

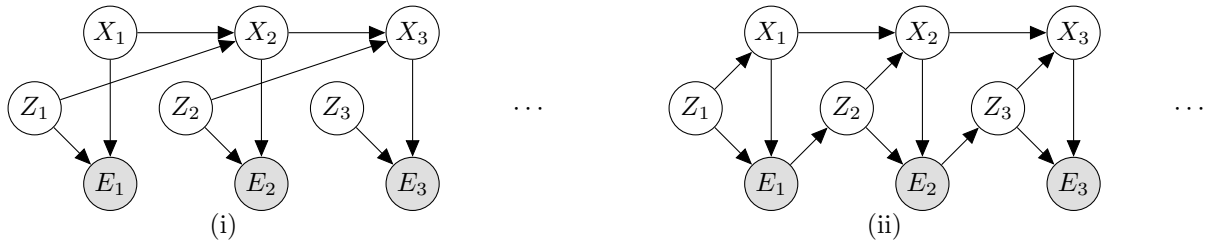
Q1. Modified HMM Updates

- (a) Recall that for a standard HMM the Elapse Time update and the Observation update are of the respective forms:

$$P(X_t | e_{1:t-1}) = \sum_{x_{t-1}} P(X_t | x_{t-1})P(x_{t-1} | e_{1:t-1})$$

$$P(X_t | e_{1:t}) \propto P(X_t | e_{1:t-1})P(e_t | x_t)$$

We now consider the following two HMM-like models:



Mark the modified Elapse Time update and the modified Observation update that correctly compute the beliefs from the quantities that are available in the Bayes' Net. (Mark one of the first set of six options, and mark one of the second set of six options for (i), and same for (ii).)

- (i) $P(X_t, Z_t | e_{1:t-1}) = \sum_{x_{t-1}, z_{t-1}} P(x_{t-1}, z_{t-1} | e_{1:t-1})P(X_t | x_{t-1}, z_{t-1})P(Z_t)$
 $P(X_t, Z_t | e_{1:t-1}) = \sum_{x_{t-1}, z_{t-1}} P(x_{t-1}, z_{t-1} | e_{1:t-1})P(X_t | x_{t-1}, z_{t-1})$
 $P(X_t, Z_t | e_{1:t-1}) = \sum_{x_{t-1}, z_{t-1}} P(x_{t-1}, z_{t-1} | e_{1:t-1})P(X_t, Z_t | x_{t-1}, z_{t-1})$
 $P(X_t, Z_t | e_{1:t-1}) = \sum_{x_{t-1}} P(x_{t-1}, z_{t-1} | e_{1:t-1})P(X_t | x_{t-1}, z_{t-1})P(Z_t)$
 $P(X_t, Z_t | e_{1:t-1}) = \sum_{x_{t-1}} P(x_{t-1}, z_{t-1} | e_{1:t-1})P(X_t | x_{t-1}, z_{t-1})$
 $P(X_t, Z_t | e_{1:t-1}) = \sum_{x_{t-1}} P(x_{t-1}, z_{t-1} | e_{1:t-1})P(X_t, Z_t | x_{t-1}, z_{t-1})$

In the elapse time update, we want to get from $P(X_{t-1}, Z_{t-1}|e_{1:t-1})$ to $P(X_t, Z_t|e_{1:t-1})$.

$$\begin{aligned} P(X_t, Z_t|e_{1:t-1}) &= \sum_{x_{t-1}, z_{t-1}} P(X_t, Z_t, x_{t-1}, z_{t-1}|e_{1:t-1}) \\ &= \sum_{x_{t-1}, z_{t-1}} P(x_{t-1}, z_{t-1}|e_{1:t-1})P(X_t|x_{t-1}, z_{t-1}, e_{1:t-1})P(Z_t|X_t, x_{t-1}, z_{t-1}, e_{1:t-1}) \\ &= \sum_{x_{t-1}, z_{t-1}} P(x_{t-1}, z_{t-1}|e_{1:t-1})P(X_t|x_{t-1}, z_{t-1})P(Z_t) \end{aligned}$$

First line: marginalization, second line: chain rule, third line: conditional independence assumptions.

- $P(X_t, Z_t | e_{1:t}) \propto P(X_t, Z_t | e_{1:t-1})P(e_t | X_t, Z_t)$
 $P(X_t, Z_t | e_{1:t}) \propto \sum_{X_t} P(X_t, Z_t | e_{1:t-1})P(e_t | X_t, Z_t)$
 $P(X_t, Z_t | e_{1:t}) \propto \sum_{Z_t} P(X_t, Z_t | e_{1:t-1})P(e_t | X_t, Z_t)$
 $P(X_t, Z_t | e_{1:t}) \propto P(X_t, Z_t | e_{1:t-1})P(e_t | X_t)P(e_t | Z_t)$
 $P(X_t, Z_t | e_{1:t}) \propto P(X_t, Z_t | e_{1:t-1})P(e_t | X_t)$
 $P(X_t, Z_t | e_{1:t}) \propto P(X_t, Z_t | e_{1:t-1}) \sum_{X_t} P(e_t | X_t)$

In the observation update, we want to get from $P(X_t, Z_t | e_{1:t-1})$ to $P(X_t, Z_t | e_{1:t})$.

$$\begin{aligned} P(X_t, Z_t | e_{1:t}) &\propto P(X_t, Z_t, e_t | e_{1:t-1}) \\ &\propto P(X_t, Z_t | e_{1:t-1}) P(e_t | X_t, Z_t, e_{1:t-1}) \\ &\propto P(X_t, Z_t | e_{1:t-1}) P(e_t | X_t, Z_t) \end{aligned}$$

First line: normalization, second line: chain rule, third line: conditional independence assumptions.

- (ii)
- $P(X_t, Z_t | e_{1:t-1}) = \sum_{x_{t-1}, z_{t-1}} P(x_{t-1}, z_{t-1} | e_{1:t-1})P(X_t | x_{t-1}, z_{t-1})P(Z_t | e_{t-1})$
 - $P(X_t, Z_t | e_{1:t-1}) = \sum_{x_{t-1}, z_{t-1}} P(x_{t-1}, z_{t-1} | e_{1:t-1})P(Z_t | e_{t-1})P(X_t | x_{t-1}, Z_t)$
 - $P(X_t, Z_t | e_{1:t-1}) = \sum_{x_{t-1}, z_{t-1}} P(x_{t-1}, z_{t-1} | e_{1:t-1})P(X_t, Z_t | x_{t-1}, e_{t-1})$
 - $P(X_t, Z_t | e_{1:t-1}) = \sum_{x_{t-1}} P(x_{t-1}, z_{t-1} | e_{1:t-1})P(X_t | x_{t-1}, z_{t-1})P(Z_t | e_{t-1})$
 - $P(X_t, Z_t | e_{1:t-1}) = \sum_{x_{t-1}} P(x_{t-1}, z_{t-1} | e_{1:t-1})P(Z_t | e_{t-1})P(X_t | x_{t-1}, Z_t)$
 - $P(X_t, Z_t | e_{1:t-1}) = \sum_{x_{t-1}} P(x_{t-1}, z_{t-1} | e_{1:t-1})P(X_t, Z_t | x_{t-1}, e_{t-1})$

In the elapse time update, we want to get from $P(X_{t-1}, Z_{t-1}|e_{1:t-1})$ to $P(X_t, Z_t|e_{1:t-1})$.

$$\begin{aligned}
P(X_t, Z_t|e_{1:t-1}) &= \sum_{x_{t-1}, z_{t-1}} P(X_t, Z_t, x_{t-1}, z_{t-1}|e_{1:t-1}) \\
&= \sum_{x_{t-1}, z_{t-1}} P(x_{t-1}, z_{t-1}|e_{1:t-1})P(Z_t|x_{t-1}, z_{t-1}, e_{1:t-1})P(X_t|Z_t, x_{t-1}, z_{t-1}, e_{1:t-1}) \\
&= \sum_{x_{t-1}, z_{t-1}} P(x_{t-1}, z_{t-1}|e_{1:t-1})P(Z_t|e_{t-1})P(X_t|x_{t-1}, Z_t)
\end{aligned}$$

First line: marginalization, second line: chain rule, third line: conditional independence assumptions.

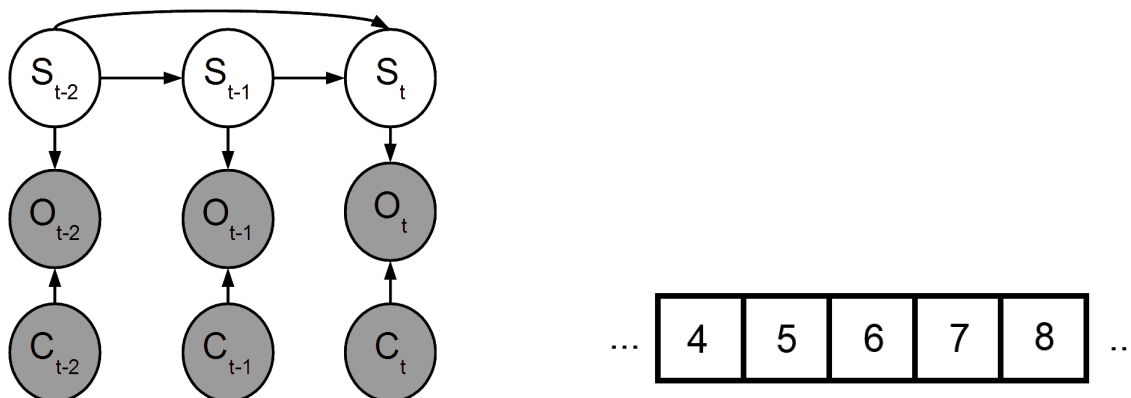
- $P(X_t, Z_t | e_{1:t}) \propto P(X_t, Z_t | e_{1:t-1})P(e_t | X_t, Z_t)$
- $P(X_t, Z_t | e_{1:t}) \propto \sum_{X_t} P(X_t, Z_t | e_{1:t-1})P(e_t | X_t, Z_t)$
- $P(X_t, Z_t | e_{1:t}) \propto \sum_{Z_t} P(X_t, Z_t | e_{1:t-1})P(e_t | X_t, Z_t)$
- $P(X_t, Z_t | e_{1:t}) \propto P(X_t, Z_t | e_{1:t-1})P(e_t | X_t)P(e_t | Z_t)$
- $P(X_t, Z_t | e_{1:t}) \propto P(X_t, Z_t | e_{1:t-1})P(e_t | X_t)$
- $P(X_t, Z_t | e_{1:t}) \propto P(X_t, Z_t | e_{1:t-1})\sum_{X_t} P(e_t | X_t)$

In the observation update, we want to get from $P(X_t, Z_t|e_{1:t-1})$ to $P(X_t, Z_t|e_{1:t})$.

$$\begin{aligned}
P(X_t, Z_t|e_{1:t}) &\propto P(X_t, Z_t, e_t|e_{1:t-1}) \\
&\propto P(X_t, Z_t|e_{1:t-1})P(e_t|X_t, Z_t, e_{1:t-1}) \\
&\propto P(X_t, Z_t|e_{1:t-1})P(e_t|X_t, Z_t)
\end{aligned}$$

First line: normalization, second line: chain rule, third line: conditional independence assumptions.

Q2. Particle Filtering



Pacman is trying to hunt a ghost in an infinite hallway with positions labeled as in the picture above. He's become more technologically savvy, and decided to locate find the ghost's actual position, \mathbf{S}_t , using some sensors he set up. From the sensors, Pacman can find, at each time step, a noisy reading of the ghost's location, \mathbf{O}_t . However, just as Pacman has gained technology, so has the ghost. It is able to cloak itself at each time step, given by \mathbf{C}_t , adding extra noise to Pacman's sensor readings.

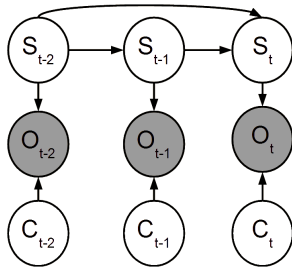
Pacman has generated an error model, given in the table below, for the sensor depending on whether the ghost is cloaked or not.

Pacman has also generated a dynamics model, given in the table below, that takes into account the position of the ghost at the two previous timesteps.

Dynamics model:			Observation model:		
$P(S_t S_{t-1}, S_{t-2}) = F(D_1, D_2)$			$P(O_t S_t, C_t) = E(C_t, D)$		
$D_1 = S_t - S_{t-1} $			$D = O_t - S_t $		
$D_2 = S_t - S_{t-2} $					
D_1	D_2	$F(D_1, D_2)$	C	D	$E(C, D)$
0	0	0.7	+	0	0.4
0	1	0.2	+	1	0.2
0	2	0	+	2	0.1
1	0	0.3	-	0	0.6
1	1	0.3	-	1	0.2
1	2	0.5	-	2	0

- (a) Assume that you currently have the following two particles: $(S_6 = 7, S_7 = 8)$ and $(S_6 = 6, S_7 = 6)$. Compute the weights for each particle given the observations $C_6 = +, C_7 = -, O_6 = 5, O_7 = 8$:

$(S_6 = 7, S_7 = 8)$	$Pr(O_6 = 5 C_6 = +, S_6 = 7) * Pr(O_7 = 8 C_7 = -, S_7 = 8) = 0.1 * 0.6 = 0.06$
$(S_6 = 6, S_7 = 6)$	$Pr(O_6 = 5 C_6 = +, S_6 = 6) * Pr(O_7 = 8 C_7 = -, S_7 = 6) = 0.2 * 0 = 0$



C	P(C)
+	0.5
-	0.5

- (b) Assume that Pacman can no longer see whether the ghost is cloaked or not, but assumes that it will be cloaked at each timestep with probability 0.5. Compute the weights for each particle given the observations $O_6 = 5$, $O_7 = 8$:

For each of the particle's states: we want to find $Pr(o_t|s_t)$, as this is the contribution to the weight of the sample. However, we have C unobserved, so: $Pr(o_t|s_t) = \sum_{c_t} Pr(o_t, c_t|s_t) = \sum_{c_t} Pr(c_t|s_t)Pr(o_t|s_t, c_t) = \sum_{c_t} Pr(c_t)Pr(o_t|s_t, c_t)$. Last equality is due to independence between C_t and S_t .

$(S_6 = 7, S_7 = 8)$	$\sum_{c_6} Pr(c_6)Pr(O_6 = 5 S_6 = 7, c_6) * \sum_{c_7} Pr(c_7)Pr(O_7 = 8 S_7 = 8, c_7) = 0.05 * 0.5 = 0.025$
$(S_6 = 6, S_7 = 6)$	$\sum_{c_6} Pr(c_6)Pr(O_6 = 5 S_6 = 6, c_6) * \sum_{c_7} Pr(c_7)Pr(O_7 = 8 S_7 = 6, c_7) = 0.2 * 0.05 = 0.01$

- (c) To prevent error propagation, assume that after weighting the particles and resampling, one of the particles you end up with is $(S_6 = 6, S_7 = 7)$.
- (i) What is the probability that after passing this particle through the dynamics model it becomes $(S_7 = 6, S_8 = 6)$?
0. It's invalid for a particle to start at $S_7 = 7$ and after one transition, become $S_7 = 6$.
- (ii) What is the probability the particle becomes $(S_7 = 7, S_8 = 8)$?
0.5. This is just $Pr(S_8 = 8|S_6 = 6, S_7 = 7) = F(D_1 = 1, D_2 = 2) = 0.5$.
- (d) To again decouple this part from previous parts, assume that you have the following three particles with the specified weights.

Particle	weight
$(S_7 = 5, S_8 = 6)$.1
$(S_7 = 7, S_8 = 6)$.25
$(S_7 = 7, S_8 = 7)$.3

What is Pacman's belief for the ghost's position at time $t = 8$?

Position	$P(S_8)$
$S_8 = 5$	$\frac{0}{.1+.25+.3} = 0$
$S_8 = 6$	$\frac{.1+.25}{.1+.25+.3} = \frac{.35}{.65} = \frac{7}{13}$
$S_8 = 7$	$\frac{.3}{.1+.25+.3} = \frac{.3}{.65} = \frac{6}{13}$
$S_8 = 8$	$\frac{0}{.1+.25+.3} = 0$