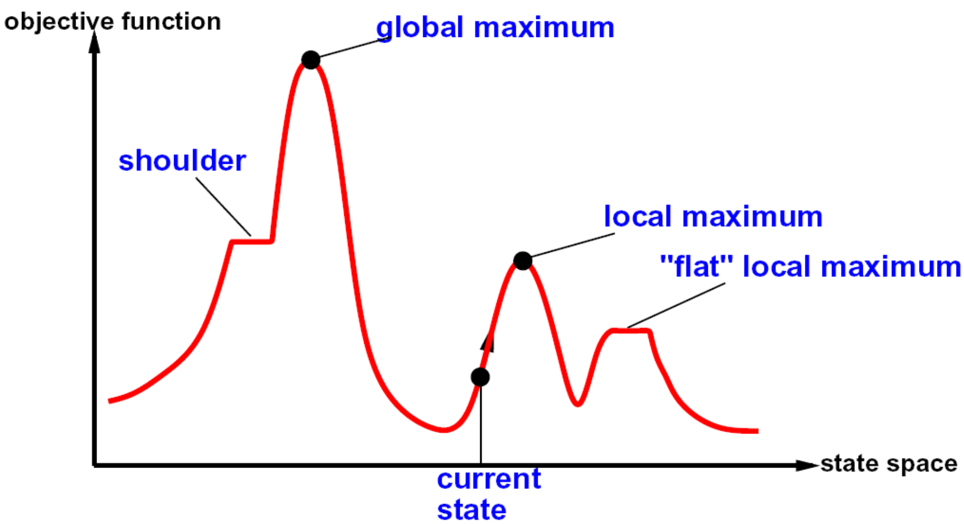



- Generally much faster and more memory efficient (but incomplete and suboptimal)

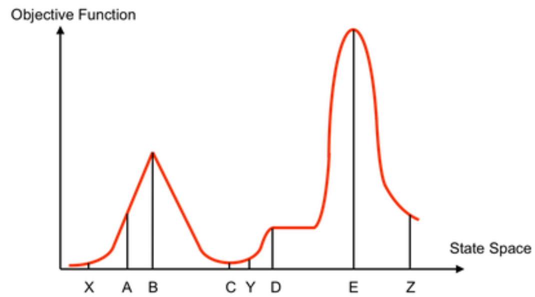
# Hill Climbing

- Simple, general idea:
  - Start wherever
  - Repeat: move to the best neighboring state
  - If no neighbors better than current, quit
- What's bad about this approach?
  - Complete?
  - Optimal?
- What's good about it?

# Hill Climbing Diagram



# Hill Climbing Quiz



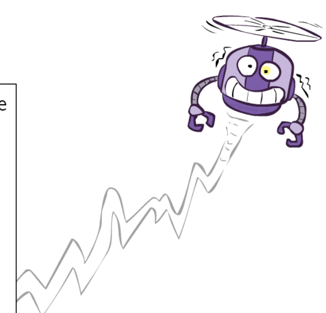

Starting from X, where do you end up ?

Starting from Y, where do you end up ?

Starting from Z, where do you end up ?

# Simulated Annealing

- Idea: Escape local maxima by allowing downhill moves
  - But make them rarer as time goes on



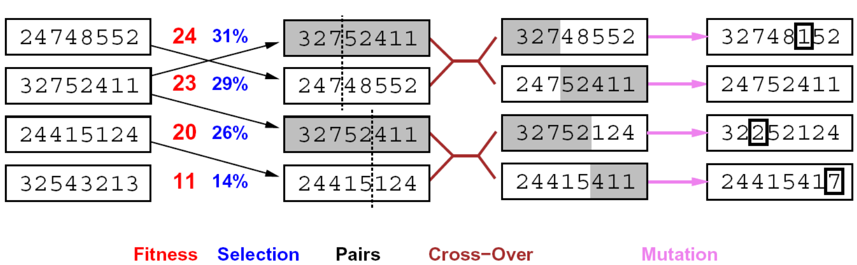
**function** SIMULATED-ANNEALING( *problem, schedule*) **returns** a solution state
  **inputs**: *problem*, a problem
        *schedule*, a mapping from time to "temperature"
  **local variables**: *current*, a node
        *next*, a node
        $T$, a "temperature" controlling prob. of downward steps

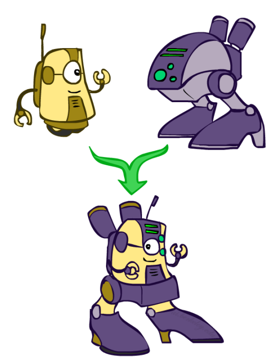
  *current* ← MAKE-NODE(INITIAL-STATE[*problem*])
  **for** $t \leftarrow 1$ **to** $\infty$ **do**
    $T \leftarrow$ *schedule*[$t$]
    **if** $T = 0$ **then return** *current*
    *next* ← a randomly selected successor of *current*
    $\Delta E \leftarrow$ VALUE[*next*] – VALUE[*current*]
    **if** $\Delta E > 0$ **then** *current* ← *next*
    **else** *current* ← *next* only with probability $e^{\Delta E/T}$

# Simulated Annealing

- Theoretical guarantee:
  - Stationary distribution: $p(x) \propto e^{\frac{E(x)}{kT}}$
  - If T decreased slowly enough, will converge to optimal state!
- Is this an interesting guarantee?
- Sounds like magic, but reality is reality:
  - The more downhill steps you need to escape a local optimum, the less likely you are to ever make them all in a row
  - People think hard about *ridge operators* which let you jump around the space in better ways

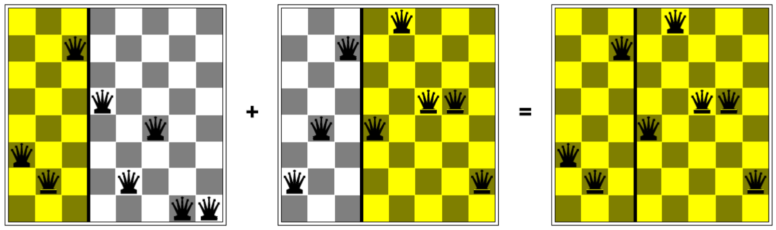
# Genetic Algorithms



- Genetic algorithms use a natural selection metaphor
  - Keep best N hypotheses at each step (selection) based on a fitness function
  - Also have pairwise crossover operators, with optional mutation to give variety
- Possibly the most misunderstood, misapplied (and even maligned) technique around

## Example: N-Queens




- Why does crossover make sense here?
- When wouldn't it make sense?
- What would mutation be?
- What would a good fitness function be?

## Next Time: Adversarial Search!