CS 188 Fall 2020 Introduction to Artificial Intelligence

# Exam Prep 11 Solutions

## Q1. Perceptron

We would like to use a perceptron to train a classifier for datasets with 2 features per point and labels +1 or -1.

Consider the following labeled training data:

| Features $(x_1, x_2)$ | Label $y^*$ |
|---|---|
| (-1,2) | 1 |
| (3,-1) | -1 |
| (1,2) | -1 |
| (3,1) | 1 |

**(a)** Our two perceptron weights have been initialized to $w_1 = 2$ and $w_2 = -2$. After processing the first point with the perceptron algorithm, what will be the updated values for these weights?

For the first point, $y = g(w_1x_1 + w_2x_2) = g(2 \cdot -1 + -2 \cdot 2) = g(-5) = -1$, which is incorrectly classified. To updathe weights, we add the first data point: $w_1 = 2 + (-1) = 1$ and $w_2 = -2 + 2 = 0$.

**(b)** After how many steps will the perceptron algorithm converge? Write "never" if it will never converge.

Note: one steps means processing one point. Points are processed in order and then repeated, until convergence.

The data is not seperable, so it will never converge.

**(c)** Instead of the standard perceptron algorithm, we decide to treat the perceptron as a single node neural network and update the weights using gradient descent on the loss function.

The loss function for one data point is $Loss(y, y^*) = (y - y^*)^2$, where $y^*$ is the training label for a given point and $y$ is the output of our single node network for that point.

**(i)** Given a general activation function $g(z)$ and its derivative $g'(z)$, what is the derivative of the loss function with respect to $w_1$ in terms of $g$, $g'$, $y^*$, $x_1$, $x_2$, $w_1$, and $w_2$?

$\frac{\partial Loss}{\partial w_1} = 2(g(w_1x_1 + w_2x_2) - y^*)g'(w_1x_1 + w_2x_2)x_1$

**(ii)** For this question, the specific activation function that we will use is:

$$g(z) = 1 \text{ if } z \geq 0 \text{ and } = -1 \text{ if } z < 0$$

Given the following gradient descent equation to update the weights given a single data point. With initial weights of $w_1 = 2$ and $w_2 = -2$, what are the updated weights after processing the first point?

Gradient descent update equation: $w_i = w_i - \alpha \frac{\partial Loss}{\partial w_1}$

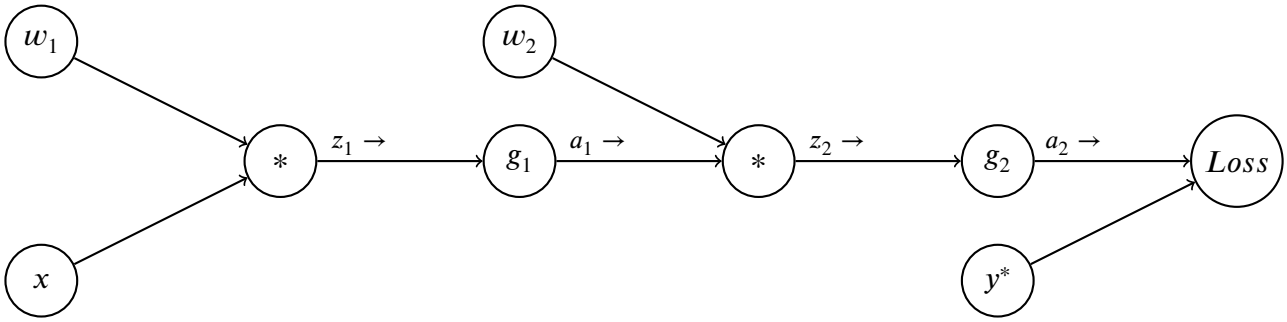
Because the gradient of $g$ is zero, the weights will stay $w_1 = 2$ and $w_2 = -2$.

**(iii)** What is the most critical problem with this gradient descent training process with that activation function?

The gradient of that activation function is zero, so the weights will not update.

# Q2. Neural Nets

Consider the following computation graph for a simple neural network for binary classification. Here $x$ is a single real-valued input feature with an associated class $y^\star$ (0 or 1). There are two weight parameters $w_1$ and $w_2$, and non-linearity functions $g_1$ and $g_2$ (to be defined later, below). The network will output a value $a_2$ between 0 and 1, representing the probability of being in class 1. We will be using a loss function $Loss$ (to be defined later, below), to compare the prediction $a_2$ with the true class $y^\star$.




1. Perform the forward pass on this network, writing the output values for each node $z_1, a_1, z_2$ and $a_2$ in terms of the node's input values:

$$z_1 = x * w_1$$
$$a_1 = g_1(z_1)$$
$$z_2 = a_1 * w_2$$
$$a_2 = g_2(z_2)$$

2. Compute the loss $Loss(a_2, y^\star)$ in terms of the input $x$, weights $w_i$, and activation functions $g_i$:

   Recursively substituting the values computed above, we have:

$$Loss(a_2, y^\star) = Loss(g_2(w_2 * g_1(w_1 * x)), y^\star)$$

3. Now we will work through parts of the backward pass, incrementally. Use the chain rule to derive $\frac{\partial Loss}{\partial w_2}$. Write your expression as a product of partial derivatives at each node: i.e. the partial derivative of the node's output with respect to its inputs. (Hint: the series of expressions you wrote in part 1 will be helpful; you may use any of those variables.)

$$\frac{\partial Loss}{\partial w_2} = \frac{\partial Loss}{\partial a_2} \frac{\partial a_2}{\partial z_2} \frac{\partial z_2}{\partial w_2}$$

4. Suppose the loss function is quadratic, $Loss(a_2, y^\star) = \frac{1}{2}(a_2 - y^\star)^2$, and $g_1$ and $g_2$ are both sigmoid functions $g(z) = \frac{1}{1+e^{-z}}$ (note: it's typically better to use a different type of loss, *cross-entropy*, for classification problems, but we'll use this to make the math easier).

   Using the chain rule from Part 3, and the fact that $\frac{\partial g(z)}{\partial z} = g(z)(1 - g(z))$ for the sigmoid function, write $\frac{\partial Loss}{\partial w_2}$ in terms of the values from the forward pass, $y^\star$, $a_1$, and $a_2$:

   First we'll compute the partial derivatives at each node:

$$\frac{\partial Loss}{\partial a_2} = (a_2 - y^\star)$$
$$\frac{\partial a_2}{\partial z_2} = \frac{\partial g_2(z_2)}{\partial z_2} = g_2(z_2)(1 - g_2(z_2)) = a_2(1 - a_2)$$
$$\frac{\partial z_2}{\partial w_2} = a_1$$

   Now we can plug into the chain rule from part 3:

$$\frac{\partial Loss}{\partial w_2} = \frac{\partial Loss}{\partial a_2}\frac{\partial a_2}{\partial z_2}\frac{\partial z_2}{\partial w_2}$$
$$= (a_2 - y^\star) * a_2(1 - a_2) * a_1$$

5. Now use the chain rule to derive $\frac{\partial Loss}{\partial w_1}$ as a product of partial derivatives at each node used in the chain rule:

$$\frac{\partial Loss}{\partial w_1} = \frac{\partial Loss}{\partial a_2}\frac{\partial a_2}{\partial z_2}\frac{\partial z_2}{\partial a_1}\frac{\partial a_1}{\partial z_1}\frac{\partial z_1}{\partial w_1}$$

6. Finally, write $\frac{\partial Loss}{\partial w_1}$ in terms of $x, y^\star, w_i, a_i, z_i$: The partial derivatives at each node (in addition to the ones we computed in Part 4) are:

$$\frac{\partial z_2}{\partial a_1} = w_2$$
$$\frac{\partial a_1}{\partial z_1} = \frac{\partial g_1(z_1)}{\partial z_1} = g_1(z_1)(1 - g_1(z_1)) = a_1(1 - a_1)$$
$$\frac{\partial z_1}{\partial a_1} = x$$

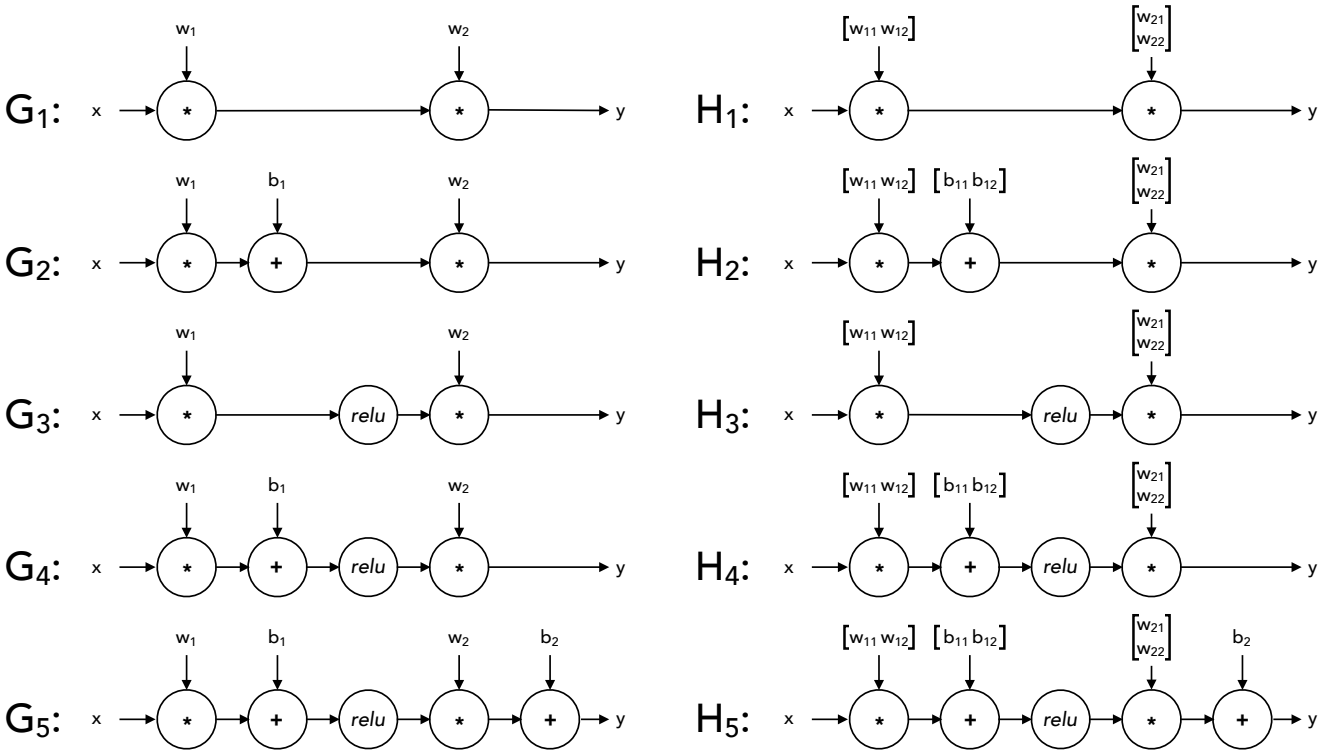
   Plugging into the chain rule from Part 5 gives:

$$\frac{\partial Loss}{\partial w_1} = \frac{\partial Loss}{\partial a_2}\frac{\partial a_2}{\partial z_2}\frac{\partial z_2}{\partial a_1}\frac{\partial a_1}{\partial z_1}\frac{\partial z_1}{\partial w_1}$$
$$= (a_2 - y^\star) * a_2(1 - a_2) * w_2 * a_1(1 - a_1) * x$$

7. What is the gradient descent update for $w_1$ with step-size $\alpha$ in terms of the values computed above?

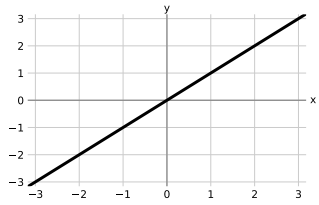
$$w_1 \leftarrow w_1 - \alpha(a_2 - y^\star) * a_2(1 - a_2) * w_2 * a_1(1 - a_1) * x$$

# Q3. [Timed: 12mins] Neural Networks: Representation

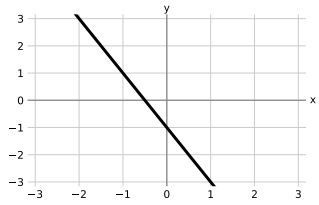



For each of the piecewise-linear functions below, mark all networks from the list above that can represent the function **exactly** on the range $x \in (-\infty, \infty)$. In the networks above, *relu* denotes the element-wise ReLU nonlinearity: $relu(z) = max(0, z)$. The networks $G_i$ use 1-dimensional layers, while the networks $H_i$ have some 2-dimensional intermediate layers.

**(a)**



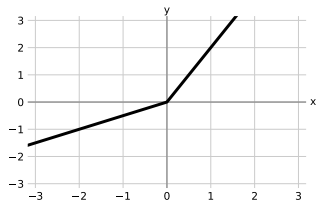
- [x] $G_1$ [x] $H_1$
- [x] $G_2$ [x] $H_2$
- [ ] $G_3$ [ ] $H_3$
- [ ] $G_4$ [x] $H_4$
- [ ] $G_5$ [x] $H_5$
- ○ None of the above



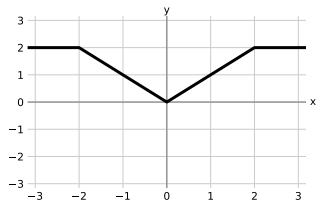
- [ ] $G_1$ [ ] $H_1$
- [x] $G_2$ [x] $H_2$
- [ ] $G_3$ [ ] $H_3$
- [ ] $G_4$ [x] $H_4$
- [ ] $G_5$ [x] $H_5$
- ○ None of the above

The networks $G_3, G_4, G_5$ include a ReLU nonlinearity on a scalar quantity, so it is impossible for their output to represent a non-horizontal straight line. On the other hand, $H_3, H_4, H_5$ have a 2-dimensional hidden layer, which allows two ReLU elements facing in opposite directions to be added together to form a straight line. The second subpart requires a bias term because the line does not pass through the origin.

**(b)**



- [ ] $G_1$ [ ] $H_1$
- [ ] $G_2$ [x] $H_2$
- [ ] $G_3$ [x] $H_3$
- [ ] $G_4$ [x] $H_4$
- [ ] $G_5$ [x] $H_5$
- ○ None of the above



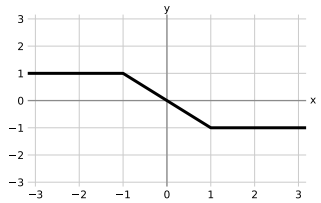
- [ ] $G_1$ [ ] $H_1$
- [ ] $G_2$ [ ] $H_2$
- [ ] $G_3$ [ ] $H_3$
- [ ] $G_4$ [ ] $H_4$
- [ ] $G_5$ [ ] $H_5$
- ● None of the above

These functions include multiple non-horizontal linear regions, so they cannot be represented by any of the networks $G_i$ which apply ReLU no more than once to a scalar quantity.

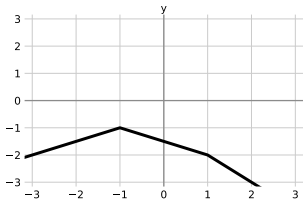
The first subpart can be represented by any of the networks with 2-dimensional ReLU nodes. The point of nonlinearity occurs at the origin, so nonzero bias terms are not required.

The second subpart has 3 points where the slope changes, but the networks $H_i$ only have a single 2-dimensional ReLU node. Each application of ReLU to one element can only introduce a change of slope for a single value of $x$.

**(c)**




☐ $G_1$ ☐ $H_1$
☐ $G_2$ ☐ $H_2$
☐ $G_3$ ☐ $H_3$
☐ $G_4$ ☐ $H_4$
☐ $G_5$ ■ $H_5$
○ None of the above




☐ $G_1$ ☐ $H_1$
☐ $G_2$ ☐ $H_2$
☐ $G_3$ ☐ $H_3$
☐ $G_4$ ■ $H_4$
☐ $G_5$ ■ $H_5$
○ None of the above

Both functions have two points where the slope changes, so none of the networks $G_i$; $H_1$, $H_2$ can represent them.

An output bias term is required for the first subpart because one of the flat regions must be generated by the flat part of a ReLU function, but neither one of them is at $y = 0$.

The second subpart doesn't require a bias term at the output: it can be represented as $-relu(\frac{-x+1}{2}) - relu(x + 1)$. Note how if the segment at $x > 2$ were to be extended to cross the x axis, it would cross exactly at $x = -1$, the location of the other slope change. A similar statement is true for the segment at $x < -1$.